The α-Theory

From Tachyonic to Bradyonic Universe

New Physical Scenarios

Vincenzo Maiella

student at the University of Naples Federico II

Department of Physics, via Cintia 6, 80126 Naples, Italy

Abstract

The present book is devoted to the construction of a new theory for the elementary particles. This theory was born in order to resolve some open issues of the Standard Model, among which the research of a general equation for the description of all quantum particles in a unique way. Starting from the Pauli equation, two relativistic partial differential equations are obtained, one to the first and other to the second order, which are able to describe particles with arbitrary spin. The analysis of the energetic spectrum concerning such equations puts in evidence that the particles they describe have an imaginary mass. Then, the probability density study of these equations shows the principle of causality is not satisfied. Therefore, these equations characterize the tachyonic universe. We can see such a universe admits spontaneous symmetry breaking. In general terms, we can establish the broken symmetry might be a global characteristic of the tachyonic universe, due to its negative square energy, which made it unstable. Hence, it is possible to assume that after the hot Big-Bang there was another catastrophic cosmological event – said Big-Break – which made to break down the tachyonic universe in a positive square energy one (bradyonic universe), and from which the four fundamental interactions are probably generated. The transition from tachyonic to bradyonic universe transforms the tachyonic equations in bradyonic ones, being then able to describe elementary particles with arbitrary spin. The first order partial differential equation gives asymmetric quantum states, while the second order partial differential equation gives symmetric quantum states and so they describe different theories. In the course of this work, both these theories will be studied. Either way, only one of the two could be the right theory of the elementary particles. Since this theory follows on from the Big-Bang, from the resultant tachyonic universe and from Big-Break, it has been called α*-Theory, i.e. the "beginning theory." The* α*-Theory may theoretically predict the experimental properties of neutrinos and anti-neutrinos and allows, thanks to the appendix A, to generalize the concepts of the Dirac sea, Pauli exclusion principle and spin-statistics theorem, thus giving rise to "s-matter" and "multi-statistics," which are able to take an interesting approach for the explanation of the Dark Matter. Furthermore, it expects on large-scale the "double inflation" mechanism, which, without the Dark Energy, could explain the acceleration of our universe (bradyonic universe). From the* α*-Theory, two string actions can be deduced too, by which new ideas can be developed. Practically, the* α*-Theory wants to be a GUT, able to describe the quantum particles and our universe in a simple and elegant way.*

Contents

Preface

The theory elaborated in this work is the result of over seven years of solitary study in theoretical physics. The version introduced here is nothing more than an abridgment of the original manuscript, which took me nearly three years of every day uninterrupted job for being written up. The sacrifice deriving from such a wide-ranging and articulated scientific work is totally repaid by the fact that the produced ideas and concepts can open new physical scenarios, able to let evolve the knowledge of the elementary particles (Standard Model). As it is known, the Standard Model, although having generated exceptional outcomes, in order to make it the human theoretical system with most elevated predictive power, has many problems, that prevented physicists to consider it a full theory, like the Einstein's Relativity. One of the greatest Standard Model problems is that bosons and fermions are described by different equations, which, in any case, can only give account of particles with fixed spin, and not about arbitrary ones. All that has given place to a spasmodic search for more general equations, which however have not carried useful expressions. A rib of this research has given life to a new theory, the Supersymmetry, which has not produced experimentally observable effects yet. Another great Standard Model problem is the unification of the strong, electromagnetic and weak interactions. The electroweak model of Glashow-Weinberg-Salam showed the electromagnetic and weak interactions can be derived from the spontaneous symmetry breaking of the group $SU(2) \otimes U(1)$. This created an idiosyncrasy among quantum interactions, because it seems strange the electromagnetic and weak interactions are unified while the strong one is not so. This condition led the theoretical physicists to believe that in principle (post Big-Bang) there was a wider group of symmetry, which, spontaneously breaking, produced all the fundamental interactions, included the gravitational one. These problematics, still unsolved, induced to consider the existence of a possible more general quantum theory, which one day will allow to solve the Standard Model incoherences. Such a theory should contain the *universal law* in order to unify the elementary interactions. This law should consist in a single equation that, case for case, can describe quantum particles which are not subjected to interactions (free elementary particles equation). The search for a more general physical model is becoming important not only from the quantum point of view, but also from the cosmological one. In fact, the Standard Cosmological Model, although having been the successfully starring of enormous theoretical and experimental triumphs, has still open problems to resolve. This incompleteness originated in 1980s the Inflationary Model, which allowed to give a coherent explanation to the horizon and flatness problems. However, also this model is plagued by numerous issues, mostly because it cannot stand on a more general theory. Moreover, the acceleration and lacking mass problems have not been still explained, if not through some hypotheses, such as Dark Energy and Dark Matter, that must be proved. At the moment, the exploration of all possible paths is tried by using Supersymmetry or Superstring theory too, since most of the physicists are convinced the Standard Model and Standard Cosmological Model problems will be resolved by a more general theory, which will connect micro and macro in an elegant and comprehensive way. The α-Theory presents itself as that theory.

