An Application of Nevanlinna's Second Main Theorem

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Abstract. This paper use Nevanlinna's Second Main Theorem of the value distribution theory, we got an important conclusion by Riemann hypothesis.

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First, we give some notations, definitions and theorems in the theory of value distribution, its contents see the references [1] and [2].

We write

$$\log^+ x = \begin{cases} \log x & 1 \le x \\ 0 & 0 \le x < 1 \end{cases}$$

It is easy to see that $\log x \le \log^+ x$.

Let $f(z)$ is a non-constant meromorphic function in the circle $|z| < R$, $0 < R \le \infty$. $n(r,f)$ represents the number of poles of $f(z)$ on the circle $|z| \le r\,(0 < r < R)$, the multiplicity of poles is included. $n(0,f)$ represents the order of pole of $f(z)$ in the origin. For arbitrary complex number $a \ne \infty$, $n(r, \frac{1}{f-a})$ represents the number of zeros of $f(z) - a$ in the circle $|z| \le r$ $(0 < r < R)$, the multiplicity of zeros is included. $n(0, \frac{1}{f-a})$ represents the order of zero of $f(z) - a$ in the origin.

We write

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\varphi}) \right| d\varphi$$

$$N(r,f) \;=\; \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt \;+\; n(0,f)\,\log r$$

and $T(r,f) \;=\; m(r,f) \,+\, N(r,f)$.
$T(r,f)$ is called the characteristic function of $f(z)$.

LEMMA 1. If $f(z)$ is a analytical function in the circle $|z| < R$ ($0 < R \leq \infty$), we have

$$T(r,f) \;\leq\; \log^+ M(r,f) \;\leq\; \frac{\rho+r}{\rho-r}\,T(\rho\,,f)(\,0<r<\rho<R)$$

where $M(r,f) \;=\; \max_{|z|=r}\,|f(z)|$
The lemma 1 follows from the References [1], page 57.

LEMMA 2. Let $f(z)$ is a non-constant meromorphic function in the circle $|z| < R$ ($0 < R \leq \infty$). a_λ ($\lambda = 1,2,...,h$) and b_μ ($\mu = 1,2,...,k$) are the zeros and poles of $f(z)$ in the circle $|z| < \rho$ ($0 < \rho < R$) respectively, each zero or pole repeated according to their multiplicity, and $z = 0$ is neither zero nor pole of the function $f(z)$, then, in the circle $|z| < \rho$, we have the following formula

$$\log\,|\,f(0)\,| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\,\left|f(\rho e^{i\varphi})\right|d\varphi \;-\; \sum_{\lambda=1}^{h}\log\frac{\rho}{|a_\lambda|} \;+\; \sum_{\mu=1}^{k}\log\frac{\rho}{|b_\mu|}$$

this formula is called Jensen formula.
The lemma 2 follows from the References [1], page 48.

LEMMA 3. Let $f(z)$ is the meromorphic function in the circle $|z| \leq R$, and

$$f(0) \;\neq\; 0,\; \infty,\; 1, \quad f'(0) \;\neq\; 0$$

when $0 < r < R$, we have

$$T(r,f) \;<\; 2\left\{N(R,\frac{1}{f}) \;+\; N(R,f) \;+\; N(R,\frac{1}{f-1})\right\}$$

$$+\ 4\ \log^{+}|f(0)|\ +\ 2\ \log^{+}\frac{1}{R|f'(0)|}\ +\ 24\ \log\frac{R}{R-r}\ +\ 2328$$

This is a form of Nevanlinna's Second Main Theorem.

The lemma 3 follows from the References [1], the theorem 3.1 of the page 75.

Now, we make some preparations.

LEMMA 4. if $f(x)$ is a function of the nonnegative degressive, we have

$$\lim_{N\to\infty}\left(\sum_{n=a}^{N} f(n)\ -\ \int_a^N f(x)\,dx\right)\ =\ \alpha$$

where $0\le\alpha\le f(a)$. in addition, if $x\to\infty$, $f(x)\to 0$, we have

$$\left|\sum_{a\le n\le\xi} f(n)\ -\ \int_a^{\xi} f(\nu)\,d\nu\ -\ \alpha\right|\ \le\ f(\xi-1)\,,\qquad (\,\xi\ge a+1\,)$$

The lemma 3 follows from the References [3], the theorem 2 of the page 91.

