Should certain linear superpositions of the potentials describing the field of magnetic monopoles be topologically forbidden, and why ?

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ABSTRACT

The impact of linear superpositions of magnetic potentials on the physical robustness of "patchings" used to build up the "wave section" of an electron orbiting around a magnetic monopole is examined. It is shown that most of these linear superpositions must be discarded if one wishes to preserve the possibility that magnetic monopoles may exist.

INTRODUCTION

The symmetry of Maxwell's equations has soon led to the speculation that magnetic monopoles, symmetric to electric charges, might exist in nature[1]. The concrete existence of genuinely *magnetic* monopoles remains however unestablished, in spite of all experimental searches.

The existence of magnetic spins, especially half-integer ones, reinforces the observational lack of symmetry between electricity and magnetism, since no symmetrically similar « electric spins », most notably half-integer ones, have ever been discovered in nature.

This basic lack of symmetry leads one to wonder whether some physical difficulties do not render the existence of magnetic monopoles more problematic than has often been assumed.

The purpose of the present note is to signal two such elementary difficulties.

1. How most solutions for the magnetic potential are usually implicitly excluded.

As proven clearly by Wu and Yang[2,3], there does not exist a vector potential for the magnetic field of a monopole which is singularity free on the sphere. I am personally indebted to M. Jean Dalibard's 2014 course at the Collège de France for making me acquainted with Wu and Yang's patching technique, whose purpose is to circumvent this difficulty. According to Wu-Yang's scheme, two overlapping regions R_a and R_b of the sphere surrounding a magnetic monopole are drawn (cf. Fig. 1).

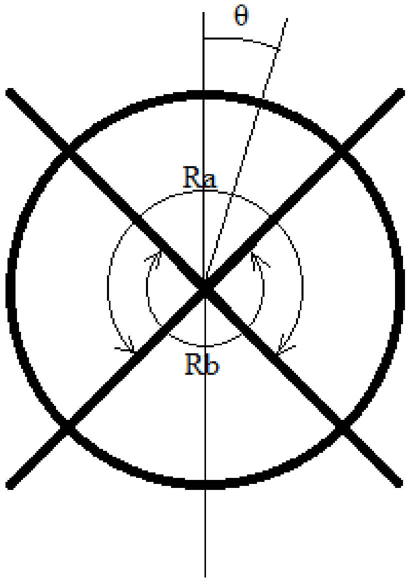

Fig.1 : Two-dimensional representation of R_a and R_b

Within R_a and R_b, two singularity free vector potentials $\mathbf{A_a}$ and $\mathbf{A_b}$ are built, "g" corresponding to the "magnetic charge" of the monopole :

$$(A_a)_r = (A_a)_\theta = 0, \qquad (A_a)_\varphi = g(1-\cos\theta)/r\sin\theta \qquad (1)$$

$$(A_b)_r = (A_b)_\theta = 0, \qquad (A_b)_\varphi = -g(1+\cos\theta)/r\sin\theta \qquad (2)$$

The motion of a single electron of charge e orbiting around the monopole can be separately described within R_a and R_b, as a function of $\mathbf{A_a}$ and $\mathbf{A_b}$, by two respective functions ψ_a and ψ_b. Since

$$(A_a)_\varphi - (A_b)_\varphi = 2g/r\sin\theta, \qquad (3)$$

one calculates that in the region $R_a \cap R_b$, ψ_a and ψ_b can be related by a simple phase factor according to the equation :

$$\psi_a = \exp^{2ieg\varphi}\, \psi_a, \qquad (4)$$

Imposing the condition that ψ_a and ψ_b are single valued along the equator leads to Dirac's so-called "quantisation" condition[4], which might perhaps better be called Dirac's "topological constraint":

$$2eg = \mathit{integer} \qquad (5)$$

As shown by Wu and Yang[3], the solutions for ψ_a and ψ_b can be explicitly calculated. Quite remarkably, all of them are singularity free on their respective domains of application. C.-N. Yang has proposed to call the solutions acting on the entire sphere $R_a \cup R_b$ "wave sections".

