Symmetry breakdown in binary elastic collisions in classical dynamics

Radhakrishnamurty Padyala
#282, Duo Marvel Layout, Yelahanka, Bangalore 560064, India.
Email: padyala1941@yahoo.com

Abstract

In classical dynamics binary elastic collision problems are solved using the two laws of conservation – the law of conservation of momentum and the law of conservation of kinetic energy. The requirement of conservation of kinetic energy is less restrictive than the Huygens' requirement (kinematics model) of reversal of relative velocity for an elastic collision. We demonstrate in this article that the dynamics model breaks symmetry and leads to paradoxical solutions in multi dimensional collisions.

Key words

Elastic collision, kinematics, relative velocity reversal, momentum conservation, Kinetic energy conservation, dynamics model, symmetry breaking.

Introduction

The problem of binary elastic collisions (to find the velocities after collision, of two bodies of known mass, given their velocities before collision), goes back almost to the beginnings of classical mechanics. Huygens solved the problem in 17th century itself, in a simple and most elegant fashion. Erlichson gives a lucid account of Huygens' solution [1]. Huygens' solution was *kinematics* based[1]. Huygens depended on symmetry considerations, Galilean velocity transformation and Torricelli principle to arrive at his solution. According to Huygens' hypothesis that, *if in a reference frame, one of the masses involved in a binary elastic collision reverses its velocity then so does the other mass. This is what symmetry meant to Huygens in the context of elastic collisions*. Huygens also noted that it was always possible to find a reference frame in which one of the masses reverses its velocity. Clearly, the relative velocity, $\mathbf{V}_{rel}$, also reverses[2] ($\mathbf{V'}_{\mathbf{rel}} = -\mathbf{V}_{rel}$) in that frame (we now call it center of mass (CM) frame) of reference. In other words, $\mathbf{V}_{rel}$ rotates through an angle of pi radians as a result of an elastic collision. Again, since the value of $\mathbf{V}_{rel}$ does not depend on the reference frame, the reversal of relative velocity applies to all reference frames that move with constant velocities and is a property of elastic collisions. This property of reversal of relative velocity played a crucial role in Huygens' solution.

---

[1] The two conditions that govern binary elastic collisions in Huygens' kinematic analysis are: 1. Conservation of momentum, and 2. Reversal of relative velocity.

[2] Velocities after collision are represented by primed **V**s: (**V′**).

However, later developments apparently eclipsed Huygens' kinematics solution, and the *dynamics* solution took precedence. In the dynamics model[3], the concept of force plays an important role. In this model of elastic collisions the masses exert equal and opposite forces on each other along the line of their centers at the instant of collision [2-8]. These forces reverse the components of the velocities parallel to this line, but leave the perpendicular components of the velocities unchanged. Thus in contrast to Huygens criterion of reversal (rotation through π radians) of $\mathbf{V}_{rel}$, it is possible in dynamics model for $\mathbf{V}_{rel}$ to rotate through an angle ≠ π radians. Since the speed of each mass remains unchanged, KE of each mass is conserved, i.e. there is no transfer of KE between the masses, in CM reference frame. In an arbitrary reference frame the velocities could change both in magnitude and direction as a result of collision and result in a transfer of KE between the masses while the sum of their KE remains constant.

The results of the dynamics model of elastic collision process, where rotation of $\mathbf{V}_{rel}$ *through an angle not equal to π radians occurs, violate Huygens' hypothesis and lead to symmetry break down. This is the origin of breakdown of symmetry in classical dynamics.*

Head-on (1-D) collisions

Consider the following diagram (Fig.1) depicting an elastic collision between bodies A, B of masses, m_A and m_B (≠ m_A) respectively.

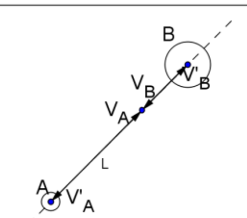

Fig. 1 Two bodies A and B of masses m_A and m_B approach each other along line L with velocities V_A, V_B respectively and collide. After collision they reverse their velocities and move away from each other with velocities V'_A (=$-V_A$), V'_B (=$-V_B$)along line L. Their momenta are equal and opposite before collision as well as after collision.

A and B move along a straight line L towards each other with speeds inversely proportional to their masses and collide. The sum of their momenta is equal to zero before collision. After the collision, they move away from each other with the signs of their velocities changed, but magnitudes remaining unchanged ($\mathbf{V_i'} = -\mathbf{V_i}$, i = A, B). The sum of their momenta is equal to zero after collision also. Collision is an instantaneous process and obeys the principle of momentum conservation. Such a collision satisfies the Huygens' condition of symmetry that – if one of the bodies involved in an elastic collision reverses its

[3] The two conditions that govern binary elastic collisions, in dynamics model are: 1. Conservation of momentum, and 2. Conservation of kinetic energy (KE).

direction of motion as a result of the collision, then the second body also reverses its velocity. Consequently, relative velocity changes sign or reverses. This is the Huygens' relative velocity reversal criterion for elastic collisions.

Oblique collisions

For finite sized masses or bodies (in contrast to point masses), oblique collisions become possible. In these collisions, two bodies move towards each other along parallel lines and collide. After collision they move away from each other along lines not parallel to the lines of approach before collision. Consequently, the relative velocity vector rotates through an angle (≠180°). Such collision processes are also known as elastic scatter. We don't deal with such processes in this article.