Let $s=\sigma+it$ is the complex number, when $\sigma>1$, Riemann Zeta function is

$$\zeta(s)\ =\ \sum_{n=1}^{\infty}\frac{1}{n^s}$$

When $\sigma>1$, we have

$$\log\zeta(s)\ =\ \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log n}$$

where $\Lambda(n)$ is Mangoldt function.

LEMMA 5. If t is any real number, we have

(1)

$$0.0426 \ \leq \ | \ \log \zeta(4+it) \ | \ \leq \ 0.0824$$

(2)

$$| \ \zeta(4+it) \ - \ 1 \ | \ \geq \ 0.0426$$

(3)

$$0.917 \ \leq \ | \ \zeta(4+it) \ | \ \leq \ 1.0824$$

(4)

$$| \ \zeta'(4+it) \ | \ \geq \ 0.012$$

PROOF.

(1)

$$| \ \log \zeta(4+it) \ | \ \leq \ \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \ \leq \ \sum_{n=2}^{\infty} \frac{1}{n^4} \ = \ \frac{\pi^4}{90} \ - \ 1 \ \leq \ 0.0824$$

$$| \ \log \zeta(4+it) \ | \ \geq \ \frac{1}{2^4} \ - \ \sum_{n=3}^{\infty} \frac{1}{n^4} \ = \ 1 \ + \ \frac{2}{2^4} \ - \ \sum_{n=1}^{\infty} \frac{1}{n^4} \ = \ \frac{9}{8} \ - \ \frac{\pi^4}{90} \ \geq \ 0.0426$$

(2)

$$| \ \zeta(4+it) - 1 \ | \ = \ \left| \ \sum_{n=2}^{\infty} \frac{1}{n^{4+it}} \ \right| \ \geq \ \frac{1}{2^4} \ - \ \sum_{n=3}^{\infty} \frac{1}{n^4}$$

$$= \ 1 + \frac{2}{2^4} \ - \ \sum_{n=1}^{\infty} \frac{1}{n^4} \ = \ \frac{9}{8} \ - \ \frac{\pi^4}{90} \ \geq \ 0.0426$$

(3)

$$| \ \zeta(4+it) \ | \ = \ \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \ \leq \ \sum_{n=1}^{\infty} \frac{1}{n^4} \ = \ \frac{\pi^4}{90} \ \leq \ 1.0824$$

$$| \ \zeta(4+it) \ | \ = \ \left| \sum_{n=1}^{\infty} \frac{1}{n^{4+it}} \right| \ \geq \ 1 - \sum_{n=2}^{\infty} \frac{1}{n^4} \ = \ 2 - \sum_{n=1}^{\infty} \frac{1}{n^4} \ = \ 2 - \frac{\pi^4}{90} \ \geq \ 0.917$$

(4)

$$| \, \zeta'(4+it) \, | \; = \; \left| \sum_{n=2}^{\infty} \frac{\log n}{n^{4+it}} \right| \; \geq \; \frac{\log 2}{2^4} \; - \; \sum_{n=3}^{\infty} \frac{\log n}{n^4}$$

by Lemma 4, we have

$$\sum_{n=3}^{\infty} \frac{\log n}{n^4} \; = \; \int_3^{\infty} \frac{\log x}{x^4} \, dx \; + \; \alpha$$

where $0 \; \leq \; \alpha \; \leq \; \frac{\log 3}{3^4}$

$$\int_3^{\infty} \frac{\log x}{x^4} \, d\,x \; = \; - \, \frac{1}{3} \int_3^{\infty} \log x \; d\,x^{-3} \; = \; \frac{\log 3}{3^4} \; + \; \frac{1}{3} \int_3^{\infty} x^{-4} \, d\,x$$

