# Randomness in quantum physics revisited.

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**Abstract.**
There has been proven that mathematical origins of randomness in quantum and Newtonian physics are coming from the same source that is dynamical ***instability***. However in Newtonian physics this instability is measured by positive finite Liapunov exponents averaged over infinite time period, while in quantum physics the instability is accompanied by a loss of the Lipchitz condition and represented by an infinite divergence of trajectories in a singular point. Although from a mathematical viewpoint such a difference is significant, from physical viewpoint it does not justify division of randomness into "deterministic "(chaos) and "true" (quantum physics). The common origin of randomness in Newtonian and quantum physics presents a support of the correspondence principle that is being searched by quantum chaos theory.

## 1. Introduction.

This work is motivated by quantum chaos that is a branch of physics which studies how chaotic classical dynamical systems can be described in terms of quantum theory. The primary question that quantum chaos seeks to answer is: "What is the relationship between quantum physics and classical chaos?" The correspondence principle states that classical mechanics is the classical limit of quantum mechanics. If this is true, then there must be quantum mechanisms underlying classical chaos, although this may not be a fruitful way of examining classical chaos. If quantum mechanics does not demonstrate an exponential sensitivity to initial conditions, how can exponential sensitivity to initial conditions arise in classical chaos, which must be the correspondence principle limit of quantum mechanics?

The objective of this work is to find more evidence in support of the correspondence principle, and in particular, to unify theory of transition from determinism to randomness in physics.

The concept of randomness entered Newtonian and quantum physics in different ways, but approximately at the same time. In 1926, Synge, J.L. introduced a new type of instability - orbital instability- in classical mechanics, [1], that can be considered as a precursor of chaos discovered a couple of decades later, [2]. In 1927, Heisenberg, W., [3] ***postulated*** randomness in quantum physics via the uncertainty principle. Since then it was a well-established opinion in the scientific community that there is a "deterministic " randomness, attributed to chaos, and a "true" randomness postulated by quantum physics. In this paper we contest this sub-division by providing a proof that the randomness in quantum physics does not have to be postulated: it follows from dynamics instability as randomness in Newtonian physics does. However in Newtonian physics this instability is measured by positive ***finite*** Liapunov exponents averaged over infinite time period, while in quantum physics the instability is accompanied by a loss of the Lipchitz condition and represented by an infinite divergence of trajectories at a singular point. Although from a mathematical viewpoint such a difference is significant, from physical viewpoint it does not justify division of randomness into "deterministic "(chaos) and "true" (quantum physics).

We start this paper with revisiting mathematical formalism of chaos in a non-traditional way that is based upon the concept of orbital instability. After that, turning to the Madelung version of the Schrödinger equation, we describe a transition from determinism to randomness in quantum mechanics. In the last part we discuss the discontinuity of the transition from quantum to Newtonian mechanics in context of randomness.

**2. Randomness in Newtonian physics.**

***a. Orbital instability as a precursor of chaos.***

Chaos is a special type of instability when the system does not have an alternative stable state and displays an irregular aperiodic motion. Obviously this kind of instability can be associated only with ignorable variables, i.e. with such variables that do not contribute into energy of the system. In order to demonstrate this kind of instability, consider an inertial motion of a particle M of unit mass on a smooth pseudosphere S having a constant negative curvature $G_0$, Fig. 1.

$$G_0 = const > 0 \qquad (1)$$

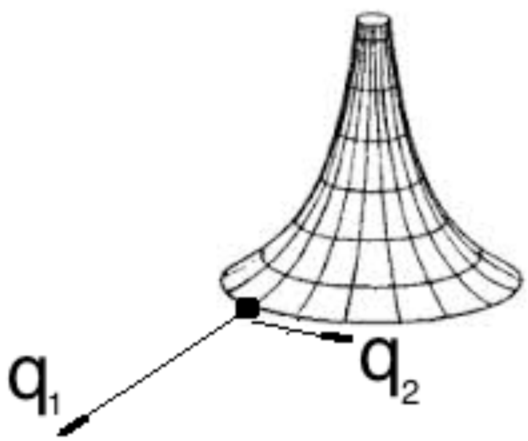



Figure 1. Inertial motion on a smooth pseudosphere.

Remembering that trajectories of inertial motions must be geodesics on S, compare two different trajectories assuming that initially they are parallel, and the distance $\varepsilon_0$ between them, are small (but not infinitesimal!),

$$0 < \varepsilon_0 << 1 \qquad (2)$$

As shown in differential geometry, the distance between these geodesics increases exponentially

$$\varepsilon = \varepsilon_0 e^{\sqrt{-G_0 t}}, \quad G_0 < 0, \qquad (3)$$

Hence no matter how small the initial distance $\varepsilon_0$, the current distance $\varepsilon$ tends to infinity.

Let us assume now that accuracy to which the initial conditions are known is characterized by the scale *L*. This means that any two trajectories cannot be distinguished if the distance between them is less than *L* i.e. if

$$\varepsilon < L \tag{4}$$

The period during which the inequality (4) holds has the order

$$\Delta t \approx \frac{1}{\sqrt{|-G_0|}} \ln \frac{L}{\varepsilon_0} \tag{5}$$

However for

$$t >> \Delta t \tag{6}$$

these two trajectories diverge such that they can be easily distinguished and must be considered as two different trajectories. Moreover the distance between them tends to infinity no matter how small is $\varepsilon_0$. That is why the motion once recorded cannot be reproduced again (unless the initial condition are known exactly), and consequently it attains stochastic features. The Liapunov exponent for this motion is positive and constant

$$\sigma = \lim_{\substack{t\to\infty \\ \varepsilon_0 \to 0}} [\frac{1}{t} \ln \frac{\varepsilon_0 e^{\sqrt{-G_0}t}}{\varepsilon_0}] = \sqrt{-G_0} = const > 0 \tag{7}$$

*Remark.* In theory of chaos, the Liapunov exponent measures divergence of initially close trajectories averaged over infinite period of time. But in this particular case, even "instantaneous" Liapunov exponent taken at a fixed time has the same value (7).