The *corpus* of this book is constituted by the discussion and analysis of a new quantum field theory, that, taking life by a global phase transition process – named Big-Break – from an unstable universe (tachyonic universe) – arisen from Big-Bang – to a stable universe (bradyonic universe), is called α-Theory, *i.e.* the "beginning theory." The chapter 1 is dedicated to the description of tachyonic universe and characteristic equations of particles composing it. It is demonstrated such a universe does not respect the principle of causality and his constituent particles are characterized by negative square energy rather than superluminal speeds. The chapter 2 begins with the study of the spontaneous symmetry breaking of the two proposed tachyonic theories. From that follows the negative square energy characterizing it is a source of instability, which probably caused a big spontaneous symmetry breaking – called Big-Break – which induced the tachyonic universe to get supercooled into the bradyonic one. The equations of elementary particles deriving by this process are a direct consequence of the tachyonic equations. In chapter 3, these results will be applied to the inequivalent representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, developing the field equations of left- and right-handed particles, that commonly are associated to the neutrinos and anti-neutrinos, respectively. It will be seen such equations are characterized to the parity violation and nonzero masses, as the physical experiences about neutrinos and anti-neutrinos prescribe. The chapter 4 will allow to describe the Lagrangian formalism from which the bradyonic equations rise (only one of these equations should be the right equation of the elementary particles, even if both will be studied) and that will give place to the Lagrangian densities individualizing the α-Theory. Furthermore, the conserved charges and currents deriving from the space-time invariance of the theory will be studied and, finally, it will be seen the connection with an external electromagnetic field and most in general with a Yang-Mills field. Also the free propagators will be calculated, for an approach to a future perturbative analysis. In chapter 5, it will be dealt, instead, with the second quantization of the α-Theory, which will carry to extend Fermi-Dirac and Bose-Einstein quantum statistics, thanks to the review of the Pauli's "spin-statistics theorem." This will concur to generalize Dirac sea as well as the exclusion principle and to define the "s-matter" and "multi-statistics," that could give new tools for resolving the lacking mass problem within our universe. Concerning the chapter 6, it will allow to put in relation the Big-Break with the "mass gap" problem of the Yang-Mills theory. This will enable to propose the mechanism of Big-Break like the true liable one for the hypothetical gluon mass, deriving from a physical process rather than from hidden mathematical properties. Then, a simple selection rule will be given, which will allow to tie the energy to the gauge groups and these last to the spin value of particles. This will help to explain the reason for which particles of high spin are not observed and to select, for each interaction, a set of spins to which we usually associate some particles. In chapter 7 it will be studied, from holistic point of view, as the α-Theory should be compared with classical and quantum fields, until the cosmology. It will be seen that α-Theory, with the "double inflation" mechanism, wants to resolve not only the horizon and flatness problems, but also the one about the acceleration of our universe. Lastly, it will be matched the α-Theory with the two most important GUT theories of the twentieth century, *i.e.* the Supersymmetry and String theory. It will be made to see that the α-Theory represents a real alternative to the Supersymmetry, exceeding it for conceptual and formal elegance. On the contrary, with regard to String theory, two string actions based on the α-Theory scenario will be presented, which could give a new impulse to the researches in this field. In the last chapter, of philosophical kind, it will be inquired into the foundations of the α-Theory. It will be seen this model could give whole justification to the two most important principles of the modern physics – the uncertainty principle and constancy of the speed of light – and, at the same time, suggesting the resolution of the quantum mechanics paradoxes. Moreover, the falsifiability concept – introduced by Karl Popper – will be studied, in order to establish the validity of a scientific theory. For this purpose, it will be seen that the α-Theory, unlike Supersymmetry or String Theory, having the typical constituent equations of a Quantum Field Theory (QFT), is falsifiable, that is, independently from its predictive power, it is a "good theory." The appendix A is an integral part of the α-Theory. It is turned to the analysis on the concepts of: physical information, indistinguishability principle of quantum particles, relation between symmetry (or anti-symmetry) concerning a wave function of a system of identical particles and the exclusion principle. The examination of these concepts will concur to operate a critical review of the "spin-statistics theorem," whose inconsistency will be shown from physical as well as mathematical point of view, highlighting that the occupation numbers play a fundamental role for quantum statistics and that it is necessary to find a theory connecting these numbers to the spin of elementary particles, in order to restore, in another way, the validity of the relation between spin and statistics. The α-Theory seems to satisfy such a wish.

In conclusion, the work introduced in this book wants to lay the foundations of the α-Theory, so allowing all the experts to deepen and develop it in the best possible way, with the opinion and aspiration that it should represent the theory which the physics will be unified by.[1]

Vincenzo Maiella

1 Introduction — The Tachyonic World

The Standard Model of particle physics is based on Klein-Gordon, Dirac and Yang-Mills equations. Each of them gives rise to a precise theory. Therefore, within the Standard Model there are three theories, able to explain dynamics of scalar, spinorial and gauge fields, respectively. This means the Standard Model is not a unified theory, because it is not devised on a unique theory, but represents a grouping of independent theories, and, although supported by important experimental confirmations, it is not in a position for explaining the fundamental interactions in a single way. This chapter was born with the attempt to find a single theory able to describe elementary particles. Such a theory can be obtained from the development of an equation that can describe the elementary particles all at once. Since the spin is an intrinsic property of quantum particles, it seems correct to identify such a general equation with the equation capable of describing the elementary particles with arbitrary spin. The search for such an equation will take us to understand the true nature of superluminal particles called "tachyons," using them not again for metaphysical or science fiction interpretations, but revealing themselves as a logical and needful characteristic of our universe, without which we could not be.

1.1 The problem of Dirac and Klein-Gordon theories

The Dirac and Klein-Gordon theories, characterized, in general terms, by the Lagrangian densities (in natural units)

$$\mathcal{L}_{Dirac} = \bar{\psi} \left(i\gamma^{\mu}\partial_{\mu} - m \right) \psi \tag{1}$$

$$\mathcal{L}_{K-G} = \partial_{\mu}\phi^{*}\partial^{\mu}\phi - m^{2}\phi^{*}\phi, \tag{2}$$

even though represent the core theories of the elementary particle physics, and therefore of the Standard Model, they only concern a limited particle classes: those having spin $s = 1/2$ and $s = 0$. This is an old problem in QFT, because the absence of an equation able to describe all the particles with arbitrary spin makes incomplete the entire theoretical framework of the high energy physics. This anomaly becomes still more obvious by the moment when an important particle class – the gauge bosons – is described by the Yang-Mills theory, through the Lagrangian density [2]

$$\mathcal{L}_{Y-M} = -\frac{1}{2g^{2}} Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{3}$$

The reason of such a conceptual disparity between particle classes is not understood, since in principle they would be described in a unitary way and not separately. This issue has gripped for a

[2] $\tilde{F}_{\mu\nu} \equiv (\partial_{\mu}\tilde{A}_{\nu} - \partial_{\nu}\tilde{A}_{\mu}) + ig[\tilde{A}_{\mu}, \tilde{A}_{\nu}]$.

