Moments Defined by Example Subdivision Curves

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Update notice: In the first version of the document, the moments defined by the dual C^2 four-point scheme were wrongly calibrated, which resulted in the statement of incorrect moment values. The update resolves this mistake, and also adds examples for two schemes that were not treated in the previous edition: 1) the quartic B-spline scheme, and 2) the dual three-point scheme. Meanwhile, the derivation of the moment formula has been published [Hakenberg et al. 2014b]. ■

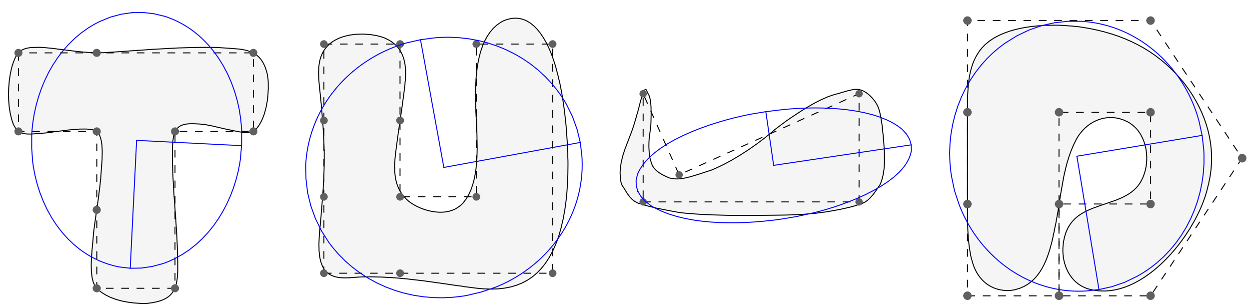

Figure: Four subdivision curves as black, continuous lines. The sequence of control points are the dots connected by dashed lines. The blue circumference marks the ellipsoid at the centroid of the area enclosed by the subdivision curve that has equivalent inertia as the area. The principal axes of the ellipsoid are also shown. ■

Abstract

We list examples of 2-dimensional domains bounded by subdivision curves together with their exact area, centroid, and inertia. We assume homogeneous mass-distribution within the space bounded by the curve. The subdivision curves that we consider are generated by 1) the low order B-spline schemes, 2) the generalized, interpolatory C^1 four-point scheme, as well as 3) the more recent dual C^2 four-point scheme.

The derivation of the $(d+2)$-linear form that computes the area moment of degree $p+q=d$ based on the initial control points for a given subdivision scheme is deferred to a publication in the near future.

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Introduction

Subdivision of curves is an iterative refinement procedure for polygons. Over the course of the iteration, the increasingly dense point cycle typically converges to a piecewise smooth curve.

Our article is restricted to subdivision of polygons with a finite number of control points $(\mathrm{px}_k, \mathrm{py}_k) \in \mathbb{R}^2$ for $k = 1, 2, \ldots, n$ in the 2-dimensional plane. If the resulting subdivision curve is compact, and not self-intersecting, we denote with $\Omega \subset \mathbb{R}^2$ the set in the interior of the curve. Then, the area moments of degree $p+q=d$ of the set Ω with respect to the x- and y-axis are well defined by the following integral

$$M_{p,q}(\Omega) = \int_\Omega x^p\, y^q\, dx\, dy$$

In a future publication, we will show that the integral $M(p, q)$ can be substituted by a $(d+2)$-linear form via the divergence theorem. The input to the multi-linear form are the coordinates of the polygon $(\mathrm{px}_k, \mathrm{py}_k)$ for $k = 1, 2, \ldots, n$. The coefficients of the multi-linear forms depend only on the subdivision rules, and subsequently apply universally to any choice of control points. The derivation of the multi-linear forms does not require the basis functions.

In [Hakenberg et al. 2014], the derivation of the trilinear forms that compute the volume enclosed by subdivision surfaces (=moment of degree 0) has been presented. That article briefly mentions moments of higher degrees of the 3-dimensional sets. However, the authors conclude that establishing the forms is not tractable by today's computational means due to the large number of unknown coefficients. Therefore, for moments of higher degree we focus on the simpler, 2-dimensional case. Here, much fewer coefficients are required, and the forms can be solved for even in the presence of a tension parameter. For instance, the form that computes the centroid (=moment of degree 1) for curves generated by the C^1 four-point scheme with parameter ω can be expressed with variable ω.

Our article is structured as follows: We review the area, centroid, and inertia for sets bounded by polygons. Then, the four families of subdivision are introduced that generate the curves in the examples. Specific curves and the stated area moments might help to verify alternative implementations of the formulas for the moments.

Moments defined by Polygons

The area moments of the 2-dimensional set $\Omega \subset \mathbb{R}^2$ enclosed by a polygon with n control points $(\mathrm{px}_k, \mathrm{py}_k) \in \mathbb{R}^2$ for $k = 1, 2, \ldots, n$ can be found for instance in [Bourke 1988]. The moment of degree 0 is the area A, which is determined by the well known alternating bilinear form, that is the determinant of 2×2 matrices

$$M_{0,0}(\Omega) = A = \frac{1}{2} \sum_{k=1}^{n} \det\begin{pmatrix} \mathrm{px}_k & \mathrm{py}_k \\ \mathrm{px}_{k+1} & \mathrm{py}_{k+1} \end{pmatrix} = \frac{1}{2} \sum_{k=1}^{n} (\mathrm{px}_k\, \mathrm{py}_{k+1} - \mathrm{px}_{k+1}\, \mathrm{py}_k)$$

The indices of the control points are taken modulo n. For instance, index $k = n$ corresponds to index $k = 0$.

The centroid (c_x, c_y) of the set Ω requires the two moments of degree 1, and corresponds to the following trilinear form

$$c_x = \frac{1}{A} M_{1,0}(\Omega) = \frac{1}{A} \frac{1}{6} \sum_{k=1}^{n} (\mathrm{px}_k + \mathrm{px}_{k+1}) \det\begin{pmatrix} \mathrm{px}_k & \mathrm{py}_k \\ \mathrm{px}_{k+1} & \mathrm{py}_{k+1} \end{pmatrix}$$

$$c_y = \frac{1}{A} M_{0,1}(\Omega) = \frac{1}{A} \frac{1}{6} \sum_{k=1}^{n} (\mathrm{py}_k + \mathrm{py}_{k+1}) \det\begin{pmatrix} \mathrm{px}_k & \mathrm{py}_k \\ \mathrm{px}_{k+1} & \mathrm{py}_{k+1} \end{pmatrix}$$

The area inertia of Ω can be found in [Juhlnet 2011]. The values are determined by 4-linear forms such as

$$M_{2,0}(\Omega) = \frac{1}{12} \sum_{k=1}^{n} \left(\mathrm{px}_k^2 + \mathrm{px}_k\, \mathrm{px}_{k+1} + \mathrm{px}_{k+1}^2\right) \det\begin{pmatrix} \mathrm{px}_k & \mathrm{py}_k \\ \mathrm{px}_{k+1} & \mathrm{py}_{k+1} \end{pmatrix}$$

$$M_{1,1}(\Omega) = \frac{1}{12} \sum_{k=1}^{n} (\mathrm{px}_k - \mathrm{px}_{k+1}) \left(3\, \mathrm{px}_{k+1}\, \mathrm{py}_{k+1}^2 + \mathrm{px}_k\, \mathrm{py}_{k+1}^2 + 2\, \mathrm{px}_{k+1}\, \mathrm{py}_k\, \mathrm{py}_{k+1} + 2\, \mathrm{px}_k\, \mathrm{py}_k\, \mathrm{py}_{k+1} + \mathrm{px}_{k+1}\, \mathrm{px}_k^2 + 3\, \mathrm{px}_k\, \mathrm{py}_k^2\right)$$

$$M_{0,2}(\Omega) = \frac{1}{12} \sum_{k=1}^{n} \left(\mathrm{py}_k^2 + \mathrm{py}_k\, \mathrm{py}_{k+1} + \mathrm{py}_{k+1}^2\right) \det\begin{pmatrix} \mathrm{px}_k & \mathrm{py}_k \\ \mathrm{px}_{k+1} & \mathrm{py}_{k+1} \end{pmatrix}$$

The coefficients in the multi-linear forms are generally not uniquely determined.

The area moments of polygons can be used to approximate the moments defined by subdivision curves. Thereby, the formulas help to validate the implementation of the exact forms.

The piecewise linear boundary of a polygon is reproduced by linear subdivision. Using our general framework to establish multi-linear forms for the computation of area moments we reproduce the forms stated above.

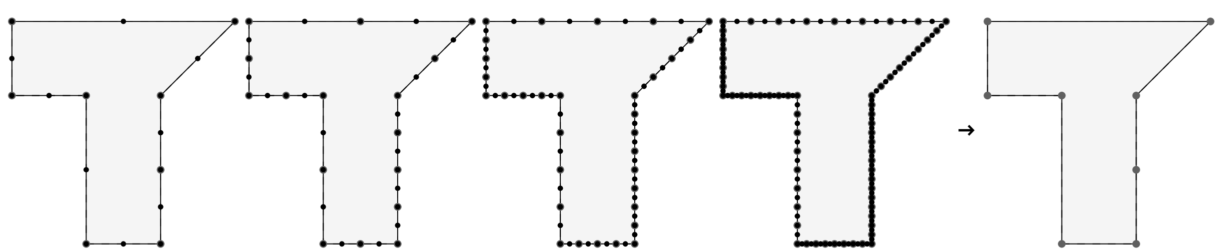


Figure: Several iterations of a T-shaped control point sequence defined by the cycle ((1, 0), (2, 0), (2, 1), (2, 2), (3, 3), (0, 3), (0, 2), (1, 2)) using linear subdivision. The enclosed area is $9/2 = 4.5$. The centroid is located at $\frac{1}{27}(37, 50)$. ■

The rules of linear subdivision are vertex interpolation, and mid-edge insertion

The basis functions that parameterize the curve between two successive control points are the linear polynomials $b_1(t) = 1 - t$, and $b_2(t) = t$ for $t \in [0, 1]$.

Schemes for Curves

We briefly review the subdivision schemes that are used in the upcoming examples.

Quadratic B-Spline

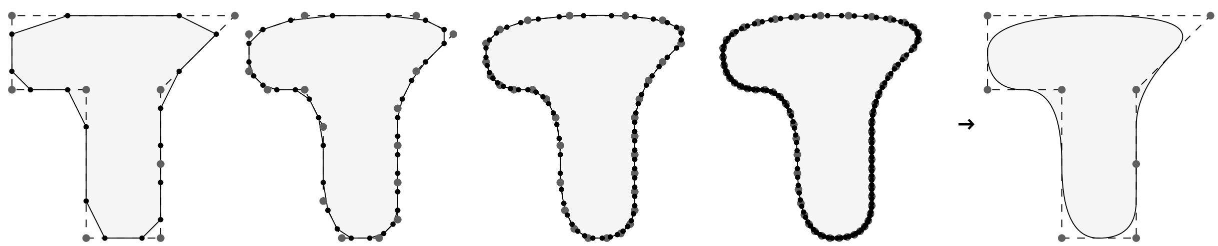


Figure: Several iterations of the T-shaped control point sequence defined above using quadratic B-spline subdivision. The area enclosed by the limit curve is $101/24 = 4.20833\ldots$, the centroid is located at $(139/101, 928/505)$. ■

Quadratic B-spline subdivision for curves is also referred to as *Chaikin's* scheme [Chaikin 1974], or *corner-cutting* scheme. The scheme is *dual*, i.e. two output control points are inserted between a pair of input control points. The weights for the insertion are symmetric

The basis functions that piecewise parametrize the curve are the quadratic polynomials

$$b_1(t) = \frac{1}{2}(t-1)^2,\ b_2(t) = \frac{1}{2} + t - t^2,\ \text{and}\ b_3(t) = \frac{1}{2}t^2\ \text{for}\ t \in [0, 1].$$

Cubic B-Spline

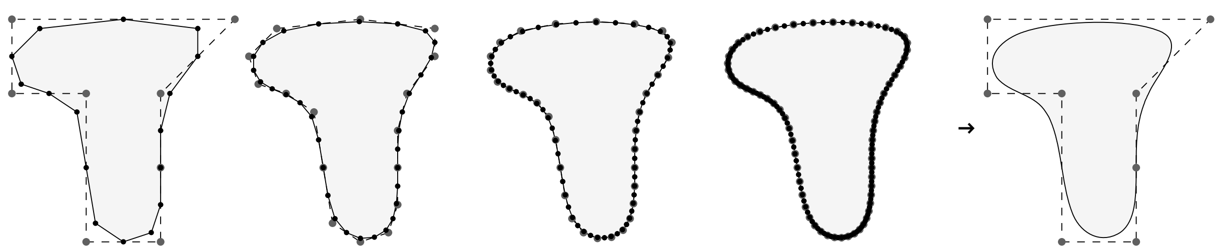
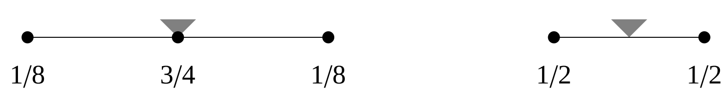


Figure: Several iterations of cubic B-spline subdivision applied to the T-shaped control point sequence with coordinates defined above. The area enclosed by the limit curve is $\frac{59}{15} = 3.93333\ldots$. The centroid is located at $\left(\frac{41077}{29736}, \frac{432751}{237888}\right)$. ■

A very popular polygon refinement algorithm is cubic B-spline subdivision with the following averaging mask and mid-edge insertion



The basis functions that parametrize the curve between a pair of successive control points are the following cubic polynomials

$$b_1(t) = -\frac{1}{6}(t-1)^3,\ b_2(t) = \frac{1}{6}\left(4 - 6t^2 + 3t^3\right),\ b_3(t) = \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right),\ \text{and}\ b_4(t) = \frac{1}{6}t^3\ \text{for}\ t \in [0, 1].$$

Quartic B-Spline

Figure: Several iterations of quartic B-spline subdivision applied to the T-shaped control point sequence with coordinates defined above. The area enclosed by the limit curve is $\frac{74573}{20160} = 3.69905...$. The centroid is located at $\left(\frac{490284845}{354370896}, \frac{212745589}{118123632}\right)$. ■

We include quartic B-spline subdivision for comparison to the upcoming dual three-point scheme.

