The Strong Finiteness of Fermat, Double Fermat and Catalan-type Fermat Primes

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Abstract

In this paper we present that so-called double Fermat numbers are an infinite subset of well-known Fermat numbers and so-called Catalan-type Fermat numbers are also an infinite subset of Fermat numbers as well as double Fermat primes and Catalan-type Fermat primes are all strongly finite as Fermat primes do. From it we get the same result that composite Fermat numbers, composite double Fermat numbers and composite Catalan-type Fermat numbers are all infinite.

Keywords: Fermat number; double Fermat number; Catalan-type Fermat number; strong finiteness.

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1. Fermat numbers and strong finiteness of Fermat primes

As well known, if n is natural number (n=0,1,2,3,…) then $F_n = 2^{2^n}+1$ is called Fermat number. There are three open problems about Fermat numbers[1]:

Is F_n composite for all n>4?

Are there infinitely many Fermat primes?

Are there infinitely many composite Fermat numbers?

As of 2014 it is known that F_n is composite for $5 \leq n \leq 32$ and there is no any new Fermat prime found for n>32. It means answer of the first problem possibly is: Yes.

Our previous papers[2,3] give an argument for Fermat primes being strongly finite by the conjecture that Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes but such primes are absolutely finite if one of them is not Mersenne prime. So-called absolute finiteness will be called strong finiteness in this paper to lead the mathematical concept to be more clear. From Corollary 2.2 (Conjecture 2.2)[3] we see Fermat primes are strongly finite, which means F_n is composite for all n>4 and is just positive answer of the first problem.

From above discussion we see Fermat primes are strongly finite i.e. every Fermat number is composite except the first five Fermat numbers F_n for n=0,1,2,3,4 being primes. Obviously answer of the second problem should be negative, that is, Fermat primes are not infinite but finite. Since Fermat primes are finite but Fermat numbers are infinite, answer of the third problem should be positive, that is, there are infinitely many composite Fermat numbers.

Considering the recurrence relations to be suitable to Fermat numbers i.e. $F_{n+1}=(F_n-1)^2+1$ with $F_0=3$ for $n\geq 0$[1] and the Fermat numbers $F_{n+1}=(F_n-1)^2+1$ to be a subset of the set of numbers of the form x^2+1, we may expect that there are infinitely many composite numbers of the form x^2+1 from existence of infinitely many composite Fermat numbers.

2. Double Fermat numbers and strong finiteness of double Fermat primes

Definition 2.1 If F_n is a Fermat number then $F_{2^n}=2^{F_n-1}+1$ is called a double Fermat number.

Double Fermat numbers F_{2^n} are a subset of the set of Fermat numbers F_n and satisfy the recurrence relations $F_{2^{n+1}}=(F_{2^n}-1)^{F_n-1}+1$ with $F_{2^0}=5$ for $n\geq 0$[2].

Definition 2.2 If double Fermat number $F_{2^n}=2^{F_n-1}+1$ is prime then the double Fermat number F_{2^n} is called double Fermat prime.

Considering all double Fermat primes to arise from double Fermat numbers of the form $2^{F_n-1}+1$, we have the following definition.

Definition 2.3 All of natural numbers are called basic sequence of number of double Fermat primes.

From Definition 2.3 we see basic sequence of number of double Fermat primes is an infinite sequence because natural numbers are infinite. Further we have the following definition.

Definition 2.4 If the first few continuous natural numbers n make $2^{F_n-1}+1$ become double Fermat primes in basic sequence of number of double Fermat primes then these natural numbers are called original continuous natural number sequence of double Fermat primes.

Lemma 2.1 The original continuous natural number sequence of double Fermat primes is n = 0,1,2.

Proof. Since the first three double Fermat numbers F_{2^n} for n=0,1,2 i.e. F_{2^0} =5, F_{2^1} =17, F_{2^2} =65537 all are double Fermat primes but the fourth double Fermat number $F_{2^3}=2^{F_3-1}+1$=F_8 is not double Fermat prime, by Definition 2.4 we can confirm there exists an original continuous natural number sequence of double Fermat primes i.e. n = 0,1,2.

Definition 2.5 Double Fermat primes are strongly finite if the first few continuous terms generated from the original continuous natural number sequence are prime but all larger terms are composite.

Conjecture 2.1 Double Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes, but such primes are strongly finite if one of them is not Mersenne prime.

Corollary 2.1 If Conjecture 2.1 is true, then double Fermat primes are strongly finite.

Proof. Since the sum of original continuous natural number sequence of double Fermat primes i.e. 0+1+2=3 is a Mersenne prime i.e. M_2=3 but the first double Fermat prime $F_{2^0} = 5$ is not a Mersenne prime, we will get the result.

Since double Fermat primes are strongly finite but double Fermat numbers are infinite, composite double Fermat numbers are infinite.

3. Catalan-type Fermat numbers and strong finiteness of Catalan-type Fermat primes

Definition 3.1 A part of Fermat numbers $F_c(n)$, which satisfy the recurrence relations $F_c(n+1) = 2^{F_c(n)-1}+1$ with $F_c(0) = F_0 = 3$ for $n \geq 0$, are called Catalan-type Fermat numbers[4].