Let us introduce a system of coordinates on the surface S: the coordinate $q_1$ along the geodesic meridians and the coordinate $q_2$ along the parallels. In differential geometry such a system is called semigeodesic. The square distance between adjacent points on the pseudosphere is

$$ds = g_{11} dq_1^2 + 2q_{12} dq_1 dq_2 + g_{22} dq_2^2 \tag{8}$$

where

$$g_{11} = 1, \quad q_{12} = 0, \quad g_{22} = -\frac{1}{G_0} e^{(-2\sqrt{-G_0}q_1)} \tag{9}$$

The Lagrangian for the inertial motion of the particle *M* on the pseudosphere is expressed via the coordinates and their temporal derivatives as

$$L = g_{ij} \dot{q}_i \dot{q}_j = \dot{q}_1^2 - \frac{1}{G_0} e^{(-2\sqrt{-G_0}q_1)} \dot{q}_2^2 \tag{10}$$

and consequently,

$$\frac{\partial L}{\partial q_2} = 0 \tag{11}$$

$$\frac{\partial L}{\partial q_1} \neq 0 \quad if \quad \dot{q}_2 \neq 0 \tag{12}$$

Hence $q_1$ and $q_2$ play the roles of position and ignorable coordinates, respectively, and therefore, the inertial motion of a particle on a smooth pseudosphere is unstable with respect to the ***ignorable*** coordinate. This instability known as orbital instability is not bounded by energy and it can persist indefinitely. As shown in [2], eventually orbital instability leads to stochasticity. Later on such motions were identified as chaotic.

***b. Randomness in chaotic systems.***

In this sub-section we present a sketch of general theory of chaos in context of origin of randomness starting with the flow generated by an autonomous ODE

$$\frac{dx_i}{dt} = V_i(\mathbf{x}), \quad i = 1,2,\ldots m \tag{13}$$

and compare two neighboring trajectories in *m*-dimensional phase space with initial conditions $x_0$ and $x_0 + \Delta x_0$ denoting $\Delta x_0 = w$. These evolve with time yielding the tangent vector $\Delta x(x_0, t)$ with its Euclidian norm

$$d(x_0, t) = \left\| \Delta x(x_0, t) \right\| \tag{14}$$

Now the Liapunov exponent can be introduced as the mean exponential rate of divergence of two initially close trajectories

$$\tilde{\lambda}(x_0, w) = \lim_{\substack{t\to\infty \\ d(0)\to 0}} \left(\frac{1}{t}\right) \ln \frac{d(x_0, t)}{d(x_0, 0)} \tag{15}$$

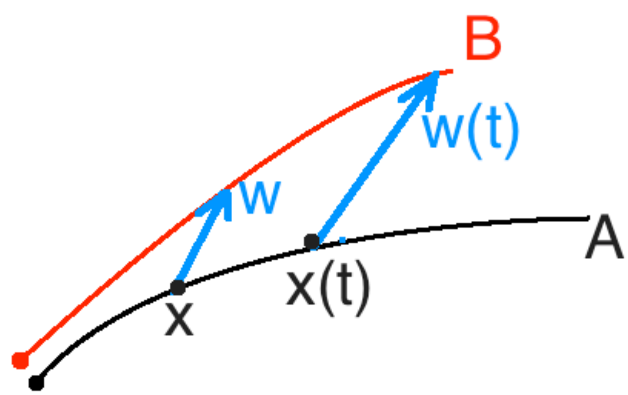
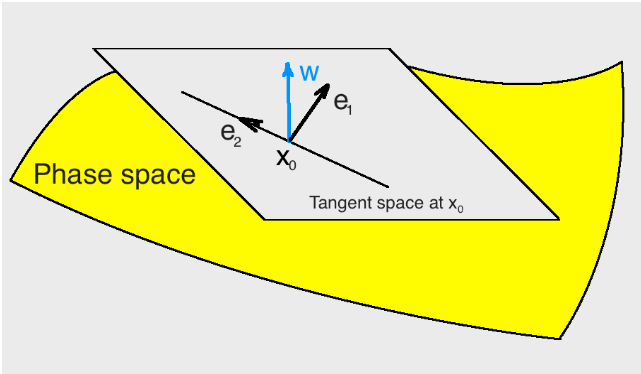



Figure 2. Two nearby trajectories that separate as time evolves.




Ficure 3. Tangent space for the Liapunov exponents.