The basis function is piecewise polynomial.

Dual Three-Point Scheme

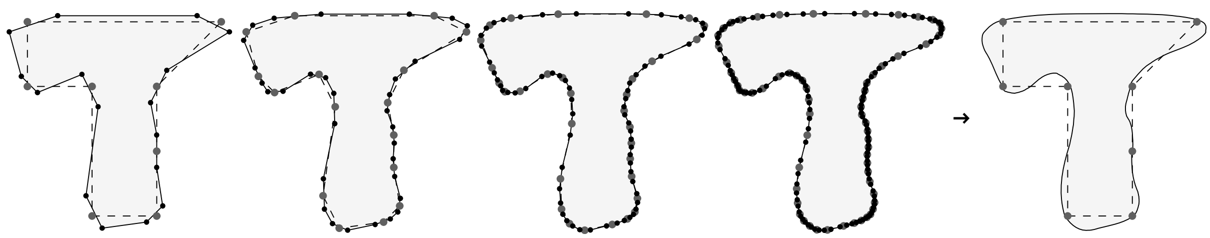


Figure: Several iterations of the dual three-point scheme. The area enclosed by the limit curve is $\frac{5517149}{1063680} = 5.18685...$. The centroid is located at $\left(\frac{33637930774117685}{24867608818761632}, \frac{234122503662743331}{124338044093808160}\right)$. ■

The dual three-point scheme was contructed by [Hormann/Sabin 2008]. The authors derive the subdivision rules as follows: "*the two new points adjacent to a given old point are taken by sampling a quadratic through three adjacent old points. It therefore has quadratic precision by construction.*" The weights are

The basis function does not have a closed form-expression.

Interpolatory C^1 Four-Point Scheme

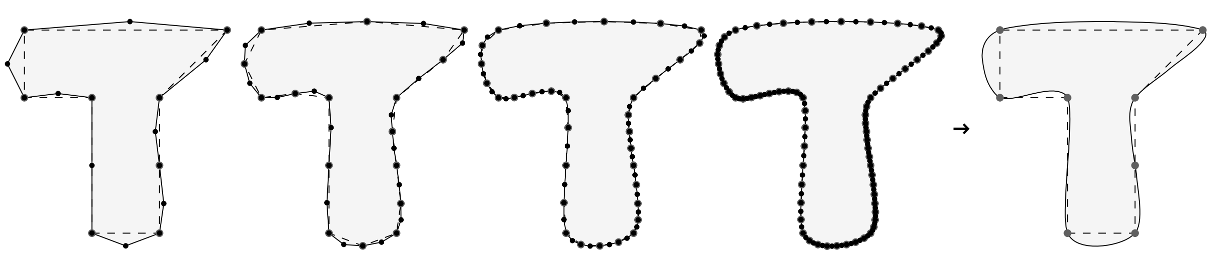
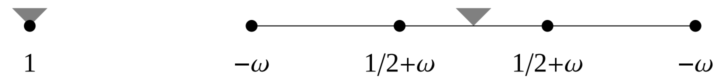


Figure: Several iterations of the C^1 four-point scheme with tension parameter $\omega = 1/16$. The area enclosed by the limit curve is $\frac{27-25\omega+171\omega^2+88\omega^3+224\omega^4+320\omega^5}{6-18\omega+54\omega^2-48\omega^3+96\omega^4}$ for general ω, and $\frac{85625}{16632} = 5.14821...$ for $\omega = 1/16$. ■

The interpolatory four-point scheme was conceived by [Dubuc 1986], and generalized later in [Dyn/Gregory/Levin 1987] who introduced the tension parameter $\omega \in \mathbb{R}$. Dubuc's original scheme corresponds to $\omega = 1/16 = 0.0625$.



[Hechler/Moessner/Reif 2008] prove that the scheme produces C^1 curves when $\omega \in (0, \omega^*)$ with ω^* as the unique real solution of the cubic polynomial $32\,\omega^3 + 4\,\omega - 1 = 0$, namely

$$\omega^* = \frac{1}{12}\sqrt[3]{27 + 3\sqrt{105}} - \frac{1}{2}\sqrt[-3]{27 + 3\sqrt{105}} = 0.192729249264812025206286592326756741813763\ldots$$

Dual C^2 Four-Point Scheme

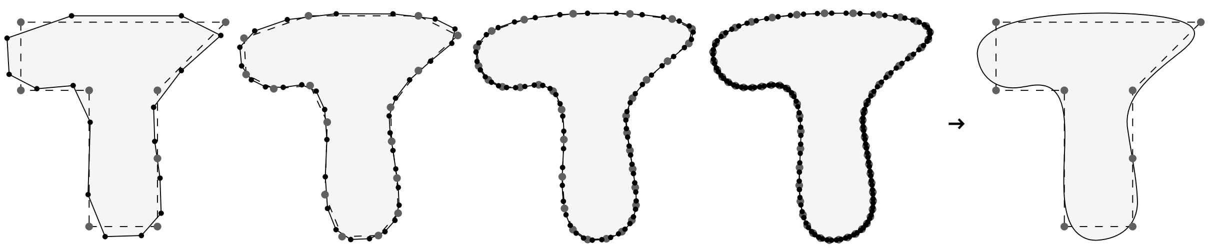


Figure: Several iterations of the C^2 four-point scheme with tension parameter $\omega = 1/128$. The area enclosed by the limit curve is the fraction $\frac{2745214799903}{544974151680} = 5.03733\ldots$. ■

The C^2 four-point scheme was introduced by [Dyn/Floater/Hormann 2005] and uses the tension parameter $\omega \in \mathbb{R}$. Smoothness is guaranteed for parameters in the interval $\omega \in (0, 1/48]$, but possibly also for values beyond $\omega > 1/48 = 0.0208333\ldots$. The default choice is $\omega = 1/128 = 0.0078125$.

The scheme is dual, i.e. the output control points are located between the input control points. The weights are

For the choice $\omega = 0.013723\ldots$ the scheme is called “tightest”. For that parameter value, the basis function sampled at the integers $k \in \mathbb{Z}$ are closest to the Kronecker sequence $\delta_{0,k}$ in the least square sense. The limit curves are almost, but not quite, entirely unlike interpolatory.

Remarks

The subdivision weights are applied coordinatewise.

In order to establish the area moments refinement through subdivision is not required. In fact, less refinement means faster evaluation of the formula. Despite that, we give a visual impression of the subdivision curves by subdiving the input polygon about 6-7 times.

For the linear, quadratic, cubic, etc. B-spline subdivision schemes, the area moments can be derived by solving the integral expression via the divergence theorem. That is because the basis functions are polynomials.

Examples

For all example curves that follow, we state the coordinates of the control points of the polygon that are input to the subdivision iteration. We apply the various subdivision schemes in turn. The limit curves are visualized and can be compared conveniently. For each contour, we state the exact area, centroid, and inertia defined by the limit curve.

The inertia is measured with respect to a) the (previously established) centroid of the area, (because that reference is the most relevant in practice), and b) the x-, and y- axis. We remark that the formula easily permits to compute the inertia with respect to any point in the plane. The principal axes are determined by eigenvalue decomposition of the inertia matrix, and are plotted in the graphics.

Whenever all weights of a subdivision scheme are rational, the coefficients in the multi-linear forms that determine the area moments are also fractions. This allows us to establish the area, centroid, and inertia in exact algebraic form given that also the coordinates of the control points are rational numbers.

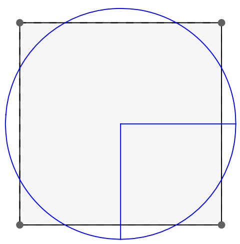
In the upcoming examples, some algebraic expressions exceed the page margins due to their large number of digits. In that case, we restate the value in full length immediately below.

Cube

Curve coordinates ↓

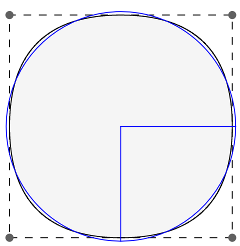
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear B-spline ↓



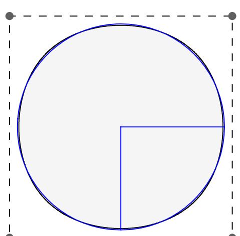
Area	1 (≈ 1.00000000000000000000)
Centroid=	$\left(\frac{1}{2}\ \ \frac{1}{2}\right)$
Centroid≈	(0.50000000000000000000 0.50000000000000000000)
Inertia =	$\left(\frac{1}{12}\ \ 0\ \ \frac{1}{12}\right)$
Inertia ≈	(0.08333333333333333333 0 0.08333333333333333333)

Quadratic B-spline ↓



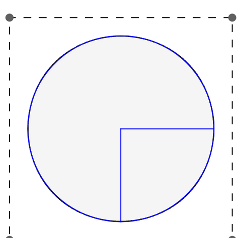
Area	$\frac{5}{6}$ (≈ 0.83333333333333333333)
Centroid=	$\left(\frac{1}{2}\ \ \frac{1}{2}\right)$
Centroid≈	(0.50000000000000000000 0.50000000000000000000)
Inertia =	$\left(\frac{31}{560}\ \ 0\ \ \frac{31}{560}\right)$
Inertia ≈	(0.05535714285714285714 3 0 0.05535714285714285714 3)

Cubic B-spline ↓



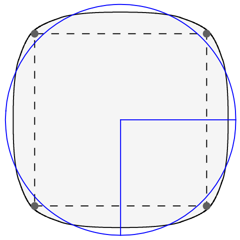
Area	$\frac{61}{90}$ (≈ 0.67777777777777777778)
Centroid=	$\left(\frac{1}{2}\ \ \frac{1}{2}\right)$
Centroid≈	(0.50000000000000000000 0.50000000000000000000)
Inertia =	$\left(\frac{27\,371}{748\,440}\ \ 0\ \ \frac{27\,371}{748\,440}\right)$
Inertia ≈	(0.03657073379295601517 8 0 0.03657073379295601517 8)

Quartic B-spline ↓



Area	$\frac{277}{504}$ (≈ 0.54960317460317460317)
Centroid=	$\left(\frac{1}{2}\ \ \frac{1}{2}\right)$
Centroid≈	(0.50000000000000000000 0.50000000000000000000)
Inertia =	$\left(\frac{3\,207\,559}{133\,436\,160}\ \ 0\ \ \frac{3\,207\,559}{133\,436\,160}\right)$
Inertia ≈	(0.02403815427542279394 1 0 0.02403815427542279394 1)

Three-Point ↓



Area $\frac{37309}{26592}$ (≈ 1.4030159446450060168)

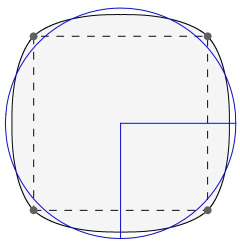
Centroid= $\left(\frac{1}{2} \quad \frac{1}{2}\right)$

Centroid≈ (0.50000000000000000000 0.50000000000000000000)

Inertia = $\begin{pmatrix} \frac{1163435159449326476495138590944569683421}{7355145083396697285318167588021721600000} \\ 0 \\ \frac{1163435159449326476495138590944569683421}{7355145083396697285318167588021721600000} \end{pmatrix}$

Inertia ≈ (0.15817977025030180374 0 0.15817977025030180374)

C^1 Four-Point ω=1/16 ↓



Area $\frac{14272}{10395}$ (≈ 1.3729677729677729678)

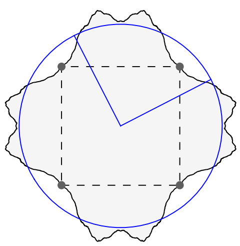
Centroid= $\left(\frac{1}{2} \quad \frac{1}{2}\right)$

Centroid≈ (0.50000000000000000000 0.50000000000000000000)

Inertia = $\begin{pmatrix} \frac{23340561324786432115362070413499461043666460891}{154520168587414501234522160187896923984378608000} \\ 0 \\ \frac{23340561324786432115362070413499461043666460891}{154520168587414501234522160187896923984378608000} \end{pmatrix}$

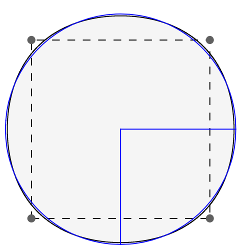
Inertia ≈ (0.15105187586940993871 0 0.15105187586940993871)

C^1 Four-Point ω=0.192729… ↓



Area 2.25279 (≈ 2.25279)
Centroid≈ (0.5 0.5)
Inertia ≈ (0.425514 0 0.425514)

C^2 Four-Point ω=1/128 ↓



Area $\frac{133808579579}{102182653440}$ (≈ 1.3095038646414700111)