There remains, however, an intriguing, nearly trivial detail concerning the authorized values of the magnetic potential in $R_a \cap R_b$ which seems to have been widely overlooked. Since $\mathbf{A_a}$ and $\mathbf{A_b}$ both represent valid solutions for the vector potential in the region $R_a \cap R_b$, in principle, any linear combination $(\alpha\mathbf{A_a} + \beta\mathbf{A_b})$ with $\alpha+\beta = 1$ also verifies the equation

$$\text{rot}\,(\alpha\mathbf{A_a} + \beta\mathbf{A_b}) = \mathbf{B} \tag{6}$$

within the same region $R_a \cap R_b$.

Therefore, within $R_a \cap R_b$, one sees no reason why it would not be possible to build a function for the orbiting electron corresponding to the vector potential $(\alpha\mathbf{A_a} + \beta\mathbf{A_b})$. Within $R_a \cap R_b$, the difference between our "new" vector potential $(\alpha\mathbf{A_a} + \beta\mathbf{A_b})$ and $\mathbf{A_a}$ would verify :

$$(\alpha A_a + \beta A_b)_\varphi - (A_a)_\varphi = g(\alpha\text{-}\beta\text{-}1)/r\sin\theta \tag{7}$$

The same reasoning that led to Dirac's constraint on the product 2eg in eq. 5 would therefore lead to the new constraint :

$$(\alpha\text{-}\beta\text{-}1).eg = \textit{integer} \tag{8}$$

Since any combinations for α and β verifying $\alpha+\beta=1$ could be considered as *a priori* acceptable for describing the magnetic field of the monopole within $R_a \cap R_b$, an infinite number of mutually incompatible constraints could be generated ! In order to avoid this, the only way to save the existence of magnetic monopoles should consist in discarding all field solutions $(\alpha\mathbf{A_a} + \beta\mathbf{A_b})$, except for $\alpha=0$ or $\beta=0$.

A highly non-rigorous way of "counting" the (infinite) number different possible gauges may perhaps help one to get a more intuitive image of the situation. Wu-Yang's analysis allows one to consider one different gauge solution for any direction in three dimensions. In other words, Wu-Yang have considered a number of different gauges which is comparable with the size of "$2^{\mathbb{R}}$". On the other hand, gauge invariance usually tells us that if a vector potential $\mathbf{A} + \nabla X$ is equivalent to $\mathbf{A}$, any field $\mathbf{A} + \alpha\nabla X$ could also be considered as equivalent to $\mathbf{A}$. For each possible direction of ∇X, an infinite number of products $\alpha\nabla X$ may be considered. Therefore, an *a priori* (highly non-rigorous!) estimate of the number of possible gauge fields, based on the number of possible $\alpha\nabla X$ combinations, would "amount" roughly to "$\mathbb{R} \times \mathbb{R}$". Perhaps this loose way of "counting" can help one to see how Wu and Yang (as well as Dirac) have only considered a subset of all *a priori* possible gauges.

Admittedly, it is not absurd to consider that all ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$), with $\alpha\beta\neq0$, may be viewed as physically "less relevant" than $\mathbf{A_a}$ and $\mathbf{A_b}$, since all fields ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$), with $\alpha\beta\neq0$, differ from $\mathbf{A_a}$ and $\mathbf{A_b}$ in a topologically fundamental way : $\mathbf{A_a}$ and $\mathbf{A_b}$ each possess only *one* singularity within the sphere, whereas all other combinations ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$) possess *two* such singularities (those of $\mathbf{A_a}$ and $\mathbf{A_b}$, both outside of $R_a \cap R_b$). In other words, a reasonable, consistent rule may exist in nature, which would enable one to explain why all fields ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$), with $\alpha\beta\neq0$ should be discarded. The difficulty raised by eq. 8 does not suffice to prove that magnetic monopoles cannot exist. At least, however, it suggests that the fundamental cost which quantum physics should pay for their existence may not be as light as one could have wished.

2. Discussion.

If all the relevant physics involved in the situation initially studied by Dirac were restricted to the existence of so-called "wave sections" and a spherically symmetric magnetic field **B**, the analytical solutions obtained by Wu and Yang for such "wave sections" would be quite reassuring : as shown by Wu and Yang, these wave sections are smooth everywhere (except at **r**=0), and form a complete set. The magnetic field **B** also looks encouragingly smooth everywhere, except at **r**=0.