long time the theorists, and in fact the development of a unique theory for quantum particles was born with the dawn of the research in this discipline. The first one that noticed the necessity of a unitary model was Ettore Majorana who, without success, tried to extend the Dirac equation to particles having arbitrary spin [1, 2, 3]. Since then, many theoretical physicists have attempted in the search for a complete theory of the elementary particles. The dream to find a unitary way in order to describe the particles with arbitrary spin is not pointless, but it is fundamental for reaching the so long wanted unification of the four fundamental interactions, which is the Saint Graal of the contemporary physics. The supergravity models are proof of that, since they use an equation of high spin, such as the Rarita-Schwinger one, in order to find an effective connection between Einstein field and particles with spin $s = 3/2$ [4]. What already said confirms that the elaboration of a theory capable of describing all the elementary particles, independently from their spin value, is fundamental into theoretical research, but also in the experimental one. In fact, necessarily, the birth of this model will bring conceptual innovations, which should concur to understand better, and maybe to resolve, modern problems, such as the existence or the non-existence of super-symmetrical particles, the Dark Matter and Dark Energy, the construction of an effective quantum gravity, and the understanding of the non-Standard properties of the Higgs boson [5, 6, 7].

1.2 Equations for particles with arbitrary spin

The problems of the Klein-Gordon and Dirac equations, developed in the past century, can thus be summarized:

1. Even if the Klein-Gordon equation usually describes particles with zero spin, it was really born as a relativistic equation for all quantum particles and in its genesis there is no reference neither zero spin nor other spin value (integer or half-integer).

2. The Dirac equation was born in order to describe the electrons (the only known elementary particles with the photons, at the time) and it is not suited for describing particles with arbitrary half-integer spin.

3. The Dirac equation is based on the strange linearization: $H \approx c\alpha_i p_i + \beta mc^2$ (with $i \in \{1,2,3\}$), which is not explained by a formal point of view.[3]

4. In the Dirac equation there are the matrices α_i and β (enclosed, in the four-dimensional expression of this equation, into matrices γ^μ), which have no direct physical meaning.

The issues 1, 2, 3, 4 tell us that Dirac and Klein-Gordon equations, taken singularly or in couples, cannot really describe all the elementary particles, but only parts of them or overabundant systems. What we need is, then, a new equation describing the elementary particles for any spin value, integer or half-integer. How can we derive such a super-equation? Let us first observe this new equation

[3]It is not sufficient to assert it is used for having an expression proportional to the amount $(p^2c^2 + m^2c^4)^{1/2}$, that still does not suffer of the problems occurring as a consequence of the square root presence.

will have to respect the Quantum Mechanics and special Relativity principles. In order to find it seems, therefore, necessary to use the non-relativistic quantum mechanics before and then to extend the found result to the case of particles with speed compared with the light one.

At this point, we can write the Schrödinger equation for a particle having arbitrary spin. For making it, we use the Pauli equation for an electron in an electromagnetic field [8, 9, 10]

$$\left\{\frac{1}{2m}\left[\vec{\sigma}\cdot(\vec{p}-\frac{e}{c}\vec{A})\right]^2+e\varphi\right\}\psi(\vec{x},t)=i\hbar\frac{\partial\psi}{\partial t}, \tag{4}$$

where the $\vec{\sigma}$ are the Pauli matrices. This equation, in absence of electromagnetic field, becomes

$$\frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}\psi(\vec{x},t)=i\hbar\frac{\partial\psi}{\partial t}, \tag{5}$$

that, generalized to an arbitrary spin $\vec{s}$, gives (for a dimensionally correct equation, we must operate the substitution $\vec{\sigma}\to\vec{s}/\hbar$) [4]

$$\frac{(\vec{s}\cdot\vec{p})^2}{2\hbar^2 m}\psi(\vec{x},t)=i\hbar\frac{\partial\psi}{\partial t}. \tag{6}$$

The expression (6) is the Schrödinger equation for a free quantum particle with spin $\vec{s}$. Naturally, for $s=0$ – being $\vec{s}=(0,0,0)$ – we are not able to write an equation of motion, since in such a case we simply have the time-independent wave function $\psi(\vec{x},t)=A\psi(\vec{x})$, where A is an arbitrary constant. Therefore, the particles having $s=0$ are described by a stationary wave function in the non-relativistic framework.

The equation (6) is important, because it will allow us, in the next pages, to write correctly the relativistic equation for particles with arbitrary spin. About that, notice for obtaining a relativistic equation for a free quantum particle, we must find an equation of the type

$$H\psi(\vec{x},t)=i\hbar\frac{\partial\psi}{\partial t}, \tag{7}$$

where H is the relativistic Hamiltonian of the generic particle that we want to study. Now the problem is as H has to be written. Since in the non-relativistic equation (6) the Hamiltonian is proportional to $(\vec{s}\cdot\vec{p})^2$, it is natural trying a relativistic Hamiltonian which is its four-dimensional generalization. Therefore, as relativistic Hamiltonian we should have an expression proportional to the scalar product between spin and momentum. Obviously, since our H is needed to have a relativistic form, this scalar product must be made in the Minkowski space, *i.e.* it will be a product between four-vectors. Hence, we must have

[4] Of course, the vector $\vec{s}$ has like components the three generators of $LieSU(2)$ for any $s\in\mathbb{N}/2$.