Centroid= $\left(\frac{1}{2} \quad \frac{1}{2}\right)$

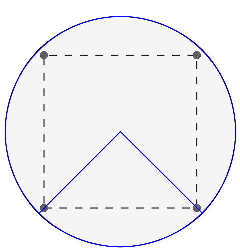
Centroid≈ (0.50000000000000000000 0.50000000000000000000)

Inertia ≈ (0.13652863906520953547 0 0.13652863906520953547)

Inertia =

{696 713 676 660 168 897 181 454 735 579 301 483 887 251 658 615 877 244 232 510 419 599 121 022 735 909 055 599 233 031 938 644 572 964 574 722 111 257 / 5 103 058 826 561 662 287 115 209 880 607 263 965 213 496 865 903 854 785 033 077 901 180 829 878 902 908 429 849 730 794 323 849 115 384 173 271 449 600, 0, 696 713 676 660 168 897 181 454 735 579 301 483 887 251 658 615 877 244 232 510 419 599 121 022 735 909 055 599 233 031 938 644 572 964 574 722 111 257 / 5 103 058 826 561 662 287 115 209 880 607 263 965 213 496 865 903 854 785 033 077 901 180 829 878 902 908 429 849 730 794 323 849 115 384 173 271 449 600}

C^2 Four-Point ω=0.013723… (Tightest) ↓



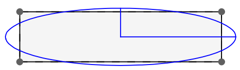
Area 1.78413 (≈ 1.78413)
Centroid≈ (0.5 0.5)
Inertia ≈ (0.253307 0 0.253307)

Rectangle

Curve coordinates ↓

$$\begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear B-spline ↓



Area 4 (≈ 4.0000000000000000000)

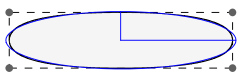
Centroid= $\left(2 \;\; \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\left(\frac{16}{3} \;\; 0 \;\; \frac{1}{3}\right)$

Inertia ≈ (5.3333333333333333333 0 0.33333333333333333333)

Quadratic B-spline ↓



Area $\frac{10}{3}$ (≈ 3.3333333333333333333)

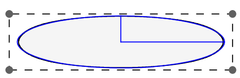
Centroid= $\left(2 \;\; \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\left(\frac{124}{35} \;\; 0 \;\; \frac{31}{140}\right)$

Inertia ≈ (3.5428571428571428571 0 0.22142857142857142857)

Cubic B-spline ↓



Area $\frac{122}{45}$ (≈ 2.7111111111111111111)

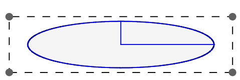
Centroid= $\left(2 \;\; \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\left(\frac{218968}{93555} \;\; 0 \;\; \frac{27371}{187110}\right)$

Inertia ≈ (2.3405269627491849714 0 0.14628293517182406071)

Quartic B-spline ↓



Area $\frac{277}{126}$ (≈ 2.1984126984126984127)

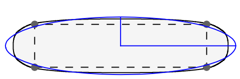
Centroid= $\left(2 \;\; \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\left(\frac{3207559}{2084940} \;\; 0 \;\; \frac{3207559}{33359040}\right)$

Inertia ≈ (1.5384418736270588122 0 0.096152617101691175765)

Three-Point ↓



Area $\frac{37309}{6648}$ (≈ 5.6120637785800240674)

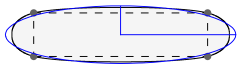
Centroid= $\left(2 \;\; \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\begin{pmatrix} \frac{1163435159449326476495138590944569683421}{114924141928073395083096368562839400000} \\ 0 \\ \frac{1163435159449326476495138590944569683421}{1838786270849174321329541897005430400000} \end{pmatrix}$

Inertia ≈ (10.123505296019315439 0 0.63271908100120721496)

C^1 Four-Point ω=1/16 ↓



Area $\frac{57\,088}{10\,395}$ (≈ 5.4918710918710918711)

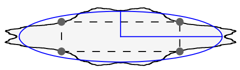
Centroid= $\left(2 \quad \frac{1}{2}\right)$

Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia = $\left(\frac{23\,340\,561\,324\,786\,432\,115\,362\,070\,413\,499\,461\,043\,666\,460\,891}{2\,414\,377\,634\,178\,351\,581\,789\,408\,752\,935\,889\,437\,255\,915\,750} \quad 0 \quad \frac{23\,340\,561\,324\,786\,432\,115\,362\,070\,413\,499\,461\,043\,666\,460\,891}{38\,630\,042\,146\,853\,625\,308\,630\,540\,046\,974\,230\,996\,094\,652\,000}\right)$

Inertia ≈ (9.6673200556422360777 0 0.60420750347763975486)

C^1 Four-Point ω=0.192729… ↓

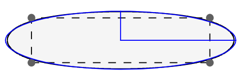


Area 9.01117 (≈ 9.01117)

Centroid≈ (2. 0.5)

Inertia ≈ (27.2329 0 1.70206)

C^2 Four-Point ω=1/128 ↓



Area $\frac{133\,808\,579\,579}{25\,545\,663\,360}$ (≈ 5.2380154585658800445)

Centroid= $\left(2 \quad \frac{1}{2}\right)$

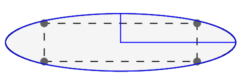
Centroid≈ (2.0000000000000000000 0.50000000000000000000)

Inertia ≈ (8.7378329001734102699 0 0.54611455626083814187)

Inertia =

{696 713 676 660 168 897 181 454 735 579 301 483 887 251 658 615 877 244 232 510 419 599 121 022 735 909 055 599 233 031 - 938 644 572 964 574 722 111 257 / 79 735 294 165 025 973 236 175 154 384 488 499 456 460 888 529 747 731 016 141 842 205 950 466 857 857 944 216 402 043 - 661 310 142 427 877 707 366 400, 0, 696 713 676 660 168 897 181 454 735 579 301 483 887 251 658 615 877 244 232 510 419 599 121 022 735 909 055 599 233 031 - 938 644 572 964 574 722 111 257 / 1 275 764 706 640 415 571 778 802 470 151 815 991 303 374 216 475 963 696 258 269 475 295 207 469 725 727 107 462 432 698 - 580 962 278 846 043 317 862 400}

C^2 Four-Point ω=0.013723… (Tightest) ↓



Area 7.13651 (≈ 7.13651)

Centroid≈ (2. 0.5)

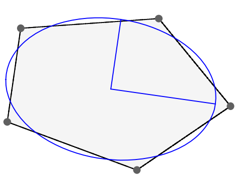
Inertia ≈ (16.2116 0 1.01323)

Pentagon

Curve coordinates ↓

$$\begin{pmatrix} 1.18841447891733 & -2.23495924064644 & -2.56969525310749 & 0.646800233496750 & 2.96943978133985 \\ 1.75075085045311 & 1.50913872078960 & -0.818051827266600 & -2.01472255459931 & -0.427115189376791 \end{pmatrix}$$

Linear B-spline ↓

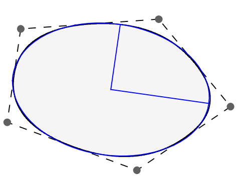


Area 14.26584774 (≈ 14.26584774)

Centroid≈ (0 0)

Inertia ≈ (24.42686118 -1.93415156 11.25823077)

Quadratic B-spline ↓

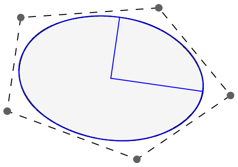


Area 12.62293802 (≈ 12.62293802)

Centroid≈ (0 0)

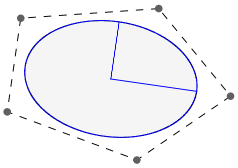
Inertia ≈ (18.81007344 -1.48940679 8.66947849)

Cubic B-spline ↓



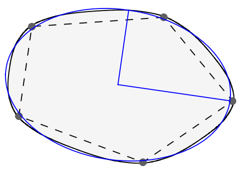
```
Area        11.05570981    (≈ 11.05570981 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 14.42631391 -1.14229484 6.64902338626 )
```

Quartic B-spline ↓



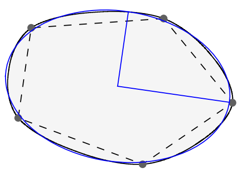
```
Area        9.67581771261    (≈ 9.67581771261 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 11.049409330762 -0.87490701725 5.09262320778 )
```

Three-Point ↓



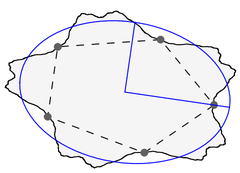
```
Area        18.02872094927    (≈ 18.02872094927 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 38.431778240108 -3.043079630885 17.7130342377 )
```

C^1 Four-Point ω=1/16 ↓



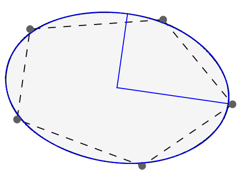
```
Area        17.8084379590    (≈ 17.8084379590 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 37.476320133803 -2.967425179423 17.272667881370 )
```

C^1 Four-Point ω=0.192729… ↓



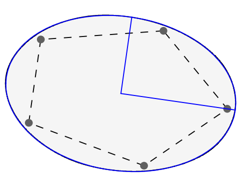
```
Area        25.4077    (≈ 25.4077 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 79.5381 -6.29793 36.6588 )
```

C^2 Four-Point ω=1/128 ↓



```
Area        17.4134720684398    (≈ 17.4134720684398 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 35.789123527692 -2.833830694326 16.495046532105 )
```

C^2 Four-Point ω=0.013723… (Tightest) ↓



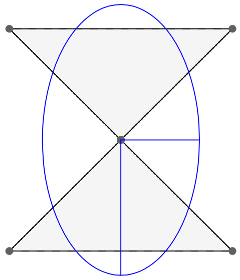
```
Area        21.8187    (≈ 21.8187 )
Centroid≈   ( 0  0 )
Inertia ≈   ( 56.1865 -4.44893 25.8961 )
```

Hourglass

Curve coordinates ↓

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

Linear B-spline ↓



Area 2 (≈ 2.0000000000000000000)

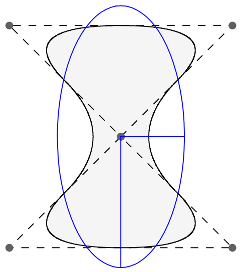
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\left(\frac{1}{3}\ \ 0\ \ 1\right)$

Inertia ≈ (0.33333333333333333333 0 1.0000000000000000000)

Quadratic B-spline ↓



Area $\frac{11}{6}$ (≈ 1.8333333333333333333)

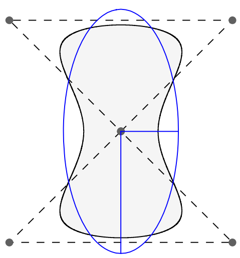
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\left(\frac{41}{240}\ \ 0\ \ \frac{1229}{1680}\right)$

Inertia ≈ (0.17083333333333333333 0 0.73154761904761904762)

Cubic B-spline ↓



Area $\frac{301}{180}$ (≈ 1.6722222222222222222)

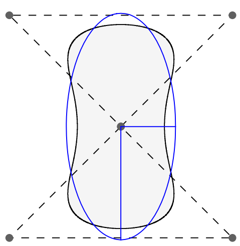
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\left(\frac{70\,657}{598\,752}\ \ 0\ \ \frac{1\,607\,567}{2\,993\,760}\right)$

Inertia ≈ (0.11800712147934370157 0 0.53697256961145850035)

Quartic B-spline ↓



Area $\frac{15\,371}{10\,080}$ (≈ 1.5249007936507936508)

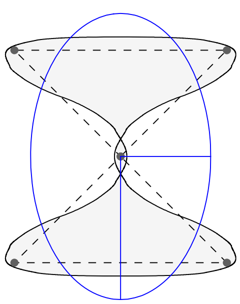
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\left(\frac{155\,315\,107}{1\,660\,538\,880}\ \ 0\ \ \frac{6\,050\,217\,457}{14\,944\,849\,920}\right)$

Inertia ≈ (0.093532954193761485428 0 0.40483628068444330018)

Three-Point ↓



Area $\frac{419\,881}{177\,280}$ (≈ 2.3684623194945848375)

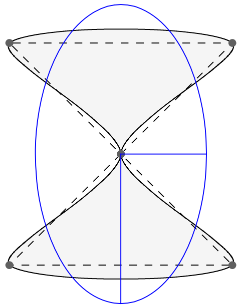
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\left(\begin{matrix}\frac{1\,572\,692\,270\,511\,747\,315\,193\,908\,332\,572\,857\,957\,566\,665\,715\,287\,363}{2\,441\,255\,180\,352\,376\,660\,998\,743\,123\,157\,186\,716\,463\,128\,780\,800\,000} \\ 0 \\ \frac{3\,969\,672\,557\,584\,467\,810\,346\,909\,110\,612\,171\,508\,784\,382\,938\,973\,001}{2\,441\,255\,180\,352\,376\,660\,998\,743\,123\,157\,186\,716\,463\,128\,780\,800\,000}\end{matrix}\right)$

Inertia ≈ (0.64421461679591443593 0 1.6260784982795103372)

C^1 Four-Point ω=1/16 ↓



Area $\frac{78\,223}{33\,264}$ (≈ 2.3515812890812890813)

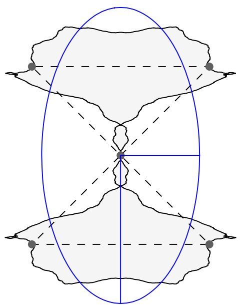
Centroid= (1 0)