Therefore in general the Lyapunov exponent cannot be analytically expressed via the parameters of the underlying dynamical system (as it was done in case of inertial motion on a pseudosphere), and that makes prediction of chaos a hard task. However some properties of the Liapunov exponents can be expressed in an analytical form. Firstly, it can be shown that in an $m$-dimensional space, there exist $m$ Liapunov exponents

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \ldots \geq \tilde{\lambda}_m \tag{16}$$

while at least one of them must vanish. Indeed, as follows from Eqs. (13) and (14), **w** grows only linearly in the direction of the flow, and the corresponding Liapunov exponent is zero. Secondly it has been proven that the sum of the Liapunov exponents is equal to the average phase space volume contraction

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \Lambda_0 \tag{17}$$

where the instantaneous phase space volume contraction

$$\Lambda = \nabla \cdot \mathbf{V} \tag{18}$$

But

$$\Lambda_0 = \Lambda \tag{19}$$

when

$$\nabla \cdot \mathbf{V} = const \tag{20}$$

Therefore in case (20), the sum of the Liapunov exponents is expressed analytically

$$\sum_{i=1}^{m} \tilde{\lambda}_i = \nabla \cdot \mathbf{V} \tag{21}$$

Thus the result we extracted from the theory of chaos, which can be used for comparison to quantum randomness is the following: the origin of randomness in Newtonian mechanics is instability of ignorable variables that leads to exponential divergence of initially adjacent trajectories; this divergence is measured by Liapunov exponents, which form a discrete spectrum of numbers that must include positive ones.

## 3. Randomness in quantum mechanics.

### *a. Background.*

Quantum mechanics has introduced randomness into the basic description of physics via the uncertainty principle. In the Schrödinger equation, randomness is included in the wave function. But the Schrödinger equation does not *simulate* randomness: it rather describes its evolution from the prescribed initial (random) value, and this evolution is fully deterministic. The main purpose of this section is to trace down the mathematical origin of randomness in quantum mechanics, i.e. to find or build a "bridge" between the deterministic and random states. In order to do that, we will turn to the Madelung equation, [1]. For a particle mass *m* in a potential *F,* the Madelung equation takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\frac{\rho}{m}\nabla S) = 0 \qquad (22)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \qquad (23)$$

Here $\rho$ and $S$ are the components of the wave function $\psi = \sqrt{\rho} e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by $2\pi$. The last term in Eq. (23) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (22) expresses continuity of the flow of probability density, and Eq. (23) is the Hamilton-Jacobi equation for the action *S* of the particle. Actually the quantum potential in Eq. (23), as a feedback from Eq. (22) to Eq. (23), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

The Madelung equations (22), and (23) can be converted to the Schrödinger equations using the ansatz

$$\sqrt{\rho} = \Psi \exp(-iS/h) \qquad (24)$$

where $\rho$ and S being real function.

Reversely, Eqs. (22), and (23) can be derived from the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 \Psi - F\Psi = 0 \qquad (25)$$

using the ansatz , which is reversed to (24)

$$\Psi = \sqrt{\rho} \exp(iS/\hbar) \qquad (26)$$

So there is one-to-one correspondence between the solutions of the Madelung and the Schrödinger equations. From the stochastic mechanics perspective, the transformation of nonlinear Madelung equation into the linear Schrödinger equation is just a suitable mathematical technique that provides an easy way of finding their solutions.

***b. Search for transition from determinism to randomness.***

In this sub-section we will apply our recent results, [5], to the problem of correspondence between quantum and Newtonian randomness. Turning to Eq. (23), we start with some simplification assuming that $F = 0$. Rewriting Eq. (23) for the one-dimensional motion of a particle, and differentiating it with respect to $x$, one obtains

$$m\frac{\partial^2 x(X,t)}{\partial t^2}-\frac{\hbar^2}{2m}\frac{\partial}{\partial X}[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}]=0 \qquad (27)$$

where $\rho(X)$ is the probability distribution of $x$ over its possible values $X$.

Let us choose the following initial conditions for the deterministic state of the system:

$$x=0,\ \rho=\delta(|x|\to 0), \qquad \dot{\rho}=0 \quad at \qquad t=0 \qquad (28)$$

We intentionally did not specify the initial velocity $\dot{x}$ expecting that the solution will comply with the uncertainty principle.

Now let us rewrite the one-dimensional version of Eqs. (22) and (23) as

$$\frac{\partial^2\rho}{\partial t^2}+\frac{\hbar^2}{2m^2}\frac{\partial^4\rho}{\partial X^4}+\xi=0 \qquad at \qquad t\to 0 \qquad (29)$$

where $\xi$ includes only lower order derivatives of $\rho$. For the first approximation, we ignore $\xi$ (later that will be justified,) and solve the equation

$$\frac{\partial^2\rho}{\partial t^2}+a^2\frac{\partial^4\rho}{\partial X^4}=0 \qquad at \qquad t\to 0 \qquad\qquad a^2=\frac{\hbar^2T^2}{2m^2L^4} \qquad (30)$$

subject to the initial conditions (28). The closed form solution of this problem is known from the theory of nonlinear waves, [6]

$$\rho=\frac{1}{\sqrt{4\pi t\frac{\hbar}{2m}}}\cos(\frac{x^2}{4t\frac{\hbar}{2m}}-\frac{\pi}{4}) \qquad at \qquad t\to 0 \qquad (31)$$

Based upon this solution, one can verify that $\xi\to 0$ at $t\to 0$, and that justifies the approximation (30) (for the proofs see the sub-section *b**). It is important to remember that the solution (31) is valid only for small times, and only during this period it is supposed to be positive and normalized.