$$H \propto s_\mu p^\mu, \tag{8}$$

where p^μ is the four-momentum given by [5]

$$p^\mu = (mc, \vec{p}). \tag{9}$$

What is instead the spin four-vector s_μ? With regard to its spatial part, it is easy to understand it is enough to take just the vectorial operator $\vec{s}$ and so the problem arises from its temporal part, which is unknown. However, when such an operator is indicated by s_0, nothing prevents us to leave it as unknown and to determine its form later on. Therefore, we can write [6]

$$s_\mu = (s_0, \vec{s}). \tag{10}$$

Naturally, since we want to write an equation for particles having arbitrary spin, namely with s variable as we like, it is clear that we do not consider the simple operatorial form of spin, but its matrix representation. Therefore, in order to simplify the calculations, since the matrix representations of the spin operator are expressed in $\hbar$ units, we divide these representations by $\hbar$, *i.e.* we place

$$\tilde{\delta} \equiv \frac{1}{\hbar} s_0 \tag{11}$$
$$\vec{\varepsilon} \equiv \frac{1}{\hbar} \vec{s}, \tag{12}$$

by which, we can thus define the (covariant) spin four-vector

$$s_\mu \equiv (\tilde{\delta}, \vec{\varepsilon}) \;\Rightarrow\; s^\mu = (\tilde{\delta}, -\vec{\varepsilon}), \tag{13}$$

that, for being precise, represents a matrix four-vector, whose elements have dimension $(2s+1) \times (2s+1)$. From what we said, it can be written

$$H \propto s_\mu p^\mu = s_0 p^0 + s_i p^i = \tilde{\delta} mc + \vec{\varepsilon} \cdot \vec{p}. \tag{14}$$

[5] Of course $p_\mu = (mc, -\vec{p})$.

[6] In principle, we could use other components for s_μ, as an example the higher-dimensional gamma matrices, but this choice does not seem to be the physical one respecting the equation (6) philosophy.

Obviously the expression (14) has not the correct dimension of an energy. We can put this expression in a right and compact fashion, if we define the "energy four-vector"

$$E^{\mu} \equiv cp^{\mu} = (mc^2, \vec{p}c), \tag{15}$$

which enables us to write [7]

$$H = s_{\mu}E^{\mu} = cs_{\mu}p^{\mu} = \tilde{\delta}mc^2 + \vec{\varepsilon} \cdot \vec{p}c. \tag{16}$$

As we can see, this Hamiltonian, with the substitutions

$$\tilde{\delta} \to \beta \tag{17}$$

$$\vec{\varepsilon} \to \vec{\alpha}, \tag{18}$$

is perfectly identical to the Dirac one. Nevertheless, it is a more power expression, because, while Dirac wrote *ad hoc* its relation, by trying to linearize the relativistic energy $(p^2c^2 + m^2c^4)^{1/2}$, such an Hamiltonian was born spontaneously, like generalization of that for a non-relativistic quantum particle previously studied. Furthermore, it does not introduce unphysical elements like the Dirac matrices $\vec{\alpha}$ and β, but only physical characteristics of a free quantum particle, such as spin four-vector and "four-energy" (*i.e.* four-momentum multiplied by c), simply coupled through their scalar product. The intrinsically physical character is, therefore, the force of this Hamiltonian, which is much better than the Dirac one from an epistemological point of view. Arriving at this point, we can write the quantum relativistic equation (to the first order) for a particle with arbitrary spin. It is

$$s_{\mu}E^{\mu}\psi(\vec{x}, t) = i\hbar\frac{\partial\psi}{\partial t}(\vec{x}, t), \tag{19}$$

and, making clear the scalar product, one obtains

$$(\tilde{\delta}mc^2 + \vec{\varepsilon} \cdot \vec{p}c)\psi(\vec{x}, t) = i\hbar\frac{\partial\psi}{\partial t}(\vec{x}, t). \tag{20}$$

Now, by remembering that

[7]Such an Hamiltonian can be defined also independently from the Pauli equation, reasoning only on the intrinsic properties of a free quantum particle, which are spin and momentum (or better the spin and energy four-vectors).

$$p^i \to -i\hbar \frac{\partial}{\partial x_i}, \tag{21}$$

we have

$$i\hbar \frac{\partial \psi}{\partial t}(\vec{x}, t) = -i\hbar c \vec{\varepsilon} \cdot \vec{\nabla}\psi + mc^2 \tilde{\delta}\psi. \tag{22}$$

The (22) represents the quantum relativistic differential equation (to the first order) for a particle having arbitrary spin. Therefore, it should be able to describe bosons as well as fermions. We call the (22) "M_α equation" to the first order.

It must be emphasized what amazing result we have just obtained. In fact, one quickly observes such an equation is identical, in form, to the Dirac one

$$i\hbar \frac{\partial \psi}{\partial t}(\vec{x}, t) = -i\hbar c \vec{\alpha} \cdot \vec{\nabla}\psi + mc^2 \beta\psi. \tag{23}$$

Let us identify the differences between these two equations:

- The Dirac equation is fixed to a four-dimensional matrix representation, which, moreover, has no immediate physical meaning.
- The Dirac equation contains the matrices $\vec{\alpha}$ and β, that have no physical meaning.
- The M_α equation changes with the spin s of particles. Then s_μ, less $\hbar$, just represents the spin four-vector of the generic particle that one wants to study. Therefore, all the quantities of M_α equation have physical meaning.
- The M_α equation admits four-dimensional representations too.

We promptly see that, formally, it is possible to pass from the M_α equation to the Dirac one and vice-versa, through the substitutions

$$\vec{\alpha} \Leftrightarrow \vec{\varepsilon} \tag{24}$$
$$\beta \Leftrightarrow \tilde{\delta} \tag{25}$$

and this could explain the huge success of the Dirac equation. In fact, it is nothing more than a special case of the M_α equation, in which the matrices $\tilde{\delta}$ and $\vec{\varepsilon}$ are arbitrarily replaced with β and $\vec{\alpha}$.

There is still a problem in the M_α equation to the first order and it is, obviously, the unknown matrix $\tilde{\delta}$, which varies with s. How can we find the general form of such a matrix? The simpler thing getting in mind is to compare the M_α equation with a well-known one, thus to characterize $\tilde{\delta}$. What equation can we take? Of course, one cannot use