Centroid≈ (1.0000000000000000000 0)

Inertia = $\begin{pmatrix} \frac{191\,993\,747\,418\,159\,360\,630\,614\,549\,273\,144\,936\,221\,187\,466\,707\,213\,503}{402\,145\,727\,814\,833\,116\,364\,899\,760\,275\,939\,474\,160\,984\,005\,367\,808\,000} \\ 0 \\ \frac{7\,683\,231\,983\,224\,363\,149\,428\,538\,701\,183\,500\,632\,071\,770\,497\,720\,770\,137}{5\,227\,894\,461\,592\,830\,512\,743\,696\,883\,587\,213\,164\,092\,792\,069\,781\,504\,000} \end{pmatrix}$

Inertia ≈ (0.47742331731685671039 0 1.4696608815785930104)

C^1 Four-Point ω=0.192729… ↓

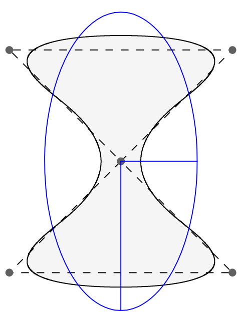


Area 3.07754 (≈ 3.07754)

Centroid≈ (1. 0)

Inertia ≈ (0.920878 0 3.23252)

C^2 Four-Point ω=1/128 ↓



Area $\frac{180\,952\,757\,381}{77\,853\,450\,240}$ (≈ 2.3242740921972528882)

Centroid= (1 0)

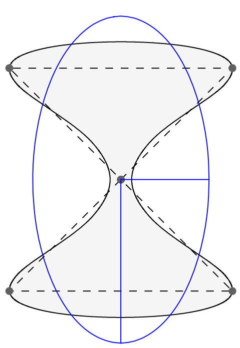
Centroid≈ (1.0000000000000000000 0)

Inertia ≈ (0.33636041699179383667 0 1.2928624843296973503)

Inertia =

{6 862 068 724 607 274 687 678 214 948 362 249 014 157 710 242 195 107 762 632 817 165 914 455 190 786 087 437 833 147 569 - 041 930 410 483 922 128 912 445 351 336 138 298 930 437 242 917 267 / 20 400 940 116 490 247 228 983 981 382 872 431 833 313 418 649 957 998 660 333 723 448 176 208 555 961 138 897 831 397 - 944 789 278 429 183 164 837 145 213 533 436 375 688 557 318 740 377 600, 0, 1 758 374 008 111 131 086 315 270 628 215 542 288 289 983 833 039 059 628 198 169 083 992 587 975 440 998 619 879 027 934 - 368 728 929 411 607 340 632 574 214 169 679 161 206 712 431 867 063 / 1 360 062 674 432 683 148 598 932 092 191 495 455 554 227 909 997 199 910 688 914 896 545 080 570 397 409 259 855 426 529 - 652 618 561 945 544 322 476 347 568 895 758 379 237 154 582 691 840}

C^2 Four-Point ω=0.013723… (Tightest) ↓



Area 2.75318 (≈ 2.75318)

Centroid≈ (1. 0)

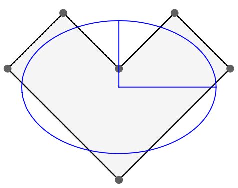
Inertia ≈ (0.568256 0 1.95662)

Axis Heart

Curve coordinates ↓

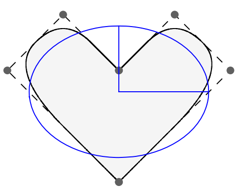
$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & 1 & \frac{3}{2} & 1 & 0 \end{pmatrix}$$

Linear B-spline ↓



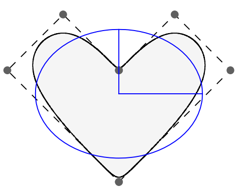
Area	$\frac{3}{2}$ (≈ 1.50000000000000000000)
Centroid=	$\begin{pmatrix} 0 & \frac{5}{6} \end{pmatrix}$
Centroid≈	(0 0.83333333333333333333)
Inertia =	$\begin{pmatrix} \frac{5}{16} & 0 & \frac{7}{48} \end{pmatrix}$
Inertia ≈	(0.31250000000000000000 0 0.14583333333333333333)

Quadratic B-spline ↓



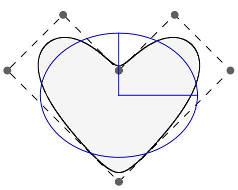
Area	$\frac{11}{8}$ (≈ 1.37500000000000000000)
Centroid=	$\begin{pmatrix} 0 & \frac{89}{110} \end{pmatrix}$
Centroid≈	(0 0.80909090909090909091)
Inertia =	$\begin{pmatrix} \frac{2161}{8960} & 0 & \frac{63\,689}{492\,800} \end{pmatrix}$
Inertia ≈	(0.24118303571428571429 0 0.12923904220779220779)

Cubic B-spline ↓



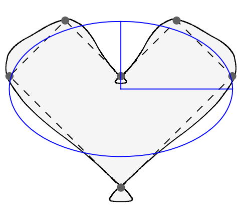
Area	$\frac{201}{160}$ (≈ 1.25625000000000000000)
Centroid=	$\begin{pmatrix} 0 & \frac{119\,963}{151\,956} \end{pmatrix}$
Centroid≈	(0 0.78945879070257179710)
Inertia =	$\begin{pmatrix} \frac{6\,063\,109}{31\,933\,440} & 0 & \frac{45\,503\,983\,271}{404\,373\,150\,720} \end{pmatrix}$
Inertia ≈	(0.18986707977593394260 0 0.11252968499510570077)

Quartic B-spline ↓



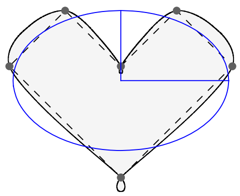
Area	$\frac{1033}{896}$ (≈ 1.15290178571428571430)
Centroid=	$\begin{pmatrix} 0 & \frac{171\,756\,779}{220\,896\,720} \end{pmatrix}$
Centroid≈	(0 0.77754336506218833851)
Inertia =	$\begin{pmatrix} \frac{193\,875\,307}{1\,265\,172\,480} & 0 & \frac{1\,160\,375\,704\,014\,139}{12\,092\,557\,978\,828\,800} \end{pmatrix}$
Inertia ≈	(0.15324021828233254015 0 0.09595783671624163826)

Three-Point ↓



Area	$\frac{253\,751}{141\,824}$ (≈ 1.78919646886281588450)
Centroid=	$\begin{pmatrix} 0 & \frac{75\,928\,267\,379\,578\,499}{85\,780\,453\,890\,717\,600} \end{pmatrix}$
Centroid≈	(0 0.88514648659133269924)
Inertia =	$\begin{pmatrix} \frac{11\,240\,120\,244\,997\,622\,785\,662\,641\,600\,058\,271\,695\,411\,779\,450\,610\,313}{23\,436\,049\,731\,382\,815\,945\,587\,933\,982\,308\,992\,478\,046\,036\,295\,680\,000} \\ 0 \\ \frac{2\,205\,953\,876\,608\,147\,787\,928\,434\,763\,463\,648\,922\,955\,119\,551\,489\,164\,122\,974\,862\,144\,417}{12\,564\,718\,646\,021\,551\,490\,288\,142\,678\,952\,340\,665\,131\,811\,167\,256\,815\,511\,506\,124\,800\,000} \end{pmatrix}$
Inertia ≈	(0.47960814104034643558 0 0.17556731183205867471)

C^1 Four-Point ω=1/16 ↓



Area $\frac{393\,023}{221\,760}$ (≈ 1.7722898629148629149)

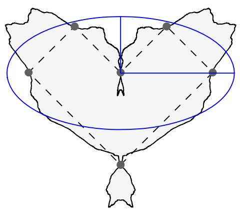
Centroid= $\left(0 \quad \frac{33\,217\,256\,278\,994\,614\,499}{38\,104\,466\,357\,332\,192\,320}\right)$

Centroid≈ (0 0.87174180494992907654)

Inertia = $\begin{pmatrix} \frac{12\,424\,508\,062\,106\,609\,074\,560\,017\,673\,148\,847\,656\,454\,406\,522\,513\,968\,101}{27\,882\,103\,795\,161\,762\,734\,633\,050\,045\,798\,470\,208\,494\,891\,038\,834\,688\,000} \\ 0 \\ \frac{2\,897\,705\,258\,480\,365\,314\,424\,264\,728\,405\,377\,800\,009\,532\,156\,130\,071\,070\,558\,820\,546\,315\,853}{15\,370\,843\,258\,599\,329\,112\,280\,627\,386\,408\,165\,719\,112\,416\,829\,668\,228\,456\,928\,954\,298\,368\,000} \end{pmatrix}$

Inertia ≈ (0.44560870131552159629 0 0.18851960232300353528)

C^1 Four-Point ω=0.192729… ↓

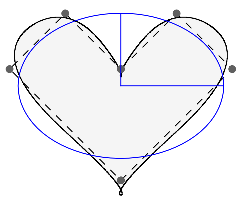


Area 2.39595 (≈ 2.39595)

Centroid≈ (0 0.993494)

Inertia ≈ (0.906361 0 0.222639)

C^2 Four-Point ω=1/128 ↓



Area $\frac{111\,340\,056\,599}{64\,114\,606\,080}$ (≈ 1.7365786582245191890)

Centroid= $\left(0 \quad \frac{53\,053\,624\,474\,195\,197\,357\,689\,384\,038\,175\,621\,434\,026\,407}{62\,427\,056\,129\,007\,134\,166\,837\,813\,962\,540\,540\,658\,412\,512}\right)$

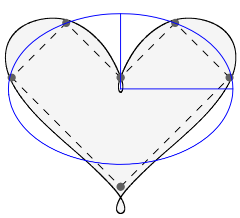
Centroid≈ (0 0.84984985299576682505)

Inertia ≈ (0.40225929542281090774 0 0.20031146587005929084)

Inertia =

{21 883 914 125 926 199 290 897 329 651 922 234 075 255 396 838 046 690 475 830 627 883 775 708 367 676 744 876 756 049 929 - 386 677 720 333 256 811 895 507 826 272 306 124 176 110 272 529 847 /
54 402 506 977 307 325 943 957 283 687 659 818 222 169 116 399 887 996 427 556 595 861 803 222 815 896 370 394 217 061 - 186 104 742 477 821 772 899 053 902 755 830 335 169 486 183 307 673 600, 0,
37 895 246 662 633 504 056 412 970 938 187 991 671 087 332 124 078 227 671 843 626 526 335 075 569 036 559 247 693 373 817 - 084 778 358 930 822 249 818 151 391 419 521 310 555 140 599 866 462 832 807 065 972 431 561 637 253 961 685 227 459 119 /
189 181 615 231 231 432 851 353 091 899 900 464 380 211 814 879 577 513 667 225 185 539 908 949 618 654 171 018 041 431 - 552 642 393 925 555 486 209 726 646 446 956 350 846 363 078 871 160 581 954 171 791 417 110 381 577 945 453 912 758 681 - 600}

C^2 Four-Point ω=0.013723… (Tightest) ↓



Area 2.0746 (≈ 2.0746)

Centroid≈ (0 0.895209)

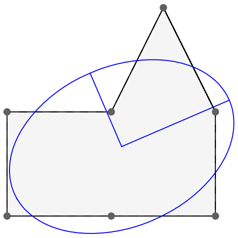
Inertia ≈ (0.592779 0 0.265731)

Dome

Curve coordinates ↓

$$\begin{pmatrix} 0 & 1 & 2 & 2 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

Linear B-spline ↓



Area $\frac{5}{2}$ $(\approx 2.50000000000000000000)$

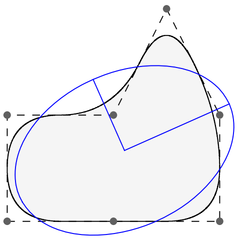
Centroid= $\left(\frac{11}{10} \quad \frac{2}{3}\right)$

Centroid≈ $(1.10000000000000000000 \quad 0.66666666666666666667)$

Inertia = $\left(\frac{63}{80} \quad \frac{1}{6} \quad \frac{17}{36}\right)$

Inertia ≈ $(0.78750000000000000000 \quad 0.16666666666666666667 \quad 0.47222222222222222222$

Quadratic B-spline ↓



Area $\frac{113}{48}$ $(\approx 2.35416666666666666667)$

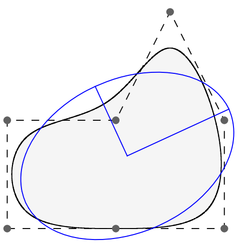
Centroid= $\left(\frac{499}{452} \quad \frac{377}{565}\right)$

Centroid≈ $(1.10398230088495575221 \quad 0.66725663716814159292)$

Inertia = $\left(\frac{265\,675}{404\,992} \quad \frac{877\,547}{6\,074\,880} \quad \frac{996\,637}{2\,531\,200}\right)$

Inertia ≈ $(0.65600061235777496839 \quad 0.14445503450273914876 \quad 0.39374091340075853350$

Cubic B-spline ↓



Area $\frac{199}{90}$ $(\approx 2.21111111111111111111)$

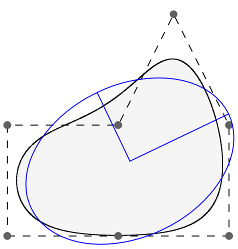
Centroid= $\left(\frac{147\,869}{133\,728} \quad \frac{538\,339}{802\,368}\right)$