An anonymous writer proposed that such numbers were all prime, however, this conjecture was refuted when Selfridge showed F_{16} is composite in 1953[5]. Catalan-type Fermat numbers are infinite and grow very quickly:

$F_c(0) = F_0 = 3$

$F_c(1) = 2^{F_c(0)-1}+1 = F_{2^0} = F_1 = 5$

$F_c(2) = 2^{F_c(1)-1}+1 = F_{2^1} = F_2 = 17$

$F_c(3) = 2^{F_c(2)-1}+1 = F_{2^2} = F_4 = 65537$

$F_c(4) = 2^{F_c(3)-1} + 1 = F_{2^4} = F_{16} = 2^{65536}+1$

$F_c(5) = 2^{F_c(4)-1} + 1 = F_{2^{16}} = F_{65536}$

$F_c(6) = 2^{F_c(5)-1} + 1 = F_{2^{65536}}$

…

Catalan-type Fermat numbers are another subset of the set of Fermat numbers but are not a subset of double Fermat numbers since the first Catalan-type Fermat number $F_c(0) = F_0 = 3$ is a Fermat number but is not a double Fermat number though $F_c(1)$ and all of the following Catalan-type Fermat numbers are double Fermat numbers.

Definition 3.2 If Catalan-type Fermat number $F_c(n)$ is prime then the Catalan-type Fermat number $F_c(n)$ is called Catalan-type Fermat prime.

Considering all Catalan-type Fermat primes to arise from Catalan-type Fermat numbers generated from the recurrence relations $F_c(n+1) = 2^{F_c(n)-1} + 1$ with $F_c(0) = F_0 = 3$ for $n \geq 0$, we have the following definition.

Definition 3.3 All of natural numbers are called basic sequence of number of Catalan-type Fermat primes.

From Definition 3.3 we see basic sequence of number of Catalan-type Fermat primes is an infinite sequence because natural numbers are infinite. Further we have the following definition.

Definition 3.4 If the first few continuous natural numbers n make $F_c(n)$ become Catalan-type Fermat primes in basic sequence of number of Catalan-type Fermat primes then these natural numbers are called original continuous natural number sequence of Catalan-type Fermat primes.

Lemma 3.1 The original continuous natural number sequence of Catalan-type Fermat primes is $n = 0,1,2,3$.

Proof. Since the first four Catalan-type Fermat numbers $F_c(n)$ for n=0,1,2,3 i.e. $F_c(0) = 3$, $F_c(1) = 5$, $F_c(2) = 17$, $F_c(3) = 65537$ are Catalan-type Fermat primes but the fifth Catalan-type Fermat number $F_c(4) = F_{16}$ is not Catalan-type Fermat prime, by Definition 3.4 we can confirm there exists an original continuous natural number sequence of Catalan-type Fermat primes i.e. $n = 0,1,2,3$.

Definition 3.5 Catalan-type Fermat primes are strongly finite if the first few continuous terms generated from the original continuous natural number sequence are prime but all larger terms are composite.

Conjecture 3.1 Catalan-type Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes, but such primes are strongly finite if one of them is not Mersenne prime.

Corollary 3.1 If Conjecture 3.1 is true, then Catalan-type Fermat primes are strongly finite.

Proof. Since the first Catalan-type Fermat prime $F_c(0) = 3$ is a Mersenne prime i.e. M_2=3 but the sum of original continuous natural number sequence of Catalan-type Fermat primes i.e. 0+1+2+3=6 is not a Mersenne prime, we will get the result.

Since Catalan-type Fermat primes are strongly finite but Catalan-type Fermat numbers are infinite, composite Catalan-type Fermat numbers are infinite.

From discussion in this paper we see if our previous Conjecture 2.2 about Fermat primes[3] is generalized to double Fermat and Catalan-type Fermat primes then we will get the same result: Fermat, double Fermat and Catalan-type Fermat primes are all strongly finite.

References

[1]. Fermat number in The On-Line Wikipedia.
http://en.wikipedia.org/wiki/Fermat_number

[2]. Pingyuan Zhou, On the Connections between Mersenne and Fermat Primes, Glob. J. Pure Appl. Math. Vol.8, No.4(2012),453-458. Full text is available at EBSCO-ASC accession 86232958.
http://connection.ebscohost.com/c/articles/86232958/connections-between-mersenne-fermat-primes

[3]. Pingyuan Zhou, On the Existence of infinitely many Mersenne Primes and Finitely Many Fermat Primes in The On-Line viXra.
http://vixra.org/pdf/1407.0081v1.pdf

[4]. Pingyuan Zhou, Catalan-type Fermat Numbers, Glob. J. Pure Appl. Math. Vol.8, No.5(2012),579-582. Full text is available at EBSCO-ASC accession 86232974. http://connection.ebscohost.com/c/articles/86232974/catalan-type-fermat-number

[5]. Fermat Number in The On-Line Wolfram MathWorld. http://mathworld.wolfram.com/FermatNumber.html