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t)=\frac{(\vec{s}\cdot\vec{p})^2}{2\hbar^2 m}\psi(\vec{x},t), \tag{26}$$

because, as previously seen, such an equation is non-relativistic. For that, being the M_α a relativistic equation, it is natural trying to confront it with another relativistic equation. Since it wants to replace itself with the Dirac equation (of which it is proposed to be the generalization), obviously we cannot use this last equation (*i.e.* the Dirac one). What is the equation left with us? The choice must go necessarily on the Klein-Gordon equation, that as we know is written out through the Schrödinger equation by inserting in place of H the relativistic energy $E=(p^2c^2+m^2c^4)^{1/2}$. But the Klein-Gordon equation has a scalar form and not a matrix one, like the M_α equation, and so we have to reduce this last equation to a scalar expression, which can be obtained for $s=0$ only, because such a point, as it is well known from the Lie algebra of the group $SU(2)$, is the only one to characterize the scalar representation. Therefore, a way of finding $\tilde{\delta}$ is to confront the M_α equation in $s=0$ with the Klein-Gordon one. However, it needs to do an observation on the form which M_α must have for $s=0$ and this concerns the circumstance that the equation we are constructing is given by the scalar product of s_μ with p^μ and so, by remembering for $s=0$ one has $\vec{s}=(0,0,0)$, we cannot simply equalize the M_α equation in $s=0$ with the ordinary Klein-Gordon one, because in general only the time derivative has to be saved (in fact, the space derivatives must be null for $s=0$, based on the product $s_\mu E^\mu$). For that reason, the equation reasonably has to be compared with the M_α one is

$$\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{m^2c^2}{\hbar^2}\Big)\psi(\vec{x},t)=0, \tag{27}$$

which is the Klein-Gordon equation of a particle moving along the t-axis. It is obvious that, the M_α equation

$$i\hbar\frac{\partial\psi}{\partial t}=-i\hbar c\vec{\varepsilon}\cdot\vec{\nabla}\psi+mc^2\tilde{\delta}\psi, \tag{28}$$

not being to the second order, will not be able to realize this wish. Therefore, it is necessary to find the M_α equation to the second order. For writing it, we can see that, given a partial differential equation of type

$$A\psi(\vec{x},t)=B\psi(\vec{x},t), \tag{29}$$

with A, B differential operators, in order to transform it to second order, we may consider

$$A^2\psi(\vec{x},t) = B^2\psi(\vec{x},t), \tag{30}$$

but also [8]

$$|A|^2\psi(\vec{x},t) = |B|^2\psi(\vec{x},t), \tag{31}$$

but also

$$(A-B)^2\psi(\vec{x},t) = 0, \tag{32}$$

but also [9]

$$|A-B|^2\psi(\vec{x},t) = 0. \tag{33}$$

Hence, we understand that, while a differential equation to the first order (with partial derivatives and not) is well-defined, its square (or better the second order differential equation we can derive by it) is not only one. This does not mean it is physically impossible to obtain a second order differential equation knowing the first order one, but it advises the physicist so that he or she understand as the Nature wants to square the first order equation, or better, what is the more suitable second order equation for the explanation of the natural phenomena. For example, Klein and Gordon, in order to find their equation, decided it had to be

$$A^2\psi(\vec{x},t) = B^2\psi(\vec{x},t), \tag{34}$$

with

$$A = i\hbar\frac{\partial}{\partial t}, \quad B = (p^2c^2 + m^2c^4)^{1/2}. \tag{35}$$

Since they wanted to eliminate the square root, these equations

$$(A-B)^2\psi(\vec{x},t) = 0, \quad |A-B|^2\psi(\vec{x},t) = 0 \tag{36}$$

[8] $|A| \equiv (A^\dagger A)^{1/2}$, $|B| \equiv (B^\dagger B)^{1/2}$.

[9] $|A-B| \equiv [(A-B)^\dagger(A-B)]^{1/2}$.

were rightly discarded (it can be mistrusted this was made intentionally). But what result they would have obtained if the form $|A|^2\psi = |B|^2\psi$ had been used? In their case, having the squared operator p already under root, they would have had simply to consider the equation

$$\overbrace{\Big(-i\hbar\frac{\partial}{\partial t}\Big)}^{A^\dagger}\overbrace{\Big(i\hbar\frac{\partial}{\partial t}\Big)}^{A}\psi = (p^\dagger p c^2 + m^2c^4)\psi, \tag{37}$$

by which, remembering that

$$\vec{p} \to -i\hbar\vec{\nabla}, \tag{38}$$

to write

$$\hbar^2\frac{\partial^2}{\partial t^2}\psi = (c^2\hbar^2\nabla^2 + m^2c^4)\psi \ \Rightarrow\ \Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{m^2c^2}{\hbar^2}\Big)\psi(\vec{x},t) = 0, \tag{39}$$

whose four-dimensional form is

$$\Big(\partial_\mu\partial^\mu - \frac{m^2c^2}{\hbar^2}\Big)\psi(\vec{x},t) = 0. \tag{40}$$

We see the obtained equation is identical to the Klein-Gordon one, apart from the sign of the constant term. What is describing this equation? Leaving out for the moment this question, thus increasing the reader suspense, we want to find the M_α equation to the second order starting from the first order one and we make it using the following way of doing the squares

$$A^\dagger A\psi(\vec{x},t) = B^\dagger B\psi(\vec{x},t), \tag{41}$$

where, in our case, we take

$$A = i\hbar\frac{\partial}{\partial t}, \quad B = -i\hbar c\vec{\varepsilon}\cdot\vec{\nabla} + mc^2\tilde{\delta}. \tag{42}$$

Obviously, this choice is reflected also in the equation to which our expression, for $s = 0$, must be reduced, that will no longer be

$$\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{m^2c^2}{\hbar^2}\Big)\psi(\vec{x},t) = 0, \tag{43}$$

but rather

$$\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{m^2c^2}{\hbar^2}\Big)\psi(\vec{x},t)=0 \;\Leftrightarrow\; \Big(-\hbar^2\frac{\partial^2}{\partial t^2}+m^2c^4\Big)\psi(\vec{x},t)=0. \tag{44}$$