Rewriting Eq. (27) in dimensionless form

$$\ddot{x}-a^2\frac{\partial}{\partial X}[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}]=0 \qquad (32)$$

and substituting Eq. (31) into Eq. (32) at $\boldsymbol{X = x}$, after Taylor series expansion, simple differentiations and appropriate approximations, one arrives at the following differential equation instead of (32)

$$\ddot{x} = c\frac{x}{t^2}, \qquad c = -\frac{3}{8\pi^2 a^2} \tag{33}$$

This is the Euler equation, and it has the following solution, [7]

$$x = C_1 t^{\frac{1}{2}+s} + C_2 t^{\frac{1}{2}-s} \qquad at \qquad 4c+1>0 \tag{34}$$

$$x = C_1\sqrt{t} + C_2\sqrt{t}\ln t \qquad at \qquad 4c+1=0 \tag{35}$$

$$x = C_1\sqrt{t}\cos(s\ln t) + C_2\sqrt{t}\sin(s\ln t) \qquad at \qquad 4c+1<0 \tag{36}$$

where $2s = \sqrt{|4c+1|}$ (37)

Thus, the qualitative structure of the solution is uniquely defined by the dimensionless constant $a^2$ via the constants *c* and *s*, (see Eqs. (33) and (37). But the cases (35) and (36) should be disqualified at once since they are in a conflict with the approximations used for derivation of Eq. (33), (see sub-section *b**).

Hence, we have to stay with the case (34). This gives us the limits

$$0 < |c| < 0.25, \tag{38}$$

In addition to that, we have to drop the second summand in Eq. (34) since it is in a conflict with the approximation used for derivation of Eq. (30) (see sub-section *b**). Therefore, instead of Eq. (34)) we now have

$$x = C_1 t^{\frac{1}{2}+s} \qquad at \qquad 4c+1>0 \tag{39}$$

For illustration, let us evaluate the constant *c* based upon the following data:

$$\hbar = 10^{-34} m^2 kg/\sec, m = 10^{-30} kg,\ L = 2.8\times 10^{-15} m, \quad L/T = \tilde{C} = 3\times 10 m/\sec$$

where *m*- mass of electron, and $\tilde{C}$-speed of light. Then,

$$c = -1.5\times 10^{-4}, \qquad i.e. \quad |c| < 0.25$$

Hence, the value of *c* is within the limit (38). Thus, for the particular case under consideration, the solution (39) is

$$x = C_1 t^{0.9998} \tag{40}$$

In the next sub-section, prior to analysis of the solution (39), we will present the proofs justifying the solution (31).

***b*. Proofs.***

**1**. Let us first justify the statement that $\xi \to 0 \ at \quad t \to 0$ (see Eq. (29)).

For that purpose, consider the solution (31)

$$\rho = \frac{1}{\sqrt{4\pi at}}\cos\left(\frac{X^2}{4at} - \frac{\pi}{4}\right) \quad at \qquad t\to 0 \tag{1*}$$

As follows from the solution (39),

$$\frac{x}{t} \approx o(t^{s-1/2}) \to \infty, \quad \frac{x^2}{t} \approx o(t^{2s}) \to 0 \qquad at \qquad t \to 0 \text{ since } 0 < s < 1/2 \qquad (2^*)$$

Then, finding the derivatives from Eq. (1') yields

$$|\frac{\partial^n \rho}{\partial X^n}| / |\frac{\partial^{n-1}\rho}{\partial X^{n-1}}| \approx o(t^{-1}) \to \infty \qquad at \qquad t \to 0 \qquad (3^*)$$

and that justifies the inequalities

$$|\frac{\partial^4 \rho}{\partial X^4}| >> |\frac{\partial^3 \rho}{\partial X^3}|, |\frac{\partial^2 \rho}{\partial X^2}|, |\frac{\partial \rho}{\partial X}|, \rho \qquad (4^*)$$

Similarly,

$$|\frac{\partial^n \rho}{\partial t^n}| / |\frac{\partial^{n-1}\rho}{\partial t^{n-1}}| \approx o(t^{-1}) \to \infty \qquad at \qquad t \to 0 \qquad (5^*)$$

and that justifies the inequalities

$$|\frac{\partial^2 \rho}{\partial t^2}| >> |\frac{\partial \rho}{\partial t}|, |\frac{\partial \rho}{\partial X}|^2$$

Also as follows from the solution (18)

$$|\frac{\partial S}{\partial x}| \approx o(t^{S-0.5}), \qquad |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{-1}),$$

$$|\frac{\partial S}{\partial x}| / |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{S+0.5}) \to 0 \qquad at \qquad t \to 0 \qquad (6^*)$$

It should be noticed that for Eq. (34), the evaluations (6*) do not go through, and that was the reason for dropping the second summand.

Finally, the inequalities (4*), (5*) and (6*) justify the transition from Eq. (29) to Eq. (31).

**2**. Next let us first prove the positivity of $\rho$ in Eq. (31) for small times. Turning to the evaluation (2*)

$\frac{x^2}{t} \approx o(t^{2s}) \to 0 \quad at \quad t \to 0$, one obtains for small times

$$\rho = \frac{1}{\sqrt{4\pi a t}} \cos(-\frac{\pi}{4}) > 0 \qquad at \qquad t \to 0 \qquad (7^*)$$

In order to prove that $\rho$ is normalized for small times, turn to Eq.(30) and integrate it over $X$

$$\int_{-\infty}^{\infty} \frac{\partial^2 \rho}{\partial t^2} dX + a^2 \int_{-\infty}^{\infty} \frac{\partial^4 \rho}{\partial X^4} dX = 0 \qquad (8^*)$$

Taking into account the initial conditions (28) and requiring that $\rho$ and all its space derivatives vanish at infinity, one obtains

$$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\rho\, dX = 0 \tag{9*}$$

But as follows from the initial conditions (28)

$$\int_{-\infty}^{\infty}\rho\, dX = 0, \quad \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\rho\, dX = 0 \qquad at \qquad t = 0 \tag{10*}$$

Combining Eqs. (9*) and (10*), one concludes that the normalization constraint is preserved during small times.