Centroid≈ $(1.10574449629097870300 \quad 0.67093777418840232911)$

Inertia = $\left(\frac{146\,945\,298\,439}{266\,899\,691\,520} \quad \frac{1\,453\,383\,887}{11\,862\,208\,512} \quad \frac{1\,092\,313\,165\,117}{3\,202\,796\,298\,240}\right)$

Inertia ≈ $(0.55056376274600798759 \quad 0.12252220027406646888 \quad 0.34104984001550386405$

Quartic B-spline ↓



Area $\frac{83}{40}$ $(\approx 2.07500000000000000000)$

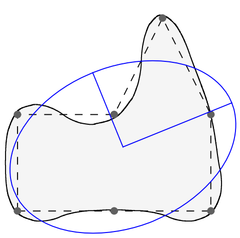
Centroid= $\left(\frac{878\,934\,515}{795\,142\,656} \quad \frac{268\,513\,307}{397\,571\,328}\right)$

Centroid≈ $(1.10537965529546436510 \quad 0.67538398292142435382)$

Inertia = $\left(\frac{1\,849\,841\,308\,004\,870\,987}{3\,961\,095\,886\,303\,395\,840} \quad \frac{205\,679\,883\,206\,147\,843}{1\,980\,547\,943\,151\,697\,920} \quad \frac{295\,885\,986\,651\,682\,043}{990\,273\,971\,575\,848\,960}\right)$

Inertia ≈ $(0.46700240567293967290 \quad 0.10384998955331727387 \quad 0.29879204658972396005$

Three-Point ↓



Area $\frac{899}{320}$ $(\approx 2.80937500000000000000)$

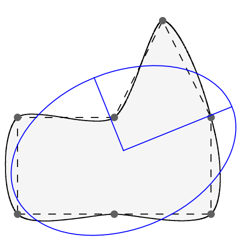
Centroid= $\left(\frac{293\,698\,903\,960\,362\,227}{269\,382\,895\,442\,895\,360} \quad \frac{268\,072\,688\,081\,383\,897}{404\,074\,343\,164\,343\,040}\right)$

Centroid≈ $(1.09026559937106864880 \quad 0.66342417581399062285)$

Inertia = $\left(\begin{array}{c}\frac{2\,457\,382\,970\,931\,809\,433\,329\,136\,850\,910\,033\,421\,515\,076\,003\,320\,397\,942\,330\,195\,339\,997}{2\,267\,697\,893\,097\,554\,955\,256\,226\,903\,469\,175\,147\,636\,580\,479\,671\,999\,186\,547\,507\,200\,000} \\ \frac{2\,222\,056\,302\,966\,153\,154\,363\,009\,120\,038\,983\,477\,146\,966\,730\,808\,341\,379\,147\,071\,360\,477}{10\,204\,640\,518\,938\,997\,298\,653\,021\,065\,611\,288\,164\,364\,612\,158\,523\,996\,339\,463\,782\,400\,000} \\ \frac{9\,701\,003\,881\,222\,499\,461\,064\,848\,860\,658\,268\,780\,943\,608\,813\,772\,748\,250\,255\,842\,624\,927}{15\,306\,960\,778\,408\,495\,947\,979\,531\,598\,416\,932\,246\,546\,918\,237\,785\,994\,509\,195\,673\,600\,000}\end{array}\right)$

Inertia ≈ $(1.08364653793242041160 \quad 0.21774959135916588456 \quad 0.63376420843165822542)$

C^1 Four-Point ω=1/16 ↓



Area $\frac{106\,445}{38\,016}$ $(\approx 2.80000526094276094280)$

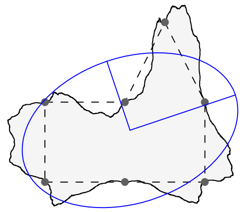
Centroid= $\left(\frac{791\,020\,381\,158\,472\,310\,627}{722\,405\,799\,402\,161\,616\,000} \quad \frac{79\,219\,953\,921\,031\,229\,627}{120\,400\,966\,567\,026\,936\,000}\right)$

Centroid≈ $(1.09498066296407611990 \quad 0.65796775706887425097)$

Inertia = $\left(\begin{array}{c}\frac{350\,393\,537\,942\,925\,339\,446\,161\,823\,337\,654\,215\,018\,247\,346\,750\,064\,503\,400\,392\,889\,234\,716\,331}{337\,051\,430\,407\,595\,004\,389\,273\,777\,993\,819\,227\,533\,009\,289\,452\,869\,396\,276\,395\,396\,300\,800\,000} \\ \frac{335\,024\,286\,482\,568\,647\,335\,019\,203\,599\,125\,072\,856\,352\,888\,480\,422\,385\,420\,093\,398\,739\,751\,791}{1\,554\,181\,595\,768\,354\,742\,461\,651\,309\,638\,166\,438\,068\,876\,168\,032\,675\,549\,496\,712\,105\,164\,800\,000} \\ \frac{376\,844\,149\,273\,869\,934\,453\,067\,101\,198\,943\,485\,006\,036\,647\,913\,216\,018\,459\,363\,679\,328\,970\,787}{635\,801\,561\,905\,236\,031\,007\,039\,172\,124\,704\,451\,937\,267\,523\,286\,094\,542\,975\,927\,679\,385\,600\,000}\end{array}\right)$

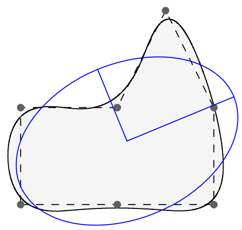
Inertia ≈ $(1.03958478241494413920 \quad 0.21556315387742115300 \quad 0.59270717760526234607)$

C^1 Four-Point ω=0.192729… ↓



Area 3.39259 (≈ 3.39259)
Centroid≈ (1.06408 0.648801)
Inertia ≈ (1.80962 0.330911 0.933294)

C^2 Four-Point ω=1/128 ↓



Area $\frac{5\,210\,379\,966\,589}{1\,868\,482\,805\,760}$ (≈ 2.7885619019489413789)

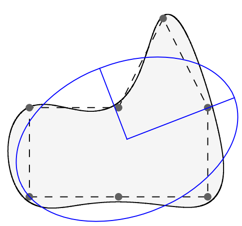
Centroid= $\left(\frac{3\,004\,463\,918\,649\,115\,412\,936\,564\,115\,865\,665\,238\,875\,561}{2\,727\,729\,761\,392\,941\,498\,447\,981\,796\,087\,725\,370\,724\,192} \;\; \frac{9\,383\,677\,212\,989\,275\,016\,629\,708\,803\,826\,316\,650\,569\,107}{14\,320\,581\,247\,312\,942\,866\,851\,904\,429\,460\,558\,196\,302\,008}\right)$

Centroid≈ (1.1014521897194306189 0.65525812471822615683)
Inertia ≈ (0.98860687812566629777 0.21037256583037583780 0.56487373032592115455

Inertia =

{85 248 860 989 728 789 489 275 214 555 858 780 363 058 878 578 578 213 318 535 356 863 794 852 748 196 299 267 189 951 858 -
379 388 544 756 186 262 549 235 613 440 039 585 272 991 843 891 893 383 339 305 903 094 397 497 602 242 991 235 359 /
86 231 304 754 175 927 014 673 898 837 571 397 340 812 975 389 839 867 599 904 146 796 042 596 995 937 610 697 322 768 -
250 850 732 542 321 797 146 671 750 444 519 475 211 226 595 347 938 221 310 483 160 636 764 459 966 691 605 584 281 600,
50 679 071 235 608 911 756 757 904 109 609 176 010 939 364 100 680 176 371 743 084 479 139 055 418 367 239 658 337 013 152 -
112 879 864 934 301 329 153 881 273 001 644 658 123 316 268 569 267 067 927 444 740 633 900 891 081 000 238 982 820 443 /
240 901 521 714 916 148 103 327 315 385 895 293 704 451 182 247 415 976 784 932 218 099 211 001 807 651 038 418 087 373 -
570 126 663 145 732 993 962 085 313 491 839 240 581 763 365 203 690 077 601 053 123 098 907 646 326 947 448 800 621 363 -
200,
714 414 441 364 551 812 354 057 443 518 199 493 357 608 458 428 171 162 629 676 806 864 471 416 687 394 427 071 491 081 -
006 912 555 931 006 334 081 893 695 135 088 924 539 659 313 670 754 044 165 328 332 571 410 936 013 320 208 044 528 788 -
319 /
1 264 732 989 003 309 777 542 468 405 775 950 291 948 368 706 798 933 878 120 894 145 020 857 759 490 167 951 694 958 711 -
243 164 981 515 098 218 300 947 895 832 156 013 054 257 667 319 372 907 405 528 896 269 265 143 216 474 106 203 262 156 -
800}

C^2 Four-Point ω=0.013723… (Tightest) ↓



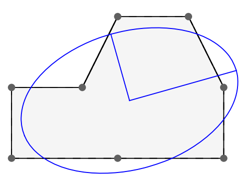
Area 3.14611 (≈ 3.14611)
Centroid≈ (1.09447 0.646055)
Inertia ≈ (1.33832 0.273627 0.734998)

Toycar

Curve coordinates ↓

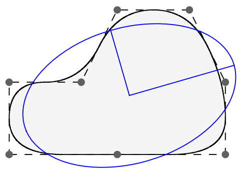
$$\begin{pmatrix} 0 & \frac{3}{2} & 3 & 3 & \frac{5}{2} & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$$

Linear B-spline ↓



Area $\frac{9}{2}$ (≈ 4.5000000000000000000)
Centroid= $\left(\frac{5}{3} \;\; \frac{22}{27}\right)$
Centroid≈ (1.6666666666666666667 0.81481481481481481481)
Inertia = $\left(\frac{45}{16} \;\; \frac{17}{36} \;\; \frac{409}{324}\right)$
Inertia ≈ (2.8125000000000000000 0.47222222222222222222 1.2623456790123456790)

Quadratic B-spline ↓



Area $\frac{205}{48}$ (≈ 4.27083333333333333)

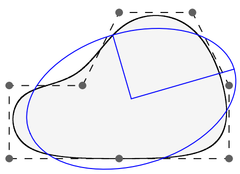
Centroid= $\left(\frac{1367}{820} \quad \frac{836}{1025} \right)$

Centroid≈ (1.66707317073170731 0.81560975609756097561)

Inertia = $\left(\frac{1\,790\,307}{734\,720} \quad \frac{1\,541\,341}{3\,673\,600} \quad \frac{5\,053\,733}{4\,592\,000} \right)$

Inertia ≈ (2.43672011106271777 00 0.41957235409407665505 1.10055161149825783 97)

Cubic B-spline ↓



Area $\frac{1457}{360}$ (≈ 4.04722222222222222)

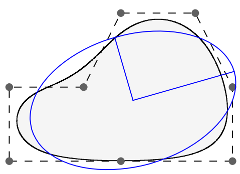
Centroid= $\left(\frac{271\,811}{163\,184} \quad \frac{150\,095}{183\,582} \right)$

Centroid≈ (1.66567187959603882 73 0.81759104923140612914)

Inertia = $\left(\frac{4\,161\,561\,944\,383}{1\,954\,134\,927\,360} \quad \frac{17\,880\,635\,975}{48\,853\,373\,184} \quad \frac{177\,794\,811\,011}{183\,200\,149\,440} \right)$

Inertia ≈ (2.12961852639581695 09 0.36600616927013135519 0.97049490163887499501)

Quartic B-spline ↓



Area $\frac{154\,723}{40\,320}$ (≈ 3.83737599206349206 35)

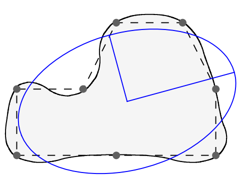
Centroid= $\left(\frac{2\,443\,969\,783}{1\,470\,487\,392} \quad \frac{201\,099\,047}{245\,081\,232} \right)$

Centroid≈ (1.66201342241770135 49 0.82054037903644943322)

Inertia = $\left(\frac{13\,758\,235\,788\,241\,543\,217}{7\,325\,404\,460\,897\,402\,880} \quad \frac{27\,817\,437\,056\,687\,699}{87\,207\,195\,963\,064\,320} \quad \frac{19\,467\,921\,964\,205\,773}{22\,609\,273\,027\,461\,120} \right)$

Inertia ≈ (1.87815374040876954 41 0.31898098258393124461 0.86105917428482214744)

Three-Point ↓



Area $\frac{3\,540\,913}{709\,120}$ (≈ 4.99339039936823104 69)

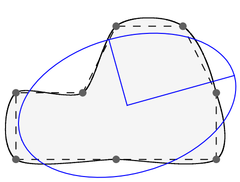
Centroid= $\left(\frac{22\,740\,372\,262\,492\,397}{13\,680\,053\,025\,099\,072} \quad \frac{584\,839\,332\,775\,567\,063}{718\,202\,783\,817\,701\,280} \right)$

Centroid≈ (1.66230147067194642 46 0.81430947631082231814)