From (41) and (42), with a few calculations, we get

$$\Big[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\varepsilon_i\varepsilon_k\partial_i\partial_k-i\hbar mc^3(\tilde{\delta}^\dagger\varepsilon_i-\varepsilon_i\tilde{\delta})\partial_i+m^2c^4\tilde{\delta}^\dagger\tilde{\delta}\Big]\psi(\vec{x},t)=0, \tag{45}$$

representing the M_α equation to the second order (according to the chosen method for raising of one order our initial equation). Now we see if a matrix representation $\tilde{\delta}$, which is able to reduce the just found equation to the previous (44), for $s=0$, exists. First of all we notice that, for $s=0$, the M_α equation to the second order is quickly written [10]

$$\Big[-\hbar^2\frac{\partial^2}{\partial t^2}+m^2c^4\tilde{\delta}_0^*\tilde{\delta}_0\Big]\psi(\vec{x},t)=0, \tag{46}$$

where, with $\tilde{\delta}_0$, we indicated the matrix $\tilde{\delta}$ for $s=0$, which obviously must be a scalar. Can a value of $\tilde{\delta}_0$ exist that reduces the (46) to (44)? It is straightforward to verify this happens, in general, for $\tilde{\delta}_0=+i$ and $\tilde{\delta}_0=-i$, *i.e.* when $\tilde{\delta}_0$ has two values.[11] Since

$$\tilde{\delta}_0^*=-\tilde{\delta}_0,\quad \tilde{\delta}_0^2=-1, \tag{47}$$

it is natural requiring in general [12]

$$\tilde{\delta}^\dagger=-\tilde{\delta},\quad \tilde{\delta}^2=-\mathbb{1}_s. \tag{48}$$

Such conditions are satisfied from the diagonal matrices having for elements the alternating sequence of $-i$ and $+i$, namely [13]

[10]Hence, unlike the non-relativistic case, the relativistic equation for a particle having $s=0$ can be written out. This means the relativistic particle description is deeper than the non-relativistic one.

[11]Note that such a result could be achieved with an inverse reasoning too, *i.e.* by observing the chosen values of $\tilde{\delta}_0$ are the only ones (with the other solution $\tilde{\delta}_0=\pm1$) able to reduce the (46) to a well-known equation, which, in this case, just coincides with the (44) one, in total agreement with the fact that $\vec{s}=(0,0,0)$ for $s=0$.

[12]The subindex s is not the matrix dimension, but the spin index.

[13]Really, there is – for the (46) – the solution $\tilde{\delta}_0=\pm1$ too, from which it can be obtained $\tilde{\delta}_0^*=\tilde{\delta}_0$, $\tilde{\delta}_0^2=1$ and so $\tilde{\delta}^\dagger=\tilde{\delta}$, $\tilde{\delta}^2=\mathbb{1}_s$. These last conditions are satisfied from the diagonal matrices having for elements the alternating sequence of $+1$ and -1 (or vice-versa). But this solution, as we will explain in the footnote 21, has been discarded because it does not allow to have the same "particle nature" for the M_α equation to the first and to the second order.

$$\text{for } s=\tfrac{1}{2} \qquad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{49}$$

$$\text{for } s=1 \qquad \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \tag{50}$$

$$\text{for } s=\tfrac{3}{2} \qquad \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tag{51}$$

.... *etc* ...... *etc* ......

This allows to define the representations of the operator $\tilde{\delta}$ for any $s \in \mathbb{N}/2$.

It is important to notate that, being the operator $\vec{\varepsilon}$ hermitian, it is better to work with a time component of spin four-vector which is hermitian too. The fact $\tilde{\delta}$ is anti-hermitian makes no problem. In fact, defining [14]

$$\tilde{\delta} \equiv i\delta, \tag{52}$$

we can consider δ in place of $\tilde{\delta}$. If into equations (28) and (45) we carry out the substitution $\tilde{\delta} = i\delta$, it can be obtained [15]

$$\hbar\frac{\partial\psi}{\partial t} = -\hbar c\vec{\varepsilon}\cdot\vec{\nabla}\psi + mc^2\delta\psi \tag{53}$$

$$\Big[-\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2c^2\varepsilon_i\varepsilon_k\partial_i\partial_k - \hbar mc^3(\varepsilon_i\delta + \delta\varepsilon_i)\partial_i + m^2c^4\mathbb{1}_s\Big]\psi(\vec{x},t) = 0, \tag{54}$$

which represent the M_α equation to the first and to the second order one written through hermitian matrices. We see now the representations of the operator δ consist of all diagonal matrices having

[14] Naturally, $\delta^\dagger = \delta$, $\delta^2 = \mathbb{1}_s$. In compact form, one has $(\delta)^s_{m,m'} = (-1)^{s-m}\delta_{m,m'}$ $\forall s \in \mathbb{N}/2 - \{0\}$, where we must not mix up the Kronecker function with the symbol delta used for our matrices.

[15] δ, unlike $\tilde{\delta}$, realizes an unitary and hermitian representation.

for elements the alternating sequence of $+1$ and -1. Before studying the equations (53) and (54) and their Lagrangian theories very well, we must observe two things. Firstly, one has to note that the adopted representation is not the only one which allows to reduce, for $s = 0$, the M_α equation to the second order in the form

$$\Big(-\hbar^2 \frac{\partial^2}{\partial t^2} + m^2 c^4\Big)\psi(\vec{x}, t) = 0. \tag{55}$$

In fact, if we take as δ the diagonal matrices having for elements the alternating sequence of -1 and $+1$ (for δ_0 we obviously take the couple ± 1 always, because it is the scalar representation), we can obtain the same results.[16] We indicate therefore, with obvious symbolic meaning, $\delta_{\pm 1}$ as the first representation and $\delta_{\mp 1}$ as the second one. These representations lead, like already explained, to the same form of the M_α equation to the first and to the second order one, but, as we will see later on, they originate some important physical differences. For now, we work with the representation $\delta_{\pm 1}$, by indicating it simply with δ. The second thing to remark concerns the wave function ψ. Since the M_α equation to the first order and to the second order one depend on the spin s, in their inside will be matrices of a $(2s+1)$-dimensional space and so it is clear that ψ will have to be a matrix column of the same dimension. Hence, it is right to add a subindex s to ψ, namely to place ψ_s in our equations, that correctly must be written [17]

$$\hbar \frac{\partial \psi_s}{\partial t} = -\hbar c \vec{\varepsilon} \cdot \vec{\nabla} \psi_s + \mu c^2 \delta \psi_s \tag{56}$$

$$\hbar^2 \frac{\partial^2 \psi_s}{\partial t^2} = \hbar^2 c^2 \varepsilon_i \varepsilon_k \frac{\partial^2 \psi_s}{\partial x^i \partial x^k} - \hbar \mu c^3 (\varepsilon_i \delta + \delta \varepsilon_i) \frac{\partial \psi_s}{\partial x^i} + \mu^2 c^4 \psi_s. \tag{57}$$