$$= \; \frac{\log 3}{3^4} \; - \; \frac{1}{3^2} \int_3^{\infty} d\,x^{-3} \; = \; \frac{\log 3}{3^4} \; + \; \frac{1}{3^5}$$

therefore

$$\sum_{n=3}^{\infty} \frac{\log n}{n^4} \; \leq \; \frac{\log 3}{3^4} \; + \; \frac{1}{3^5} \; + \; \frac{\log 3}{3^4}$$

$$| \, \zeta'(4+it) \, | \; \geq \; \frac{\log 2}{2^4} \; - \; \frac{2 \log 3}{3^4} \; - \; \frac{1}{3^5} \; \geq \; 0.012$$

This completes the proof of Lemma 5.

Let $\delta = \frac{1}{100}$, c_1, c_2, ... , is the positive constant.

LEMMA 6. When $\sigma \; \geq \; \frac{1}{2}$, $|t| \geq 2$, we have

$$| \, \zeta(\sigma + it)| \; \leq \; c_1 \, |t|^{\frac{1}{2}}$$

The lemma 6 follows from the References [4], the theorem 2 of the page 140.

LEMMA 7. If $f(z)$ is the analytic function in the circle $|z - z_0| \; \leq \; R$, $0 < r < R$, in the circle $|z - z_0| \; \leq \; r$, we have

$$| f(z) - f(z_0) | \le \frac{2r}{R-r} \; (A(R) - Ref(z_0))$$

where $A(R) = \max_{|z-z_0| \le R} Ref(z)$.

The lemma 6 follows from the References [4], the theorem 2 of the page 61.

Now, we assume that Riemann hypothesis is correct, and abbreviation as RH. In other words, when $\sigma > \frac{1}{2}$, the function $\zeta(\sigma + it)$ has no zeros. The function $\log \zeta(\sigma + it)$ is a multi-valued analytic function in the region $\sigma > \frac{1}{2}, t \ge 1$. we choose the principal branch of the function $\log \zeta(\sigma + it)$, therefore, if $\zeta(\sigma + it) = 1$, then $\log \zeta(\sigma + it) = 0$.

LEMMA 8. If RH is correct, when $\delta = \frac{1}{100}$, $\sigma \ge \frac{1}{2} + 2\delta$, $t \ge 16$, we have

$$| \log \zeta(\sigma + it) | \le c_2 \log t + c_3$$

proof. In Lemma 7, we choose $f(z) = \log \zeta(z + 4 + it)$, $z_0 = 0$, $R = \frac{7}{2} - \delta$, $r = \frac{7}{2} - 2\delta$, $t \ge 16$. Because $\log \zeta(z + 4 + it)$ is the analytic function in the circle $|z| \le R$, by Lemma 7, in the circle $|z| \le r$, we have

$$| \log \zeta(z + 4 + it) - \log \zeta(4 + it) | \le \frac{7}{\delta} \; (A(R) - Re \log \zeta(4 + it))$$

therefore

$$| \log \zeta(z + 4 + it) | \le \frac{7}{\delta} \; (A(R) + | \log \zeta(4 + it) |) + | \log \zeta(4 + it) |$$

by Lemma 6, we have

$$A(R) = \max_{|z-z_0| \le R} \log | \zeta(z + 4 + it) | \le \frac{1}{2} \log t + \log c_1$$

by Lemma 5, we have

$$| \log \zeta(z + 4 + it) | \le c_2 \log t + c_3$$

therefore, when $\sigma \ge \frac{1}{2} + 2\delta$, we have

$$| \log \zeta(\sigma + it) | \le c_2 \log t + c_3$$

This completes the proof of Lemma 8.