Inertia = $\left(\frac{10\,772\,811\,743\,404\,090\,964\,192\,238\,481\,521\,033\,726\,578\,147\,773\,854\,805\,204\,826\,016\,720\,511}{2\,946\,750\,027\,795\,733\,318\,276\,158\,180\,326\,725\,338\,506\,353\,321\,863\,620\,913\,050\,419\,200\,000} \quad \frac{1\,316\,881\,837\,123\,537\,275\,075\,665\,610\,060\,263\,244\,211\,966\,608\,036\,078\,742\,888\,117\,375\,871}{2\,210\,062\,520\,846\,799\,988\,707\,118\,635\,245\,044\,003\,879\,764\,991\,397\,715\,684\,787\,814\,400\,000} \quad \frac{74\,942\,025\,555\,958\,499\,761\,104\,307\,461\,019\,854\,016\,154\,759\,196\,475\,333\,781\,599\,220\,903\,917}{46\,411\,312\,937\,782\,799\,762\,849\,491\,340\,145\,924\,081\,475\,064\,819\,352\,029\,380\,544\,102\,400\,000} \right)$

Inertia ≈ (3.65582816383733478 06 0.59585727765699829120 1.61473616694323780 91)

C^1 Four-Point ω=1/16 ↓



Area $\frac{275\,923}{55\,440}$ (≈ 4.97696608946608946 61)

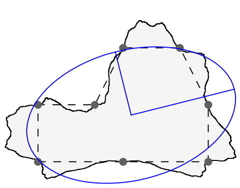
Centroid= $\left(\frac{133\,677\,220\,478\,758\,063\,129}{80\,254\,071\,675\,557\,184\,960} \quad \frac{1\,278\,434\,647\,295\,477\,753}{1\,573\,609\,248\,540\,336\,960} \right)$

Centroid≈ (1.66567524472994253 14 0.81242192016940673344)

Inertia = $\left(\frac{10\,092\,248\,407\,806\,152\,956\,461\,123\,542\,608\,424\,161\,655\,105\,717\,425\,146\,318\,920\,630\,209\,453\,079}{2\,846\,016\,936\,866\,907\,195\,782\,977\,027\,577\,073\,752\,699\,363\,296\,313\,003\,251\,872\,125\,386\,752\,000} \quad \frac{567\,785\,884\,388\,458\,520\,800\,089\,068\,714\,978\,392\,325\,544\,492\,137\,146\,471\,608\,953\,869\,278\,793}{952\,160\,078\,142\,972\,628\,000\,922\,461\,432\,035\,704\,028\,095\,808\,692\,953\,293\,824\,865\,478\,656\,000} \quad \frac{996\,358\,406\,196\,662\,173\,448\,888\,363\,430\,852\,719\,293\,611\,019\,101\,946\,616\,444\,527\,154\,648\,459}{634\,773\,385\,428\,648\,418\,667\,281\,640\,954\,690\,469\,352\,063\,872\,461\,968\,862\,549\,910\,319\,104\,000} \right)$

Inertia ≈ (3.54609569502998104 49 0.59631347440635083658 1.56962851478696352 82)

C^1 Four-Point ω=0.192729… ↓

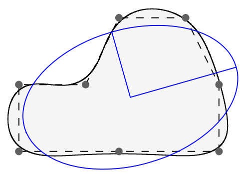


Area 5.97624 (≈ 5.97624)

Centroid≈ (1.64085 0.819887)

Inertia ≈ (5.6933 0.874739 2.41909)

C^2 Four-Point ω=1/128 ↓



Area $\frac{3\,236\,186\,052\,947}{653\,968\,982\,016}$ (≈ 4.9485314165371584815)

Centroid= $\left(\frac{396\,531\,895\,788\,692\,440\,640\,423\,645\,026\,926\,099\,588\,059\,141}{237\,188\,410\,313\,386\,344\,247\,470\,376\,165\,140\,904\,582\,586\,240}\;\; \frac{288\,101\,768\,284\,716\,691\,056\,251\,028\,722\,761\,195\,174\,174\,357}{355\,782\,615\,470\,079\,516\,371\,205\,564\,247\,711\,356\,873\,879\,360}\right)$

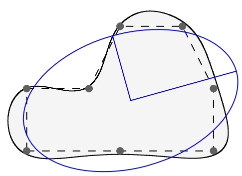
Centroid≈ (1.6718013129932138710 0.80976909988719045254)

Inertia ≈ (3.4196314721867912093 0.58593463773240902914 1.5249255849301451544)

Inertia =

{18 803 568 592 762 397 901 465 214 688 155 761 569 655 800 270 403 459 487 662 229 476 397 707 662 478 902 489 308 066 385 -
905 286 701 593 137 394 451 765 903 769 367 398 397 265 353 853 811 934 419 207 047 017 470 637 049 875 369 525 553 859 -
927 /
5 498 711 994 464 673 367 122 792 585 421 464 871 342 552 422 937 226 623 402 045 648 444 817 913 453 422 301 049 263 553 -
220 688 035 821 828 618 572 110 497 953 157 544 425 385 289 112 175 656 576 726 595 296 910 816 140 588 084 986 957 004 -
800,
80 547 145 511 787 767 802 229 631 212 364 701 096 079 667 387 562 075 681 212 370 956 509 739 656 855 257 094 685 988 662 -
454 141 691 507 403 357 034 396 029 985 413 660 555 496 835 516 983 264 547 378 225 129 344 824 542 725 553 153 937 199 /
137 467 799 861 616 834 178 069 814 635 536 621 783 563 810 573 430 665 585 051 141 211 120 447 836 335 557 526 231 588 -
830 517 200 895 545 715 464 302 762 448 828 938 610 634 632 227 804 391 414 418 164 882 422 770 403 514 702 124 673 925 -
120,
23 036 062 100 333 645 930 730 524 439 226 448 786 625 110 757 894 403 609 834 050 906 597 214 205 677 118 312 221 087 535 -
912 825 451 054 506 047 195 021 657 610 500 352 232 708 473 713 896 920 585 893 529 530 169 541 200 607 243 138 770 393 /
15 106 351 633 144 707 052 535 144 465 443 584 811 380 638 524 552 820 393 961 663 869 353 895 366 630 281 046 838 636 -
135 221 670 428 081 946 754 318 984 884 486 696 550 619 190 354 703 779 276 309 688 448 617 886 857 529 088 145 568 563 -
200}

C^2 Four-Point ω=0.013723… (Tightest) ↓



Area 5.51887 (≈ 5.51887)

Centroid≈ (1.66872 0.808129)

Inertia ≈ (4.4216 0.746355 1.94727)

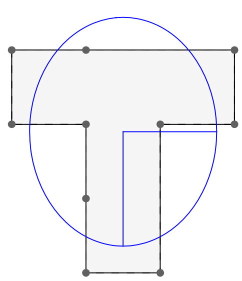
Letter T

The symmetry of the otherwise symmetric T shape is broken deliberately by inserting additional control points to make the resulting curve more interesting, and the values of the area moments less trivial.

Curve coordinates ↓

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 1 \\ 3 & 2 & 2 & 1 & 0 & 0 & 2 & 2 & 3 & 3 \end{pmatrix}$$

Linear B-spline ↓



Area 5 (≈ 5.0000000000000000000)

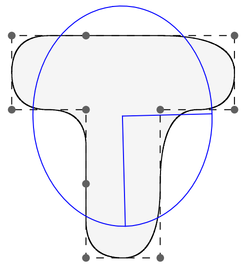
Centroid= $\left(\frac{3}{2}\;\; \frac{19}{10}\right)$

Centroid≈ (1.5000000000000000000 1.9000000000000000000)

Inertia = $\left(\frac{29}{12}\;\; 0\;\; \frac{217}{60}\right)$

Inertia ≈ (2.4166666666666666667 0 3.6166666666666666667)

Quadratic B-spline ↓



Area $\frac{115}{24}$ (≈ 4.7916666666666666667)

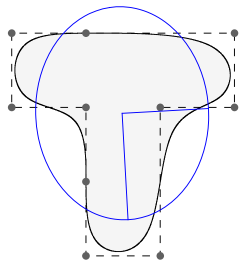
Centroid= $\left(\frac{343}{230} \quad \frac{2201}{1150}\right)$

Centroid≈ (1.4913043478260869565 1.9139130434782608696)

Inertia = $\left(\frac{317\,137}{154\,560} \quad -\frac{41\,999}{1\,545\,600} \quad \frac{12\,020\,011}{3\,864\,000}\right)$

Inertia ≈ (2.0518698240165631470 −0.027173266045548654244 3.1107688923395445135

Cubic B-spline ↓



Area $\frac{3307}{720}$ (≈ 4.5930555555555555556)

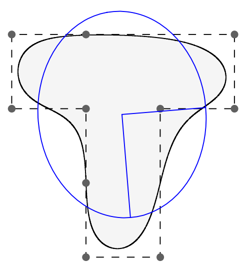
Centroid= $\left(\frac{309\,716}{208\,341} \quad \frac{800\,305}{416\,682}\right)$

Centroid≈ (1.4865820937789489347 1.9206613196634363855)

Inertia = $\left(\frac{296\,697\,146\,167}{166\,326\,120\,576} \quad -\frac{85\,372\,363\,553}{1\,663\,261\,205\,760} \quad \frac{2\,251\,341\,722\,633}{831\,630\,602\,880}\right)$

Inertia ≈ (1.7838277303619854014 −0.051328296035132073566 2.7071415058998940912

Quartic B-spline ↓



Area $\frac{5567}{1260}$ (≈ 4.4182539682539682540)

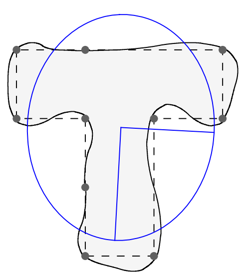
Centroid= $\left(\frac{209\,468\,327}{141\,090\,048} \quad \frac{50\,851\,139}{26\,454\,384}\right)$

Centroid≈ (1.4846428218665004636 1.9222197349218186294)

Inertia = $\left(\frac{92\,957\,590\,217\,000\,611}{58\,571\,377\,571\,266\,560} \quad -\frac{731\,470\,725\,596\,911}{10\,982\,133\,294\,612\,480} \quad \frac{39\,382\,778\,600\,971\,291}{16\,473\,199\,941\,918\,720}\right)$

Inertia ≈ (1.5870821905101126031 −0.066605522440321279314 2.3907181810350911222

Three-Point ↓



Area $\frac{1\,456\,499}{265\,920}$ (≈ 5.4772074308062575211)

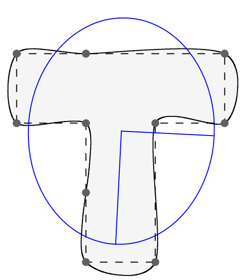
Centroid= $\left(\frac{598\,073\,505\,431\,557\,139}{393\,895\,260\,507\,745\,920} \quad \frac{184\,267\,405\,356\,569\,543}{98\,473\,815\,126\,936\,480}\right)$

Centroid≈ (1.5183566937581775486 1.8712325212447783555)

Inertia = $\begin{pmatrix} \frac{140\,772\,356\,225\,439\,869\,691\,982\,022\,197\,024\,975\,874\,720\,092\,827\,355\,456\,718\,417\,372\,064\,151}{43\,271\,948\,035\,385\,264\,430\,216\,683\,734\,254\,616\,280\,740\,925\,947\,007\,226\,459\,859\,845\,120\,000} \\ \frac{4\,329\,939\,125\,816\,508\,253\,338\,961\,060\,497\,398\,549\,393\,625\,305\,086\,077\,744\,673\,627\,219\,997}{54\,089\,935\,044\,231\,580\,537\,770\,854\,667\,818\,270\,350\,926\,157\,433\,759\,033\,074\,824\,806\,400\,000} \\ \frac{25\,694\,988\,685\,802\,561\,142\,134\,221\,579\,876\,442\,985\,934\,513\,768\,199\,724\,739\,333\,555\,847\,291}{5\,408\,993\,504\,423\,158\,053\,777\,085\,466\,781\,827\,035\,092\,615\,743\,375\,903\,307\,482\,480\,640\,000} \end{pmatrix}$

Inertia ≈ (3.2532012681824373280 0.080050736283483011475 4.7504195863409161963)

C^1 Four-Point ω=1/16 ↓



Area $\frac{1\,209\,407}{221\,760}$ (≈ 5.4536751443001443001)

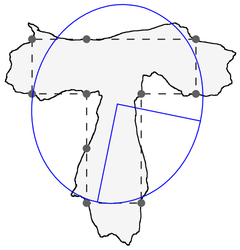
Centroid= $\left(\frac{531\,132\,116\,051\,023\,421\,491}{351\,764\,209\,808\,245\,736\,640} \quad \frac{73\,698\,801\,254\,482\,029\,211}{39\,084\,912\,200\,916\,192\,960}\right)$

Centroid≈ (1.5099094826632732252 1.8856074404269596632)

Inertia = $\begin{pmatrix} \frac{3\,511\,953\,009\,278\,964\,672\,443\,332\,726\,489\,749\,752\,644\,148\,310\,086\,453\,380\,371\,891\,928\,787\,218\,111}{1\,135\,176\,644\,594\,510\,071\,341\,126\,197\,225\,937\,849\,613\,152\,860\,817\,930\,543\,916\,813\,570\,572\,288\,000} \\ \frac{16\,862\,929\,564\,216\,989\,173\,285\,281\,818\,984\,463\,606\,999\,172\,611\,961\,227\,339\,511\,778\,156\,495\,299}{227\,035\,328\,918\,902\,014\,268\,225\,239\,445\,187\,569\,922\,630\,572\,163\,586\,108\,783\,362\,714\,114\,457\,600} \\ \frac{1\,036\,445\,598\,463\,172\,877\,534\,857\,207\,623\,597\,743\,838\,420\,702\,748\,655\,127\,893\,779\,358\,698\,083\,967}{227\,035\,328\,918\,902\,014\,268\,225\,239\,445\,187\,569\,922\,630\,572\,163\,586\,108\,783\,362\,714\,114\,457\,600} \end{pmatrix}$

