On the Existence of Infinitely Many Mersenne Primes and Finitely Many Fermat Primes

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Abstract

From existence of the intersection of the set of Mersenne primes and the set of Fermat primes being a set to contain only one element 3 to be the first Mersenne prime and also the first Fermat prime we fell there are connections between Mersenne and Fermat primes. In this paper, it is presented that two symmetric conjectures related to Mersenne and Fermat primes themselves will lead us to expect Mersenne primes to be infinite but Fermat primes to be finite.

Keywords: Mersenne prime; Fermat prime; basic sequence of number; original continuous prime (natural) number sequence; absolute finiteness.

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1. Introduction

Of many kinds of prime numbers in the list of prime numbers[1], Mersenne primes and Fermat primes are two important and interesting kinds of prime numbers. Let x be positive integer, it has been proved that primes of the form $2^x \pm 1$ only appear among Mersenne numbers $M_p=2^p-1$ and Fermat numbers $F_n= 2^{2^n}+1$, where p is prime number and n is natural number[2,3]. M_p is Mersenne prime if M_p is prime, and F_n is Fermat prime if F_n is prime. We have known every Mersenne prime must lead to appearing of a corresponding perfect number $P_p=(M_p^2+M_p)/2$ and the largest known prime has almost always been a Mersenne prime in modern times[2]. It is surprising that Gauss-Wantzel theorem shows existence of connections between Fermat primes and constructible polygons[4]. Obviously, such two kinds of primes belong to a common origin $2^x \pm 1$, so that there may be some unknown connections between Mersenne and Fermat primes.

Although there are only 48 known Mersenne primes to this day, it has been generally believed that Mersenne primes are infinite. Lenstra, Pomerance and Wagstaff have conjectured that there is an infinite number of Mersenne primes in studying the number of primes p less than x with 2^p-1 being prime[5], however, it has been an open problem because the conjecture has not been proved. In contrast, since Euler showed that the sixth Fermat number F_5 is composite in 1732, any new Fermat prime has not been found so that many mathematicians begin believe that Fermat primes are finite. Hardy and Wright give an argument for the conjecture that the number of Fermat primes is finite, however, as Hardy and Wright note, the Fermat numbers do not behave “ randomly ” in that they are pairwise relatively prime[6]. P. A. Panzone gives an argument for existence of infinite set of Mersenne and Fermat primes[7], however, from which we have three possible results: Mersenne primes are infinite but Fermat primes are finite, Fermat primes are infinite but Mersenne primes are finite, or Mersenne and Fermat primes are all infinite. From above statements we see the two problems are all very difficult. Perhaps it will lead us to a new way to discover connections between Mersenne and Fermat primes that Gauss-Wantzel theorem shows existence of connections between Fermat primes and constructible polygons, that is, we should note that the intersection of the set of Mersenne primes and the set of Fermat primes is a set to contain only one element 3 to be the first Mersenne prime and also the first Fermat prime, so that we may present conjectures related to Mersenne and Fermat primes themselves. Basing on such conjectures, we will discuss whether Mersenne primes are infinite but Fermat primes are finite.

2. Two Symmetric Conjectures Related to Mersenne and Fermat Primes Themselves

Definition 2.1 If p is a prime number then $M_p=2^p-1$ is called a Mersenne number.

Definition 2.2 If Mersenne number $M_p=2^p-1$ is prime then the number M_p is called Mersenne prime.

Considering all Mersenne primes to arise from Mersenne numbers of the form 2^p-1, we have the following definition.

Definition 2.3 All of primes are called basic sequence of number of Mersenne primes.

From Definition 2.3 we see basic sequence of number of Mersenne primes is an infinite sequence because prime numbers are infinite. Further we have the following definition.

Definition 2.4 If the first few continuous prime numbers p make $M_p=2^p-1$ become Mersenne primes in basic sequence of number of Mersenne primes then these prime numbers are called original continuous prime number sequence of Mersenne primes.

Lemma 2.1 The original continuous prime number sequence of Mersenne primes is p =2,3,5,7.

Proof. Since M_p for p = 2,3,5,7 are Mersenne primes but M_{11} is not Mersenne prime, by Definition 2.4 we can confirm there exists an original continuous prime number sequence of Mersenne primes i.e. p =2,3,5,7.

Definition 2.5 If n is a natural number (n = 0,1,2,3, …) then $F_n = 2^{2^n} +1$ is called a Fermat number.

Definition 2.6 If Fermat number $F_n = 2^{2^n} +1$ is prime then the Fermat number F_n is called Fermat prime.

Considering all Fermat primes to arise from Fermat numbers of the form $2^{2^n} +1$, we have the following definition.

Definition 2.7 All of natural numbers are called basic sequence of number of Fermat primes.

From Definition 2.7 we see basic sequence of number of Fermat primes is an infinite sequence because natural numbers are infinite. Further we have the following definition.

Definition 2.8 If the first few continuous natural numbers n make $F_n = 2^{2^n} +1$ become Fermat primes in basic sequence of number of Fermat primes then these natural numbers are called original continuous natural number sequence of Fermat primes.

Lemma 2.2 The original continuous natural number sequence of Fermat primes is n = 0,1,2,3,4.

Proof. Since F_n for n = 0,1,2,3,4 are Fermat primes but F_5 is not Fermat prime, by Definition 2.8 we can confirm there exists an original continuous natural number sequence of Fermat primes i.e. n = 0,1,2,3,4.

Definition 2.9 Mersenne primes are absolutely finite if the first few continuous terms generated from the original continuous prime number sequence are prime but all larger terms are composite.

Definition 2.10 Fermat primes are absolutely finite if the first few continuous terms generated from the original continuous natural number sequence are prime but all larger terms are composite.

Conjecture 2.1 Mersenne primes are infinite if both the sum of corresponding original continuous prime number sequence and the first such prime are Fermat primes, but such primes are absolutely finite if one of them is not Fermat prime.

Conjecture 2.2 Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Mersenne primes, but such primes are absolutely finite if one of them is not Mersenne prime.

Conjecture 2.1 and Conjecture 2.2 are a simplification of our previous two symmetric conjectures[8].

Corollary 2.1 If Conjecture 2.1 is true, then Mersenne primes are infinite.

Proof. Since the sum of original continuous prime number sequence of Mersenne primes i.e. 2+3+5+7=17 is a Fermat prime i.e. F_2 and the first Mersenne prime M_2=3 is a Fermat prime i.e. F_0, we will get the result.

Corollary 2.2 If Conjecture 2.2 is true, then Fermat primes are absolutely finite.

Proof. Since the first Fermat prime F_0=3 is a Mersenne prime i.e. M_2 but the sum of original continuous natural number sequence of Fermat primes i.e. 0+1+2+3+4=10 is not a Mersenne prime, we will get the result.

Lemma 2.3 If a prime number sequence is absolutely finite then the prime number sequence is finite.

Proof. Since absolute finiteness is a special case of finiteness, a prime number sequence to be absolutely finite will be finite.

From Corollary 2.1, Corollary 2.2 and Lemma 2.3 we see if Conjecture 2.1 and Conjecture 2.2 are true then Mersenne primes are infinite but Fermat primes are finite.

By our previous another conjecture basing on connections between the number of Mersenne primes within finite *p*-value and composite Fermat number[9] we got the same result: Mersenne primes are infinite but Fermat primes are finite.

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