We now calculate the probability densities of our equations. Concerning the M_α equation to the first order, we have [18]

$$\frac{\partial (\psi_s^\dagger \psi_s)}{\partial t} = -\vec{\nabla} \cdot (c \psi_s^\dagger \vec{\varepsilon} \psi_s) + \frac{2\mu c^2}{\hbar} \psi_s^\dagger \delta \psi_s, \tag{58}$$

from which, if we put

$$\begin{cases} \rho(\vec{x}, t) \equiv \psi_s^\dagger \psi_s \equiv |\psi_s|^2 \\ \vec{j}(\vec{x}, t) \equiv c \psi_s^\dagger \vec{\varepsilon} \psi_s, \end{cases} \tag{59}$$

[16] In this case, we have $(\delta)^s_{m,m'} = (-1)^{s-m\pm 1} \delta_{m,m'}$ $\forall s \in \mathbb{N}/2 - \{0\}$. Note that such a representation corresponds to $\tilde{\delta}$ having for diagonal elements the alternating sequence of i and $-i$, which is the other choice based on the conditions (48).

[17] Rename also the mass m in μ for reasons which will be clear in the next pages.

[18] For obtaining the (58), we proceed as the Dirac equation, but, unlike it, we consider the sum and not the difference of the equations calculated by the hermitian conjugate of the (56).

we get the non-continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \frac{2\mu c^2}{\hbar}(\psi_s^\dagger \delta \psi_s), \tag{60}$$

which becomes a continuity equation for massless particles only ($\mu = 0$). The (60) is a true tragedy for the M_α equation to the first order. In fact, if we interpret it as an equation for single-particle, we will have that ρ is a probability density which does not conserve itself, while, if we interpret it as a field equation, we will have ρ is a charge density that also does not conserve itself. Therefore, we must sadly admit that the M_α equation to the first order does not describe elementary particles. Now we see if at least the M_α equation to the second order admits a continuity equation. We can show it has the following non-continuity equation too

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \frac{\mu c^3}{\hbar}\left[\frac{\partial \psi_s^\dagger}{\partial x_i}(\varepsilon_i \delta + \delta \varepsilon_i)\psi_s - \psi_s^\dagger(\varepsilon_i \delta + \delta \varepsilon_i)\frac{\partial \psi_s}{\partial x^i}\right], \tag{61}$$

where we have defined

$$\begin{cases} \rho(\vec{x}, t) \equiv (\psi_s^\dagger \dot{\psi}_s - \dot{\psi}_s^\dagger \psi_s) \\ \vec{j}(\vec{x}, t) \equiv c^2 \left(\frac{\partial \psi_s^\dagger}{\partial x^k}\varepsilon_k \vec{\varepsilon} \psi_s - \psi_s^\dagger \vec{\varepsilon} \varepsilon_k \frac{\partial \psi_s}{\partial x^k}\right). \end{cases} \tag{62}$$

Also the (61) becomes a continuity equation in a massless regime only ($\mu = 0$). Hence, the M_α equation to the second order, like the first order one, does not describe elementary particles, but rather *objects* that work in a strange way. In order to discover some other information on these curious objects, we study the energetic spectrum of the first and second order M_α equation. To do this, let us set in the classical limit ($v \ll c$), thanks to which we can simplify our equations in such a way (if $v \ll c$ will be $cp_i \ll mc^2$, $c^2 p_i p_k \ll m^2 c^4$)

$$\hbar \frac{\partial \psi_s}{\partial t} = \mu c^2 \delta \psi_s \tag{63}$$

$$\hbar^2 \frac{\partial^2 \psi_s}{\partial t^2} = \mu^2 c^4 \psi_s\,. \tag{64}$$

By replacing in the above equations the generic plane-wave solution [19]

$$\psi_s(\vec{x}, t) = e^{i(\vec{k}\cdot\vec{x} - \omega t)} u_s(k), \tag{65}$$

[19] Naturally, $u_s(k)$ is a column vector.

we have, with s varying, the following energetic spectra:

1. First Order M_α Equation

for $s = 0$ $$E = \pm i\mu c^2 \Rightarrow E_1^{s=0} = i\mu c^2, E_2^{s=0} = -i\mu c^2. \tag{66}$$

for $s = \frac{1}{2}$ $$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} i\mu c^2 & 0 \\ 0 & -i\mu c^2 \end{pmatrix} \Rightarrow E_1^{s=1/2} = i\mu c^2, E_2^{s=1/2} = -i\mu c^2. \tag{67}$$

for $s = 1$ $$\begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix} = \begin{pmatrix} i\mu c^2 & 0 & 0 \\ 0 & -i\mu c^2 & 0 \\ 0 & 0 & i\mu c^2 \end{pmatrix} \Rightarrow E_1^{s=1} = i\mu c^2, E_2^{s=1} = -i\mu c^2, E_3^{s=1} = i\mu c^2. \tag{68}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, etc \ldots\ldots \, etc \ldots\ldots$$

2. Second Order M_α Equation

for $s = 0$ $$E^2 = -\mu^2 c^4 \Rightarrow E_1^{s=0} = i\mu c^2, E_2^{s=0} = -i\mu c^2. \tag{69}$$

for $s = \frac{1}{2}$ $$\begin{pmatrix} E^2 & 0 \\ 0 & E^2 \end{pmatrix} = \begin{pmatrix} -\mu^2 c^4 & 0 \\ 0 & -\mu^2 c^4 \end{pmatrix}$$ and so

$$E_1^{s=1/2} = i\mu c^2, E_2^{s=1/2} = -i\mu c^2, E_3^{s=1/2} = i\mu c^2, E_4^{s=1/2} = -i\mu c^2. \tag{70}$$

for $s = 1$ $\quad \begin{pmatrix} E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} = \begin{pmatrix} -\mu^2 c^4 & 0 & 0 \\ 0 & -\mu^2 c^4 & 0 \\ 0 & 0 & -\mu^2 c^4 \end{pmatrix}$ $\quad$ and so

$$E_1^{s=1} = i\mu c^2, E_2^{s=1} = -i\mu c^2, E_3^{s=1} = i\mu c^2, E_4^{s=1} = -i\mu c^2, E_5^{s=1} = i\mu c^2, E_6^{s=1} = -i\mu c^2. \tag{71}$$

.... *etc* ...... *etc* ......