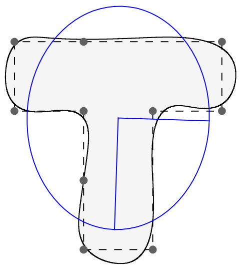
Inertia ≈ (3.0937502335008391878 0.074274473688808579130 4.5651291514784276430)

C^1 Four-Point ω=0.192729… ↓



```
Area          6.56749    (≈ 6.56749 )
Centroid≈     ( 1.56173  1.81404 )
Inertia ≈     ( 5.51705  0.408631  7.43572 )
```

C^2 Four-Point ω=1/128 ↓



Area $\frac{5\,036\,901\,605\,833}{934\,241\,402\,880}$ (≈ 5.3914347943750595855)

Centroid= $\begin{pmatrix} \frac{2\,595\,192\,806\,879\,677\,216\,806\,254\,291\,916\,252\,312\,428\,731}{1\,730\,472\,612\,928\,113\,491\,882\,092\,552\,374\,416\,198\,817\,147} \\ \frac{1\,473\,655\,849\,938\,034\,908\,325\,932\,516\,629\,081\,348\,622\,423\,799}{775\,251\,730\,591\,794\,844\,363\,177\,463\,463\,738\,457\,070\,081\,856} \end{pmatrix}$

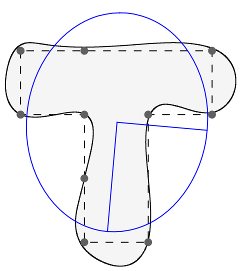
Centroid≈ (1.4997017505456964657 1.9008739894241931993)

Inertia ≈ (2.8977510114631261544 0.047924795226393687272 4.3366519636169417484)

Inertia =

{12 400 009 816 773 174 860 159 783 643 349 405 143 545 841 537 621 604 266 638 385 623 343 017 894 309 041 358 176 665 810 - 992 419 816 897 795 597 462 332 554 649 385 821 530 022 679 448 182 242 621 109 761 680 955 221 720 855 949 068 213 296 - 057 /
4 279 184 018 130 042 444 128 351 482 559 579 446 516 423 211 197 901 207 683 775 527 389 826 701 855 597 675 242 997 363 - 235 203 453 114 169 694 308 129 045 361 292 485 296 017 559 256 848 820 732 018 538 617 563 648 516 251 807 649 470 873 - 600,
526 518 659 319 483 481 011 938 103 374 056 998 809 840 101 419 182 292 257 779 216 392 667 213 259 478 986 219 553 482 - 668 407 949 058 374 697 033 022 866 678 207 377 579 834 321 696 832 998 599 744 490 480 659 193 863 093 050 276 749 513 /
10 986 351 779 537 978 033 705 652 073 323 695 626 486 324 033 884 213 626 915 983 382 258 861 878 961 739 859 417 194 - 770 822 088 454 721 873 412 857 840 938 026 424 865 971 803 746 487 416 741 288 879 431 624 040 175 908 220 302 052 556 - 800,
43 300 440 808 104 394 657 221 077 282 063 941 634 118 855 527 908 850 741 648 039 067 855 153 280 190 894 900 992 565 718 - 247 820 986 179 468 033 459 367 600 733 995 872 403 654 267 471 948 141 604 231 740 256 182 733 326 028 985 252 367 746 - 917 /
9 984 762 708 970 099 036 299 486 792 639 018 708 538 320 826 128 436 151 262 142 897 242 928 970 996 394 575 566 993 847 - 548 808 057 266 395 953 385 634 439 176 349 132 357 374 304 932 647 248 374 709 923 440 981 846 537 920 884 515 432 038 - 400}

C^2 Four-Point ω=0.013723… (Tightest) ↓



```
Area          5.94558    (≈ 5.94558 )
Centroid≈     ( 1.51182  1.87934 )
Inertia ≈     ( 3.86144  0.15817  5.63274 )
```

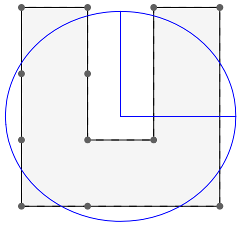
Letter U

The symmetry of the otherwise symmetric letter U is broken deliberately by inserting additional control points to make the result more interesting, and challenging to compute.

Curve coordinates ↓

$$\begin{pmatrix} 0 & 1 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \end{pmatrix}$$

Linear B-spline ↓



Area $7 \quad (\approx 7.0000000000000000000)$

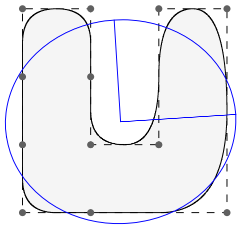
Centroid= $\left(\frac{3}{2} \quad \frac{19}{14}\right)$

Centroid≈ $(1.5000000000000000000 \quad 1.3571428571428571429)$

Inertia = $\left(\frac{79}{12} \quad 0 \quad \frac{457}{84}\right)$

Inertia ≈ $(6.5833333333333333333 \quad 0 \quad 5.4404761904761904762)$

Quadratic B-spline ↓



Area $\frac{157}{24} \quad (\approx 6.5416666666666666667)$

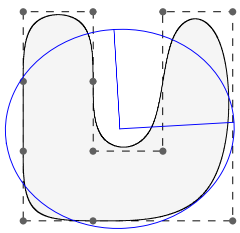
Centroid= $\left(\frac{2261}{1570} \quad \frac{2101}{1570}\right)$

Centroid≈ $(1.4401273885350318471 \quad 1.3382165605095541401)$

Inertia = $\left(\frac{10\,013\,307}{1\,758\,400} \quad \frac{823\,757}{10\,550\,400} \quad \frac{23\,597\,411}{5\,275\,200}\right)$

Inertia ≈ $(5.6945558462238398544 \quad 0.078078271913861085836 \quad 4.4732732408249924173)$

Cubic B-spline ↓



Area $\frac{293}{48} \quad (\approx 6.1041666666666666667)$

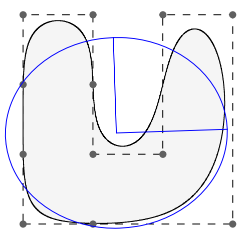
Centroid= $\left(\frac{382\,973}{276\,885} \quad \frac{243\,647}{184\,590}\right)$

Centroid≈ $(1.3831482384383408274 \quad 1.3199360745435830760)$

Inertia = $\left(\frac{5\,444\,786\,198\,543}{1\,105\,236\,316\,800} \quad \frac{50\,678\,792\,867}{736\,824\,211\,200} \quad \frac{41\,852\,421\,617}{11\,164\,003\,200}\right)$

Inertia ≈ $(4.9263547675553543664 \quad 0.068780032057393935972 \quad 3.7488722340208573211)$

Quartic B-spline ↓



Area $\frac{230\,551}{40\,320} \quad (\approx 5.7180307539682539683)$

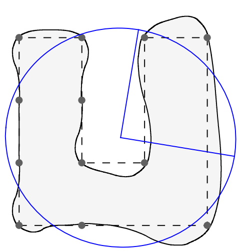
Centroid= $\left(\frac{182\,774\,609}{136\,947\,294} \quad \frac{357\,949\,645}{273\,894\,588}\right)$

Centroid≈ $(1.3346346879990195352 \quad 1.3068883456726059881)$

Inertia = $\left(\frac{167\,294\,945\,284\,979\,339}{38\,983\,938\,205\,335\,552} \quad \frac{97\,684\,244\,547\,773\,189}{2\,728\,875\,674\,373\,488\,640} \quad \frac{4\,345\,721\,331\,764\,429\,707}{1\,364\,437\,837\,186\,744\,320}\right)$

Inertia ≈ $(4.2913813479748039070 \quad 0.035796517029014197786 \quad 3.1849903405820391740)$

Three-Point ↓



Area $\frac{17\,119\,703}{2\,127\,360} \quad (\approx 8.0473934830024067389)$

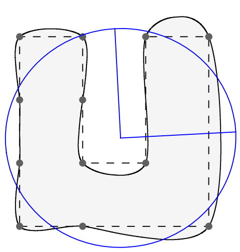
Centroid= $\left(\frac{6\,165\,865\,203\,308\,803}{3\,807\,441\,292\,464\,390} \quad \frac{2\,876\,765\,949\,909\,289}{2\,052\,237\,859\,768\,040}\right)$

Centroid≈ $(1.6194248918590438701 \quad 1.4017702364356749680)$

Inertia = $\begin{pmatrix} \frac{599\,517\,261\,658\,452\,215\,463\,963\,354\,983\,923\,337\,588\,761\,683\,441\,949\,192\,152\,085\,593\,169}{70\,445\,829\,062\,775\,717\,493\,598\,203\,388\,040\,660\,099\,374\,490\,857\,354\,646\,326\,771\,712\,000} \\ -\frac{12\,654\,145\,193\,311\,721\,575\,262\,515\,392\,450\,763\,973\,680\,030\,751\,982\,100\,060\,568\,350\,409}{94\,927\,003\,701\,612\,668\,963\,004\,671\,232\,111\,527\,793\,483\,356\,474\,449\,523\,419\,054\,080\,000} \\ \frac{34\,818\,486\,511\,102\,920\,696\,974\,223\,259\,150\,210\,274\,955\,357\,819\,756\,291\,249\,939\,309\,020\,359}{4\,509\,032\,675\,826\,601\,775\,742\,721\,883\,525\,297\,570\,190\,459\,432\,536\,352\,362\,405\,068\,800\,000} \end{pmatrix}$

Inertia ≈ $(8.5103301307478422235 \quad -0.13330395672329375779 \quad 7.7219414926328849784)$

C^1 Four-Point ω=1/16 ↓



Area $\frac{1\,063\,193}{133\,056} \quad (\approx 7.9905678811928811929)$

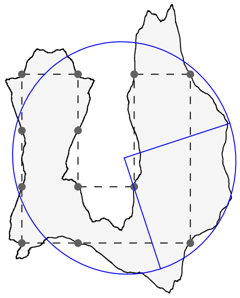
Centroid= $\left(\frac{412\,867\,896\,656\,157\,348\,923}{257\,697\,398\,751\,797\,512\,800} \quad \frac{720\,224\,570\,454\,200\,798\,299}{515\,394\,797\,503\,595\,025\,600}\right)$

Centroid≈ $(1.6021422748384551851 \quad 1.3974230511109825990)$

Inertia = $\begin{pmatrix} \frac{398\,116\,069\,029\,356\,818\,404\,123\,802\,723\,767\,844\,491\,757\,011\,170\,194\,856\,506\,058\,957\,892\,714\,163}{47\,977\,735\,174\,551\,332\,336\,020\,001\,196\,360\,176\,617\,380\,254\,472\,439\,720\,648\,001\,807\,216\,640\,000} \\ \frac{37\,331\,984\,531\,379\,784\,584\,533\,963\,303\,409\,571\,975\,227\,176\,209\,542\,711\,341\,029\,838\,061\,782\,233}{831\,614\,076\,358\,889\,760\,491\,013\,354\,070\,243\,061\,367\,924\,410\,855\,621\,824\,565\,364\,658\,421\,760\,000} \\ \frac{37\,175\,983\,161\,761\,761\,444\,322\,119\,853\,595\,111\,432\,975\,415\,218\,837\,047\,773\,147\,894\,340\,666\,714\,417}{4\,989\,684\,458\,153\,338\,562\,946\,080\,124\,421\,458\,368\,207\,546\,465\,133\,730\,947\,392\,187\,950\,530\,560\,000} \end{pmatrix}$

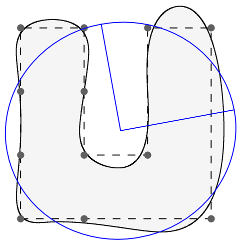
Inertia ≈ $(8.2979337724246762127 \quad 0.044890996428094208483 \quad 7.4505679614699399214)$

C^1 Four-Point ω=0.192729… ↓



Area   10.2258   (≈ 10.2258)
Centroid≈   (1.82232 1.51285)
Inertia ≈   (12.7238 −0.372037 13.7363)

C^2 Four−Point ω=1/128 ↓



Area $\frac{1\,255\,719\,303\,781}{159\,504\,629\,760}$ (≈ 7.8726197833280999304)

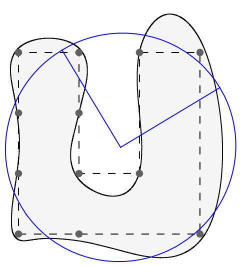
Centroid= $\left(\frac{222\,840\,653\,204\,984\,960\,598\,307\,199\,540\,645\,452\,699\,630\,123}{141\,503\,661\,457\,704\,814\,986\,413\,509\,506\,082\,937\,293\,977\,912}\quad \frac{1\,566\,370\,393\,661\,265\,123\,659\,659\,877\,792\,646\,163\,614\,634\,589}{1\,132\,029\,291\,661\,638\,519\,891\,308\,076\,048\,663\,498\,351\,823\,296}\right)$

Centroid≈ (1.5748048559972536353 1.3836836247956827277)
Inertia ≈ (7.9931976765272596750 0.17602946590013285366 7.0657278878916655095)