**3**. The solutions (34), (35) and (36) have been derived under assumption that

$$\frac{x^2}{t} \to 0 \qquad at \qquad t \to 0 \tag{11*}$$

since this assumption was exploited for expansion of $\rho$ in Eq. (31) in Taylor series. However, in the cases (35) and (36),

$$\frac{x^2}{t} \approx o(1) \qquad at \qquad t \to 0,$$

and that disqualify their derivation. Actually these cases require an additional analysis that is out of scope of this paper. For the same reason, Eq. (13) has been truncated to the form (39).

***c. Analysis of solution.***

Turning to the solution (39), we notice that it satisfies the initial condition (28) i.e. $x=0$ at $t=0$ for *any* values of $C_1$: all these solutions co-exist in a superimposed fashion; it is also consistent with the sharp initial condition for the solution (31) of the corresponding equation (22). The solution (31) describes the simplest *irreversible* motion: it is characterized by the "beginning of time" where all the trajectories intersect (that results from the violation of the Lipchitz condition at *t=0,* Fig.5); then the solution splits into a continuous set of random samples representing a stochastic process with the probability density $\rho$ controlled by Eq. (31). The irreversibility of the process follows from the fact that the backward motion obtained by replacement of $t$ with *(-t)* in Eqs. (31) and (39) leads to imaginary values. Actually Fig.4 illustrates a jump from determinism to a coherent state of superimposed solutions that is ***lost in solutions of the Schrödinger equation.***

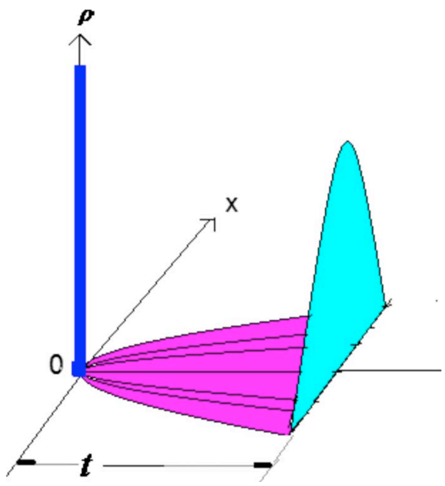



**Figure 4.** Hidden statistics of transition from determinism to randomness.

Let us show that this jump is triggered by instability of the deterministic state. Indeed, turning to the solution represented by Eq. (39) with $|C_1| \le 0.25$, we observe that for fixed values of $C_1$, the solution (39) is ***unstable*** since

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} \qquad > 0 \tag{41}$$

and therefore, an initial error always grows generating ***randomness***. Initially, at *t=0*, that growth is of ***infinite rate*** since the Lipchitz condition at this point is violated (such a point represents a ***terminal*** repeller)

$$\frac{d\dot{x}}{dx} \to \infty \qquad at \qquad t \to 0 \tag{42}$$

This means that an *infinitesimal* initial error becomes finite in a bounded time interval. That kind of instability (similar to blow-up, or Hadamard, instability) has been analyzed in [5]. Considering first Eq.(39) at fixed $C_1$ as a sample of the underlying stochastic process (54), and then varying $C_1$, one arrives at the whole ensemble of one-parametrical random solutions characterizing that process, (see Fig.5). It should be stressed again that this solution is valid only during a small initial period representing a "bridge" between deterministic and random states, and that was essential for the derivation of the solutions (39), and (31).

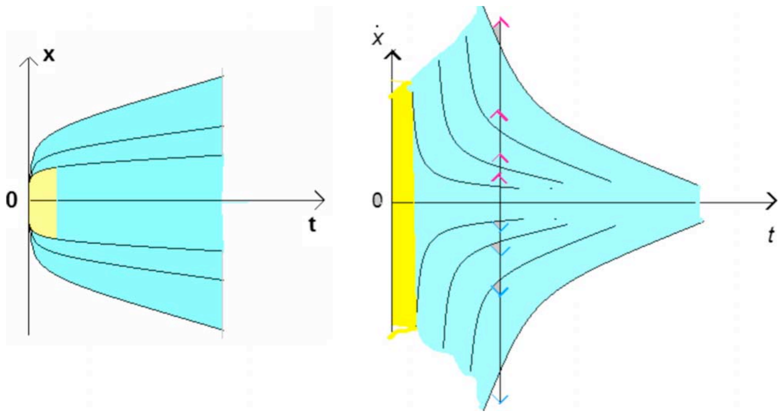



**Figure 5.** Family of random trajectories and particle velocities

Returning to the quantum interpretation of Eqs. (22) and (23), one notice that during this transitional period, the quantum postulates are preserved. Indeed, as follows from Eq. (41),

$$\dot{x} \to \infty \quad at \quad t \to 0 \tag{43}$$

i.e. the initial velocity is *not defined*, (see the flat area in Fig. 5), and ***that confirms the uncertainty principle.*** It is interesting to note that an enforcement of the initial velocity would "blow-up" the solution (39); at the same time, the qualitative picture of the solution is not changed if the initial velocity is not enforced: the solution is composed of superposition of a family of random trajectories with the singularity (43) at the origin. Next, the solution (39) justifies the *belief* sheared by the most physicists that particle trajectories do not exist, although, to be more precise, as follows from Eq. (39), ***deterministic*** trajectories do not exist: each run of the solution (39) produces different trajectory that occurs with probability governed by Eq. (31). It is easily verifiable that the transition of motion from one trajectory to another is very sensitive to errors in initial conditions in the neighborhood of the deterministic state. Indeed, as follows from Eq. (39),

$$C_1 = x_0 t_0^{-(s+0.5)}, \qquad \frac{\partial C_1}{\partial x_0} = t_0^{-(s+0.5)} \to \infty \quad as \qquad t_0 \to 0 \tag{44}$$

where $x_0$ and $t_0$ are small errors in initial conditions.