2-D collisions

Let us now consider the collision depicted in Fig. 2 between the same bodies A and B as those in Fig. 1. Before collision B moves with the same velocity $\mathbf{V_B}$, as in the collision in Fig. 1. Mass A moves with a velocity $\mathbf{V_A}$; now it has an additional component of velocity to the velocity it had in the collision shown in Fig. 1, along a direction perpendicular to line L. The total momentum is ($m_A \mathbf{V_A} + m_B \mathbf{V_B}$). But, the sum of the momenta parallel to L is zero. Therefore, the total momentum is in a direction perpendicular to L.

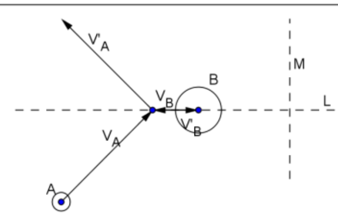

Fig. 2. Body B moves with the same velocity V_B as in Fig.1, before collision and body A moves with velocity V_A (having an additional component of velocity in a direction parallel to M (perpendicular to L)). After collision, B reverses its velocity and moves with velocity V'_B and A moves with velocity V'_A (with the component parallel to L reversed but the component parallel to M unchanged).

After the collision, B moves away with velocity $\mathbf{V'_B}$ (= $-\mathbf{V_B}$). Body A moves away with its velocity component parallel to L reversed, and the velocity component perpendicular to L unchanged, obeying the law of conservation of momentum. It moves away with velocity $\mathbf{V'_A}$.

Thus though body B reverses the direction of its velocity as a result of the collision, but body A does not do so. Evidently, this 2-D collision breaks symmetry demanded by Huygens' hypothesis that, if one of the bodies involved in an elastic collision reverses its direction of motion as a result of collision, the other too, necessarily reverses its direction of motion.

Discussion

In classical mechanics, such a result (Fig. 2) is not considered symmetry breaking or violative of any natural principle, since this collision process satisfies both the laws of conservation of momentum and of energy (total energy is equal to KE in elastic collisions). These two laws are less restrictive than the symmetry restriction which demands conservation of momentum and the reversal of relative velocity. The conservation laws merely demand the restoration of relative speed – but not reversal of relative velocity.

An important example of symmetry breaking collision is this: One of the bodies (the target) has a very large mass compared to the other (projectile). Its velocity is zero before and after collision. Since, +**0** = **-0** = **0**, we can treat the velocity of the target to have changed from +**0** to **-0** as a result of the collision. The projectile which collides with this massive target at an angle ≠ 90° with a velocity **V** leaves the collision with velocity **V′**(≠ **-V**). Here, though the target reverses its velocity due to collision, the projectile does not, thereby breaking symmetry.

In kinetic theory of gases, for example, molecules of negligible mass collide elastically with the massive walls of the container. In these collisions the wall (target) changes its velocity from +**0** to **-0** but the molecules (projectiles) don't reverse the direction of motion. Therefore, these collisions break symmetry.

As a matter of fact, if one considers two instants of time before (or after) collision, both the above laws of conservation, as well as restoration of relative speed are satisfied. Therefore, these criteria are not indicative of a collision at all! In other words, these criteria do not enable us to distinguish two instants of time as the ones corresponding to '*before'* and *'after'*. On the other hand reversal of relative velocity is a criterion that enables us to distinguish two instants of time as '*before*' and '*after*' (and is also a consequence of symmetry of the process).

One might be tempted to argue that Huygens' criterion of reversal of relative velocity applies to 1-D collisions only. If that were to be the case, then in multidimensional collisions, relative velocity reverses only in one direction (along the line of centers at the instant of collision) and remains unchanged in other directions, with the duel result that '*the collision occurred'* (reversal of relative velocity in one direction) and '*the collision did not occur'* (no change in relative velocity in other directions). Such a situation leads to a paradox.

We see that collisions depicted in the above diagrams represent elastic collision processes, in accordance with the dynamics model, but they fail to satisfy the crucial symmetry property of an elastic collision, viz., the reversal of relative velocity. *Any elastic collision that does not satisfy the reversal of relative velocity principle breaks symmetry*.

In 1D-collisions the relative velocity necessarily changes sign. Therefore, both dynamics model and kinematics model yield results that satisfy the symmetry criterion. This is also the reason why the dynamics model gives unique solutions to 1D-collision problems, as does the kinematics model. However, in multi dimensional collision problems, this need not be the case. In such cases, where the relative velocity is not parallel to the line of centers at the instant of collision, the dynamics model does not yield unique solutions (due to violation of symmetry of the process); whereas the kinematics model, which satisfies the symmetry, leads to unique solutions.

References:

1. H. Erlichson, Am. J. Phys. **65** (2), (1997), pp149-154.
2. J. C. Maxwell, The dynamical Theory of Gases in Selected readings in physics: Kinetic theory vol. 1, Stephen Brush, Pergamon Press, New York (1966), pp. 151-152.
3. M. N. Saha and B. N. Srivatsava, A Treatise on Heat (1950), pp. 825-827.
4. L. D. Landau and E. M. Lifshitz, Mechanics, 3rd Edn. Pergamon Press, New York (1988), pp. 41-47.
5. H. Y. Carr and R. T. Weidner, Physics from ground up, McGraw Hill, New Delhi (1971), Ch. 8.
6. R. Stephen Berry, S. A. Rice, and John Ross, Physical Chemistry, 2nd Edn., Oxford Univ. Press, (2000), p. 825.
7. F. S. Crawford, Am. J. Phys. **57**(2) (1989), p. 121.
8. A. J. Rica da Silva and Jose P. Lemos, Am. J. Phys. **74** (7), 584-590 (2006).
9. R. P. Feynman, R. B. Leighton and M. Sands, The Feynman lectures on Physics, 11th Reprint, Narosa Pub., New Delhi (2003), Vol. 1, pp. 208-210; 661.