Inertia =

{20 565 797 494 975 045 139 207 319 456 261 334 780 008 039 022 079 257 407 738 066 718 939 224 718 104 012 464 336 329 643 -
636 768 193 716 583 778 617 736 213 770 332 632 616 351 342 676 793 756 202 588 442 198 109 233 176 780 233 074 824 908 -
421 /
2 572 912 409 681 590 887 044 036 144 669 341 924 550 291 894 689 985 784 806 920 809 740 770 889 751 489 934 594 822 164 -
937 238 699 473 611 216 023 562 484 201 991 552 539 986 696 862 152 959 766 807 709 935 812 363 407 735 430 147 840 409 -
600,
1 710 987 278 628 724 958 602 908 196 865 559 872 935 679 755 703 566 498 340 128 830 517 733 387 102 802 364 072 446 985 -
457 141 205 378 243 387 857 711 503 154 976 252 974 030 316 307 294 261 073 181 211 040 950 237 468 435 354 553 520 025 -
513 /
9 719 891 325 463 787 795 499 692 102 084 180 603 856 658 268 828 835 187 048 367 503 465 134 472 394 517 530 691 550 400 -
874 012 864 678 086 816 089 013 829 207 523 642 928 838 632 590 355 625 785 718 015 313 068 928 429 222 736 114 063 769 -
600,
44 150 211 775 032 281 026 679 134 601 992 637 828 156 303 597 811 692 158 996 499 470 521 781 604 745 998 891 148 015 218 -
028 841 465 133 086 760 466 574 647 270 017 858 559 783 667 066 050 517 564 606 323 195 658 578 446 056 292 404 524 661 -
067 /
6 248 501 566 369 577 868 535 516 351 339 830 388 193 566 029 961 394 048 816 807 680 799 015 017 967 904 126 873 139 543 -
419 008 270 150 198 667 485 794 604 490 550 913 311 396 263 808 085 759 433 675 866 986 972 882 561 643 187 501 898 137 -
600}

C^2 Four−Point ω=0.013723… (Tightest) ↓



Area   9.0947   (≈ 9.0947)
Centroid≈   (1.68885 1.43218)
Inertia ≈   (10.2341 0.167818 10.0566)

Letter D

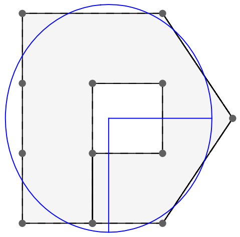
The symmetry of the otherwise symmetric shape D is broken deliberately by inserting additional control points to make the outcome more interesting, and challenging to compute.

Any self-overlapping region contributes double, because the winding number inside these areas is 2.

Curve coordinates ↓

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & \frac{3}{2} & 3 & 3 & 2 & 1 \end{pmatrix}$$

Linear B-spline ↓



Area $\frac{13}{2}$ ($\approx$ 6.5000000000000000000)

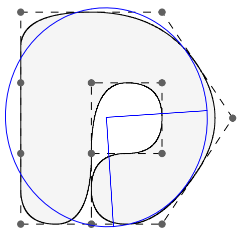
Centroid= $\left(\frac{16}{13} \quad \frac{3}{2} \right)$

Centroid≈ (1.2307692307692307692 1.5000000000000000000)

Inertia = $\left(\frac{635}{156} \quad 0 \quad \frac{239}{48} \right)$

Inertia ≈ (4.0705128205128205128 0 4.9791666666666666667)

Quadratic B-spline ↓



Area $\frac{97}{16}$ ($\approx$ 6.0625000000000000000)

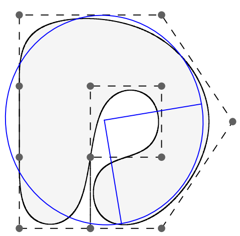
Centroid= $\left(\frac{588}{485} \quad \frac{8801}{5820} \right)$

Centroid≈ (1.2123711340206185567 1.5121993127147766323)

Inertia = $\left(\frac{7\,695\,031}{2\,172\,800} \quad -\frac{1\,066\,703}{26\,073\,600} \quad \frac{46\,525\,183}{11\,174\,400} \right)$

Inertia ≈ (3.5415275220913107511 -0.040911228215513009327 4.1635508841638029782

Cubic B-spline ↓



Area $\frac{8123}{1440}$ ($\approx$ 5.6409722222222222222)

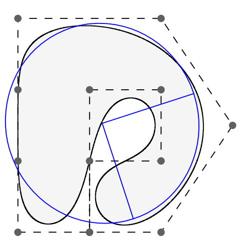
Centroid= $\left(\frac{116\,437}{97\,476} \quad \frac{6\,230\,683}{4\,093\,992} \right)$

Centroid≈ (1.1945196766383520046 1.5219089338718785967)

Inertia = $\left(\frac{80\,317\,569\,839}{25\,939\,533\,312} \quad -\frac{25\,457\,520\,163}{389\,092\,999\,680} \quad \frac{113\,727\,245\,285\,989}{32\,683\,811\,973\,120} \right)$

Inertia ≈ (3.0963382753630320980 -0.065427854481928262483 3.4796199837253128602

Quartic B-spline ↓



Area $\frac{212\,041}{40\,320}$ ($\approx$ 5.2589533730158730159)

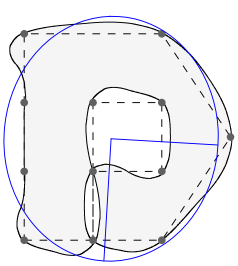
Centroid= $\left(\frac{74\,140\,700}{62\,976\,177} \quad \frac{1\,540\,193\,585}{1\,007\,618\,832} \right)$

Centroid≈ (1.1772816885978963124 1.5285478358348109953)

Inertia = $\left(\frac{487\,423\,680\,528\,048\,661}{179\,270\,383\,581\,020\,160} \quad -\frac{191\,034\,029\,632\,076\,791}{2\,509\,785\,370\,134\,282\,240} \quad \frac{7\,348\,947\,751\,588\,183\,973}{2\,509\,785\,370\,134\,282\,240} \right)$

Inertia ≈ (2.7189303151558243408 -0.076115683797238728219 2.9281180132128149776

Three-Point ↓



Area $\frac{15\,903\,293}{2\,127\,360}$ ($\approx$ 7.4756002745186522262)

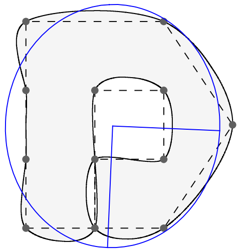
Centroid= $\left(\frac{56\,588\,225\,140\,677\,427}{44\,800\,864\,229\,667\,140} \quad \frac{1\,585\,771\,801\,453\,735\,793}{1\,075\,220\,741\,512\,011\,360} \right)$

Centroid≈ (1.2631056590913863348 1.4748337157482383205)

Inertia = $$\begin{pmatrix} \frac{1\,040\,867\,992\,942\,949\,041\,699\,943\,397\,662\,623\,157\,151\,224\,019\,501\,851\,366\,613\,689\,485\,913\,737}{196\,866\,615\,392\,889\,109\,466\,208\,002\,578\,496\,248\,914\,690\,066\,850\,531\,764\,728\,441\,241\,600\,000} \\ \frac{12\,625\,676\,249\,360\,199\,972\,726\,767\,904\,095\,816\,909\,238\,196\,544\,230\,100\,727\,035\,037\,924\,991}{131\,244\,410\,261\,926\,072\,977\,472\,001\,718\,997\,499\,276\,460\,044\,567\,021\,176\,485\,627\,494\,400\,000} \\ \frac{4\,076\,202\,442\,789\,656\,236\,969\,572\,923\,938\,402\,512\,733\,270\,017\,263\,166\,372\,626\,072\,695\,669\,197}{590\,599\,846\,178\,667\,328\,398\,624\,007\,735\,488\,746\,744\,070\,200\,551\,595\,294\,185\,323\,724\,800\,000} \end{pmatrix}$$

Inertia ≈ (5.2871737082779427678 0.096199725566696392125 6.9018007186485618241)

C^1 Four-Point ω=1/16 ↓



Area $\frac{3\,295\,183}{443\,520}$ (≈ 7.4296153499278499278)

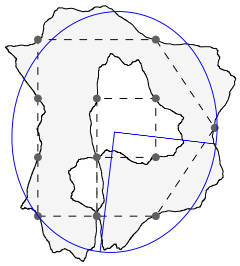
Centroid= $\left(\frac{15\,864\,686\,104\,274\,849\,177}{12\,610\,872\,019\,886\,438\,160} \;\; \frac{946\,681\,651\,373\,640\,981\,599}{638\,950\,849\,007\,579\,533\,440} \right)$

Centroid≈ (1.2580165811894197507 1.4816188957946137757)

Inertia = $\left(\begin{array}{c} \frac{5\,119\,570\,881\,316\,376\,495\,134\,772\,393\,219\,119\,571\,246\,714\,354\,845\,017\,257\,106\,656\,498\,413\,223}{980\,638\,226\,812\,736\,897\,685\,221\,616\,789\,466\,994\,709\,668\,756\,047\,878\,282\,149\,877\,792\,768\,000} \\ \frac{124\,958\,625\,130\,966\,927\,882\,288\,581\,332\,916\,613\,831\,180\,896\,757\,546\,744\,913\,811\,638\,514\,343}{1\,961\,276\,453\,625\,473\,795\,370\,443\,233\,578\,933\,989\,419\,337\,512\,095\,756\,564\,299\,755\,585\,536\,000} \\ \frac{2\,003\,207\,903\,488\,258\,125\,181\,061\,728\,470\,929\,832\,124\,562\,792\,440\,822\,450\,066\,888\,032\,425\,594\,177}{298\,114\,020\,951\,072\,016\,896\,307\,371\,503\,997\,966\,391\,739\,301\,838\,554\,997\,773\,562\,849\,001\,472\,000} \end{array} \right)$

Inertia ≈ (5.2206519604645312051 0.063712907428209547703 6.7196031139274551357)

C^1 Four-Point ω=0.192729… ↓

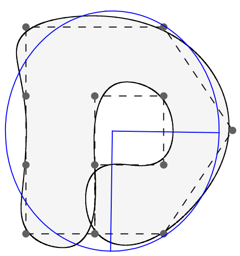


Area 9.42559 (≈ 9.42559)

Centroid≈ (1.29856 1.42927)

Inertia ≈ (8.48196 0.394433 11.7425)

C^2 Four-Point ω=1/128 ↓



Area $\frac{19\,213\,630\,039\,321}{2\,615\,875\,928\,064}$ (≈ 7.3450081608193611076)

Centroid= $\left(\begin{array}{c} \frac{7\,767\,710\,635\,010\,057\,552\,868\,129\,067\,995\,612\,818\,896\,849}{6\,194\,500\,418\,531\,362\,252\,372\,386\,332\,384\,572\,458\,905\,280} \\ \frac{1\,049\,426\,597\,935\,791\,509\,788\,626\,934\,265\,861\,376\,059\,256\,499}{704\,108\,214\,239\,731\,509\,352\,994\,579\,781\,046\,402\,828\,900\,160} \end{array} \right)$

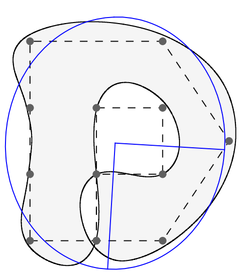
Centroid≈ (1.2539688611162744189 1.4904336815171532621)

Inertia ≈ (5.1031222143008463907 0.019331459652967635314 6.4493339101949551101)

Inertia =

{1 791 389 136 723 680 781 226 680 819 426 651 144 046 703 948 885 285 860 946 951 561 398 597 104 622 449 358 211 712 180-
167 398 473 118 841 289 211 000 915 072 810 292 727 757 510 840 877 086 854 612 971 936 808 461 218 836 301 919 123 637-
403 /
351 037 866 916 755 818 454 658 460 450 873 001 765 606 056 811 939 520 078 780 277 908 587 859 612 129 213 092 382 742-
221 010 206 203 062 281 913 600 181 815 906 637 867 202 912 966 395 320 633 613 306 268 190 243 011 602 352 748 416 204-
800,
81 432 892 331 581 048 605 684 265 328 720 238 081 151 928 275 584 242 375 709 620 368 609 340 144 416 612 018 047 895 501-
575 466 329 418 486 192 453 178 296 628 093 822 537 337 023 658 102 325 228 361 128 392 676 095 115 327 098 594 239 113 /
4 212 454 403 001 069 821 455 901 525 410 476 021 187 272 681 743 274 240 945 363 334 903 054 315 345 550 557 108 592 906-
652 122 474 436 747 382 963 202 181 790 879 654 406 434 955 596 743 847 603 359 675 218 282 916 139 228 232 980 994 457-
600,
842 193 275 819 170 191 210 089 934 503 164 290 224 698 050 925 595 790 191 806 894 858 169 263 098 501 944 729 815 084-
782 614 485 131 281 006 821 596 045 418 419 505 068 860 691 076 410 022 542 498 784 956 735 523 627 222 195 100 808 895-
573 969 /
130 586 086 493 033 164 465 132 947 287 724 756 656 805 453 134 041 501 469 306 263 381 994 683 775 712 067 270 366 380-
106 215 796 707 539 168 871 859 267 635 517 269 286 599 483 623 499 059 275 704 149 931 766 770 400 316 075 222 410 828-
185 600}

C^2 Four-Point ω=0.013723… (Tightest) ↓



Area 8.4803 (≈ 8.4803)

Centroid≈ (1.28296 1.4678)

Inertia ≈ (6.748 0.127723 8.89297)

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