Actually Eq. (39) represents a hidden statistics of the underlying Schredinger equation. As pointed out above, the cause of the randomness is non-Lipchitz instability of Eq. (39) at *t=0*. Therefore, trajectories of quantum particles have the same "status" as trajectories of classical particles in a chaotic motion with the only difference that the random "choice" of the trajectory is made only at $t_0 \to 0$. It should be emphasized again that the transition (39) is *irreversible*. However, as soon as the difference between the current probability density and its initial sharp value becomes finite, one arrives at the conventional quantum formalism described by the Schrödinger, as well as the Madelung equations. Thus, in the conventional quantum formalism, the *transition* from the classical to the quantum state has been ***lost*****,** and that created a major obstacle to interpretation of quantum mechanics as an extension of the Newtonian mechanics. However, as demonstrated above, the quantum and classical worlds can be reconciled via the more subtle mathematical treatment of the ***same equations***. This result is generalizable to multi-dimensional case as well as to case with external potentials.

***d. Comments on equivalence of Schrödinger and Madelung equations.***

Equivalence of Schrödinger and Madelung equations was questioned by some quantum physicists on the ground that to recover the Schrödinger equation from the Madelung equation, one must add by hand a quantization condition, as in the old quantum theory. However, this argument has been challenged by other physicists. We will not go into details of this discussion since this paper is focused on mathematical rather than physical equivalence of Schrödinger and Madelung equations. Firstly we have to notice that the Schrödinger equation is more attractive for computations due to its linearity, while the Madelung equations have a methodological advantage: they allow one to trace down the Newtonian origin of the quantum physics. Indeed, if one drops the Planck's constant, the Madelung equations degenerate into the Hamilton-Jacobi equation supplemented by the Liouville equation. However despite the fact that these two forms of the same governing equations of quantum physics can be obtained from one another without a violation of any of mathematical rules, there is more significant difference between them, and this difference is associated with the concept of stability. Indeed, as demonstrated above, the solution of the Madelung equations with deterministic initial condition (28) is unstable, and it describes the jump from the determinism to randomness. This illuminates the origin of randomness in quantum physics. However the Schrödinger equation does not have such a solution; moreover, it does not "allow" posing such a problem and that is why the randomness in quantum mechanics had to be postulated. So what happens with mathematical equivalence of Schrödinger and Madelung equations? In order to answer this question, let us turn to the concept of stability. It should be recalled that stability is not an invariant of a physical model. It is an attribute of mathematical description: it depends upon the frame of reference, upon the class of functions in which the motion is presented, upon the metrics of configuration space, and in particular, upon the way in which the distance between the basic and perturbed solutions is defined, [9]. As an example, consider an inviscid stationary flow with a *smooth* velocity field, [2]

$$v_x = A\sin z + C\cos y,\ v_y = B\sin x + A\cos z,\ v_z = C\sin y + b\cos x \tag{45}$$

Surprisingly, the trajectories of individual particles of this flow are unstable (Lagrangian turbulence). It means that this flow is stable in the Eulerian representation, but unstable in the Lagrangian one. The same happens with stability in Hilbert space (Schrödinger equalion), and stability in physical space (Madelung equations). One should recall that stability analysis is based upon a departure from the basic state into a perturbed state, and such departure requires an expansion of the basic space. However, Schrödinger and Madelung equations in the ***expanded*** spaces are not necessarily equivalent any more, and that explains the difference in the concept of stability of the same solution as well as the interpretation of randomness in quantum mechanics.

There is another "mystery" in quantum mechanics that can be clarified by transfer to the Madelung space: a ***belief*** that a particle trajectory does not exist. Indeed, let us turn to Eq. (39). For any particular value of the arbitrary constant $C_1$, it presents the corresponding particle's trajectory. However as a result of non-Lipchitz instability at $t = 0$, this constant is supersensitive to infinitesimal disturbances, and actually it becomes random at $t=0$. That makes random the choice of the whole trajectory, while the randomness is controlled by Eq. (30). Actually this provides a justification for the ***belief*** that a particle can occupy any place at any time: it is due to randomness of its trajectory. However it should be emphasized that the particle makes random choice only once: at $t = 0$. After that it stays on the chosen trajectory. Therefore in our interpretation, this belief does not mean that a trajectory does not exist: it means only that the trajectory exists, but it is unstable. Based upon that, we can extract some deterministic information about the particle trajectory by posing the following question: find such a trajectory that has the highest probability to appear. The solution of this problem is straight forward: in the process of collecting statistics for the arbitrary constant $C_1$ find such its value that has the highest frequency to appear. Then the corresponding trajectory will have the highest probability to appear as well.

Thus, strictly speaking, the Schrödinger and Madelung equations are equivalent only in the open time interval

$$t > 0 \qquad (46),$$

since the Schrödinger equation does not include the infinitesimal area around the singularity at

$$t = 0 \qquad (47)$$

while the Madelung equation exists in the closed interval

$$t \geq 0 \qquad (48)$$

But all the "machinery" of randomness emerges precisely in the area around the singularity (47). That is why the source of randomness is missed in the Schrödinger equation, and the randomness had to be postulated.