Still one issue would remain problematical, however, since quantum physics usually resort to the magnetic potential **A**, not the magnetic field **B**, in order to achieve the quantization of a system. What is more, **A** can exhibit some crucial information that **B** only contains in an indirect way, as has been clearly demonstrated by the Aharonov-Bohm experiment. Inasmuch as all magnetic potential linear solutions ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$) may be considered as *a priori* relevant in the region

$R_a \cap R_b$, discarding all of them except $\mathbf{A_a}$ and $\mathbf{A_b}$ would hint at a certain form of restriction placed upon the universal application of linear principles in quantum mechanics... which is a heavy price to pay indeed. A supplementary postulate should be added to those already laying the foundations of quantum mechanics !

The status of all ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$) fields, with $\alpha\beta\neq 0$, remains intriguing, if not worrying, from yet another point of view. If the reason for eliminating all ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$) fields, with $\alpha\beta\neq 0$, rests on the rejection of their double angular singularities, such an elimination paradoxically implies, topologically speaking, that the singularities of $\mathbf{A_a}$ and $\mathbf{A_b}$ themselves are also "still there" to a certain extent, albeit in a hidden way. If such angular singularities could be attributed to some classical objects, such as "Dirac strings", everything would be fine : at the classical level, linear combinations of different objects have no physical meaning ; this could account for the fact that all superpositions ($\alpha\mathbf{A_a} + \beta\mathbf{A_b}$), with $\alpha\beta\neq 0$, might also be meaningless. As a matter of fact, this certainly explains why all these superpositions are usually ignored in the literature. Unfortunately, the quantum "vacuum", to which only two particles (one magnetic monopole and one electron) have been added in the system considered by Dirac and his followers, is not known to exhibit any classical feature (apart from the *a priori* basic topological features of space-time itself).

Still one more question – or doubt - may be raised concerning the robustness of the patching technique followed by Wu and Yang. This technique indeed provides a complete set of remarkably smooth solutions for the motion of *one* electron around a magnetic monopole (under the condition that Dirac's constraint of eq. 5 is obeyed), which is a remarkable achievement. But what happens if two, or more electrons orbit around the same monopole ? If we further suppose that such electrons interact with each other by means of a separate Hamiltonian which is unrelated to the monopole, the simple phase factor found in eq. 4, which allows one to "patch" together different parts of a "wave section", looses its validity. If a more complicated "patching" solution could exist, everything would remain fine, at least at the level of principle. Unfortunately, it is *a priori* far from obvious that eq. 4 can be replaced by any equivalent equation at all when several particles orbit around the same magnetic monopole.

One may pay attention, in particular, to two different sub-scenarios : (i) two electric charges orbit around the monopole within the same circle defined by at constant angle θ (for instance $\theta=\pi/2$) and a constant radius r ; (ii) the same charges both orbit within another circle defined by the same constant angle θ and a larger constant radius r'. Two different "quantisation" conditions for the monopole, both analogous to eq.5, may *a priori* be obtained from (i) and (ii). In general, nothing would guarantee their mutual compatibility. Modifying the environment in such a way as to adjust the interaction strength between different particles orbiting around the monopole at different distances would, in fact, inevitably lead to incompatible constraints for the magnitude of its magnetic charge, except with a small set of tuning parameters.

CONCLUSION

The quest for magnetic monopoles will only come to a conclusion one day if a *positive* experimental result is obtained ; thousands of negative experiments will never suffice to prove that magnetic monopoles are totally absent from our world. Notwithstanding the reservations formulated above, the possibility of their existence remains a quite reasonable hypothesis, especially if their life-time is limited. Several eminent physicists have already demonstrated this with much more advanced tools than I use here.

The only question is whether nature itself is presently willing - or perhaps *would* be willing in a future experiment - to pay the price for the existence of stable magnetic monopoles. Maybe the price is greater than we think.

Many physical equations hint at a greater degree of symmetry than humans usually encounter in practice. Paradoxically, however, this may just show how nature has reached a quite fine compromise for the benefit and the joy of living human beings: it has kept a sufficient amount of symmetry within its equations to let our minds dream of beautiful, ideal, symmetric configurations, while it lets us dwell in an imperfect, yet arguably more hospitable world than the symmetric world of our dreams.

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