LEMMA 9. If RH is correct, when $\delta = \frac{1}{100}$, $t \geq 16$, $\rho = \frac{7}{2} - 2\delta$, in the circle $|z| \leq \rho$, we have

$$N\left(\rho\,,\ \frac{1}{\zeta(z+4+it)\ -1}\right) \ \leq\ \log\log t\ +\ c_4$$

proof. In Lemma 2, we choose $f(z) = \log\zeta(z+4+it)$, $R = \frac{7}{2} - \delta$, $\rho = \frac{7}{2} - 2\delta$, a_λ $(\lambda = 1, 2, ..., h)$ are the zeros of the function $\log\zeta(z+4+it)$ in the circle $|z| < \rho$, each zero repeated according to their multiplicity. Because the function $\log\zeta(z+4+it)$ has no poles in the the circle $|z| < \rho$, and $\log\zeta(4+it)$ is not equal to zero, we have

$$\log|\log\,\zeta(4+it)\,| = \frac{1}{2\pi}\int_0^{2\pi}\log\,\big|\log\,\zeta(4+it+\rho e^{i\varphi})\,\big|\,d\varphi - \sum_{\lambda=1}^{h}\log\,\frac{\rho}{|a_\lambda|}$$

by Lemma 5 and Lemma 8, we have

$$\sum_{\lambda=1}^{h}\log\,\frac{\rho}{|a_\lambda|}\ \leq\ \log\log t\ +\ c_4$$

because $z = 0$ is neither zero nor pole of the function $\log\zeta(z+4+it)$, if r_0 is a sufficiently small positive number, we have

$$\sum_{\lambda=1}^{h}\log\,\frac{\rho}{|a_\lambda|}\ =\ \int_{r_0}^{\rho}\left(\log\frac{\rho}{t}\right)\ d\,n(t,\frac{1}{f})\ =\ \left[\left(\log\frac{\rho}{t}\right)\ n(t,\frac{1}{f})\right]\Bigg|_{r_0}^{\rho}$$

$$+\ \int_{r_0}^{\rho}\frac{n(t,\frac{1}{f})}{t}\ d\,t\ =\ \int_{0}^{\rho}\frac{n(t,\frac{1}{f})}{t}\ d\,t\ =\ N\left(\rho\,,\frac{1}{f}\right)$$

$$=\ N\left(\rho\,,\ \frac{1}{\log\zeta(z+4+it)}\right)\ \geq\ N\left(\rho\,,\ \frac{1}{\zeta(z+4+it)-1}\right)$$

This completes the proof of Lemma 9.

THEOREM. If RH is correct, when $\sigma \geq \frac{1}{2} + 4\delta$, $\delta = \frac{1}{100}$, $t \geq 16$, we have

$$|\zeta(\sigma + it)| \leq c_8 \,(\log t)^{c_6}$$

proof. In Lemma 3, we choose $f(z) = \zeta(z + 4 + it)$, $t \geq 16$, $R = \frac{7}{2} - 2\delta$, $r = \frac{7}{2} - 3\delta$. by Lemma 5, we have $f(0) = \zeta(4 + it) \neq 0, \infty, 1$, and $|f'(0)| = |\zeta'(4+it)| \geq 0.012$, $|f(0)| = |\zeta(4 + it)| \leq 1.0824$. because $\zeta(z + 4 + it)$ is the analytic function, and it have neither zeros nor poles in the circle $|z| \leq R$, we have

$$N\left(R, \frac{1}{f}\right) = 0 , \qquad N(R, f) = 0$$

therefore, by Lemma 9, we have

$$T\,(r, \zeta(z + 4 + it)) \leq 2 \log\log t + c_5$$

In Lemma 1, we choose $R = \frac{7}{2} - 2\delta$, $\rho = \frac{7}{2} - 3\delta$, $r = \frac{7}{2} - 4\delta$. by the maximal principle, in the circle $|z| \leq r$, we have

$$\log^+ |\zeta(z + 4 + it)| \leq c_6 \log\log t + c_7$$

therefore, when $\sigma \geq \frac{1}{2} + 4\delta$, we have

$$\log^+ |\zeta(\sigma + it)| \leq c_6 \log\log t + c_7$$

$$\log |\zeta(\sigma + it)| \leq c_6 \log\log t + c_7$$

$$|\zeta(\sigma + it)| \leq c_8 \,(\log t)^{c_6}$$

This completes the proof of Theorem.

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