*Remark.* An example of fundamental difference between stability in open and closed intervals is given in [9].

Hence although historically the Schrödinger equation was proposed first, and only after a couple of months, Madelung introduced its hydrodynamic version that bears his name, strictly speaking, the foundations of quantum mechanics would be saved of many paradoxes had it be based upon the Madelung equation.

**4. Randomness in physics and the correspondence principle.**

The discovery of the origin of randomness in quantum mechanics opens up a strong support to the correspondence principle: the randomness in physics (both quantum and Newtonian) is caused by dynamical instability. However this support comes with some complications: the types of dynamical instability in Newtonian and quantum physics are qualitatively different.

Indeed in Newtonian physics it is Liapunov instability of ignorable variables, i.e. such variables that do not contribute into energy of the system. For an $m$-dimensional system, this instability manifests itself in appearance of positive Liapunov exponents in the spectrum of $m$ Liapunov exponents, while each of these exponents measures the averaged divergence of adjacent trajectories.

In quantum physics, the instability has a different nature: it is caused by the loss of uniqueness of the solution in a singular point due to failure of Lipchitz condition at this point. In context of terminal dynamics, [8], this point represents a terminal repeller that is characterized by infinite divergence of trajectories. As a result of that, quantum system makes a random choice of the trajectory only once – at the beginning of the transition from determinism to randomness, while a Newtonian system may change trajectories continuously during its chaotic motion.

***a. Terminal repeller.***

In order to capture the fundamental properties of the effects associated with failure of the Lipchitz condition, let us turn to a simple ODE

$$m\dot{v} = \alpha v^k, \quad k = \frac{N}{N+2} < 1, \quad m, \alpha > 0 \tag{49}$$

where N is a natural number.

One can verify that that for Eq. (49), the equilibrium point $v = 0$ becomes a terminal repeller, and since

$$\frac{d\dot{v}}{dv} = k\frac{\alpha}{m}v^{k-1} \to \infty \quad at \quad v \to 0 \tag{50}$$

it is infinitely unstable. If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period

$$t_0 = \int_{v_0}^{0} \frac{mdv}{\alpha v^k} = \frac{mv_0^{1-k}}{\alpha(1-k)} < \infty \tag{51}$$

while for a regular repeller the time period would be infinite. Here the motion is irreversible since the inversion of time in the solution of Eq. (50)

$$v = \pm[\frac{\alpha}{m}(1-k)t]^{1/1-k} \tag{52}$$

leads to imaginary values of $v$ since $k < 1$.

But in addition to that, terminal repellers possess even more surprising characteristics: the solution (52) becomes totally unpredictable. Indeed, two different motions described by Eq. (52) are possible for "almost the same" initial conditions:

$$v_0 = +\varepsilon \to 0 \quad or \quad v_0 = -\varepsilon \to 0 \quad at \quad t = 0 \tag{53}$$

The most essential property of this result in that the divergence of these two solutions is characterized by an unbounded rate

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{\alpha t^{1/(1-k)}}{m | v_0 |}\right) \to \infty \quad at \quad | v_0 | \to 0 \tag{54}$$

In contrast to the classical case where $t_0 \to \infty$, here $\sigma$ can be defined within an arbitrarily small time interval $t_0$ since during this interval the initial infinitesimal distance between the solutions becomes finite. Thus a terminal repeller represents a vanishingly small, but infinitely powerful "pulse of randomness" that is pumped into the system via terminal repeller., Figs.6,7. Obviously, failure of the uniqueness of the solution here results from the violation of the Lipchitz condition(50) at $v = 0$.

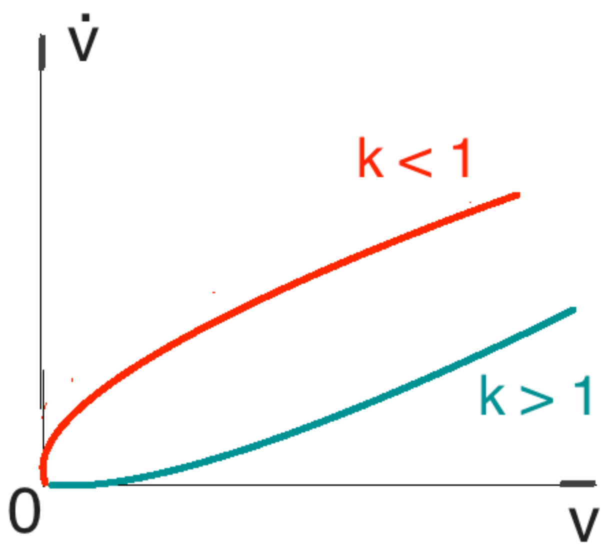



Figure 6.Terminal repeller in phase space ($k < 1$), classical repeller in phase space ($k > 1$).

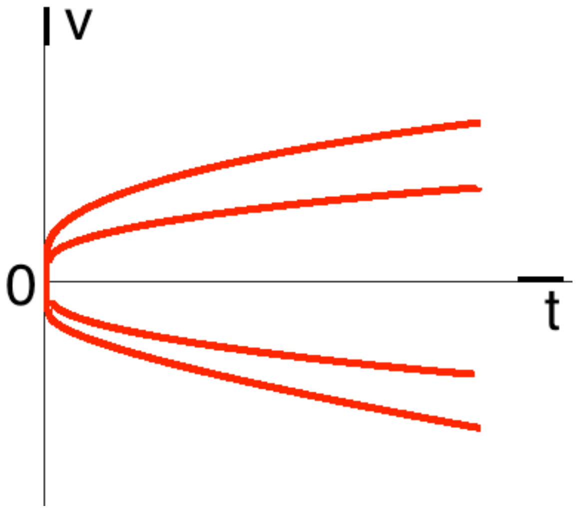



Figure 7. Terminal repeller in physical space.

Now one can verify that the solution (39) that describes transition from determinism to randomness in quantum physics belong to the same class as the solution (52) that starts with the terminal repeller , (compare Fig. 7 with Fig. 5 that present qualitative description of solutions).

***b. Discontinuous transition from quantum to Newtonian physics.***

As follows from the comparison between the mechanisms of instability in quantum and Newtonian physics performed in the previous subsections, the transition from quantum to Newtonian randomness is not smooth. Let us take a deeper look into the transition from quantum to Newtonian physics in terms of the mathmatical formalism. For that purpose, start with the Madelung equations (22),(23) and let the Planck constant to be zero. As a result, we arrive at the system of the Hamilton-Jacoby and Liouville equations that desribes Newtonian mechanics.

$$\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+F=0 \tag{55}$$

$$\frac{\partial \rho}{\partial t}+\nabla\bullet(\frac{\rho}{m}\nabla S)=0 \tag{56}$$

The most important mathematical property of this system of PDE is its hyperbolic type since the order of the highest temporal and space derivatives is the same. This property provide existence of weak discontinuities of $S$ and $\rho$ that propagate with a finite speed [10]

$$\lambda_S = \frac{\Delta_n S}{m} = \lambda_\rho \tag{57}$$

where $\nabla_n S$ is the projection od the vector $\nabla S$ onto the normal **n** to the front of the discontinuity.

Returning to the Madelung equations (22), (23), one can see that they are of parabolic type since the quantum potential brings the space derivatives of higher order than the temporal ones. As a result, all the discontinuities dissapeared, and any changes propagate instantaneously. Indeed in order for the second derivatives $\nabla^2\rho, \nabla^2 S$ to exist, the first derivatives $\nabla\rho, \nabla S$ must be continuous. This qualitative difference in mathematical formalism reflects the corresponding difference in mechanisms of randomness formation in quantum and Newtonian dynamics. But does this difference "inflicts any damage" to the correspondence principle?

In order to answer this question, let us turn to fluid dynamics in which the continuity of transition from the Euler to the Navier-Stokes (NS) equations is mathematically similar to transition from Newtonian to quantum phyaics.

The Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F} \tag{58}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{59}$$

where **v** velocity**,** p pressure, and $\rho$ mass dencity are of hyperbolic type, and therefore, they include a capability to transport discontinuities in the form of sound or shock waves.

The Navier-Stokes equations bring to Eq. (58) additional terms with higher order space derivative, and that makes the system parabolic. These "viscose" terms play the same role as quantum potential in Eq. (23), while the role of the Planks constant is played by viscosity coefficient. As a result, the NS equations have different formulation of boundary conditions: slip boundary conditions on a rigid wall for NS equations, instead of no-slip ones for the Euler equations. For a mathematician, this difference is significant: it affects existence, uniqueness and satbility of solutions. Indeed the criteria of stability are different even for a vanishingly small viscosity, and that resembles the mismatch between the mechanisms of randomness in quantum and Newtonian physics discussed above. However there is something more that can allert physicists: the sound and shock waves that are of enormous importance in physics and in engineering applications dissapear in NS equations that are supposed to be more precise than the Euler equations. However a detailed analysis of NS equations shows that such allert is highly exaggurated. It turns out that "shock"waves exist in NS equation, but without sharp discontinuities that are slightly smoothed out due to viscosity. Actually they are closer to reality than original shock waves found from the Euler equations. However it does not seem reasonable to study shock waves with NS equations since the results will be imbedded in enormous amount of mathematical details that does not have physical importance. For the same reason, it seems impractical to study chaos using limiting case of quantum mechanics.

Therefore our conclusion is more optimistic than the beginning of this discussion: the discovery of the origin of randomness in quantum physics supports the correspondence principle rather than raises more doubts.

## 5. Conclusion.

This work is motivated by quantum chaos that is a branch of physics which studies how chaotic classical dynamical systems can be described in terms of quantum theory. The primary question that quantum chaos seeks to answer is: "What is the relationship between quantum physics and classical chaos?" The correspondence principle states that classical mechanics is the classical limit of quantum mechanics. If this is true, then there must be quantum mechanisms underlying classical chaos, although this may not be a fruitful way of examining classical chaos. If quantum mechanics does not demonstrate an exponential sensitivity to initial conditions, how can exponential sensitivity to initial conditions arise in classical chaos, which must be the correspondence principle limit of quantum mechanics? The objective of this work is to find more evidence in support of the correspondence principle, and in particular, to unify theory of transition from determinism to randomness in physics.
As a result, there has been proven that mathematical origins of randomness in quantum and Newtonian physics are coming from the same source that is dynamical ***instability***. However in Newtonian physics this instability is measured by positive finite Liapunov exponents averaged over infinite time period, while in quantum physics the instability is accompanied by a loss of the Lipchitz condition and represented by an infinite divergence of trajectories in a singular point. Although from a mathematical viewpoint such a difference is significant, from physical viewpoint it does not justify division of randomness into "deterministic "(chaos) and "true" (quantum physics). The common origin of randomness in Newtonian and quantum physics presents a support of the correspondence principle that is being searched by quantum chaos theory.

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