Therefore, the M_α equation, to the first as well as to the second order, does not describe real particles, but with imaginary mass ones, *i.e.* it always describes tachyons. It is important to realize that, for every s, there are tachyons with positive and negative (imaginary) energy, just like it happens for bradyonic particles. Nevertheless, the distribution of such tachyonic solutions introduces a substantial difference between the first and second order M_α equation. In fact, regarding the first order equation we have that, for $s = 0$ and half-integer spins, the states with positive and negative (imaginary) energy are perfectly symmetric, while, for s integer, an asymmetry is recorded, in the sense the solutions with positive (imaginary) energy are greater than negative (imaginary) energy ones.[20] On the contrary, concerning the second order M_α equation, we have a perfect symmetry, for every s, between positive and negative (imaginary) energy solutions, namely we always have $2s + 1$ positive and $2s + 1$ negative solutions.[21]

[20] This is true for the representation $\delta_{\pm 1}$. It is straightforward to prove that for the representation $\delta_{\mp 1}$ is exactly the opposite.

[21] All that happens since we chose in (46) the solution $\tilde{\delta}_0 = \pm i$. If we choose $\tilde{\delta}_0 = \delta_0 = \pm 1$ and so $\tilde{\delta}^\dagger = \delta^\dagger = \delta$, $\tilde{\delta}^2 = \delta^2 = \mathbb{1}_s$, we have the equations

$$i\hbar \frac{\partial \psi_s}{\partial t} = -i\hbar c \vec{\varepsilon} \cdot \vec{\nabla} \psi_s + \mu c^2 \delta \psi_s$$
$$\hbar^2 \frac{\partial^2 \psi_s}{\partial t^2} = \hbar^2 c^2 \varepsilon_i \varepsilon_k \frac{\partial^2 \psi_s}{\partial x^i \partial x^k} - i\hbar \mu c^3 (\delta \varepsilon_i - \varepsilon_i \delta) \frac{\partial \psi_s}{\partial x^i} + \mu^2 c^4 \psi_s.$$

As it simply to prove, in the classical limit the first describes bradyons, while the second one describes tachyons and so the "particle nature" of our equations is not conserved.

The question now we ask ourselves is what physical usefulness these equations have. The idea they could describe real particles with arbitrary spin was powerful and fascinating. Must we leave this dream, or the found equations are giving precious informations we have to discover and analyze? Thinking this is the just road to follow for any logical-deductive theory, we want to explore more closely the meaning of our equations, by trying to describe in the best way the stage which they show us.

1.3 The tachyonic universe

From the first and second order M_α equation emerged that the universe described by them is made of strange objects with imaginary energetic spectrum. The special Relativity tells us this characteristic belongs to a space-time region made of events connected from space-like intervals (the absolute elsewhere cone), that is, in terms of particles, from the region of superluminal particles, called tachyons. Therefore, our equations describe the tachyonic universe, since they describe the particles faster-than-light and whose speed cannot be inferior to c. But this vision is in direct contrast with the method in which we have found the energetic spectra of the equations (56) and (57), because these spectra have been found in the classical limit, *i.e.* in the range of speeds much smaller than c. And this is evidently an abuse, if we deal with particles whose minimal speed must be c. This brings to following "tachyonic paradox" (TP)

TP: The first and second order M_α equation, if put in the $v \ll c$ limit, describe tachyons, namely particles with $v > c$. But this breaks the hypothesis to consider particles having $v \ll c$ only, since them naturally cannot be tachyons.

The tachyonic paradox reaches what in philosophical speak is called "vicious circle." How can we resolve the tachyonic paradox? It is the special Relativity (SR) which can help us, since we can note within it there is a double vision on the tachyonic world. In fact, as it is well known the energy concerning a bradyon with (real) mass m is

$$E_B = \frac{mc^2}{(1 - \frac{v^2}{c^2})^{1/2}},$$

where it needs to impose $v < c$ for $E_B \in \mathbb{R}$. If $v > c$, the above expression becomes [22]

$$\frac{mc^2}{\mp i(\frac{v^2}{c^2} - 1)^{1/2}}. \tag{72}$$

[22]Naturally, one can choose the plus or minus sign only, but we have a more general discussion with the double sign.

Only through the correspondence

$$m \to \mp i\mu \tag{73}$$

a real energy is still obtained. It is given by

$$E_T = \frac{\mu c^2}{(\frac{v^2}{c^2} - 1)^{1/2}}, \tag{74}$$

representing the energy of a tachyon with (real) mass μ. This viewpoint leads us to think that the tachyon is a particle having $v > c$ and $E^2 > 0$ (real energy). It is against our results leading to the tachyonic paradox (TP).

Nevertheless, the TP suggests us another way in order to characterize a tachyon, being always in agreement with the SR. In fact, starting from the expression [23]

$$E_T = \frac{\mu c^2}{(\frac{v^2}{c^2} - 1)^{1/2}},$$

and supposing now $v < c$, we have

$$E_T = \frac{\mu c^2}{\mp i(1 - \frac{v^2}{c^2})^{1/2}},$$

from which, multiplying up and down by $\pm i$, we obtain

$$E_T = \frac{\pm i\mu c^2}{(1 - \frac{v^2}{c^2})^{1/2}}, \tag{75}$$

i.e., since $\mu \in \mathbb{R}$, the energy E_T must be imaginary and so the tachyonic universe is characterized from $E^2 < 0$, on this view. This trait of tachyonic universe resolves the TP, because now the tachyons are not seen like particles having speed v (in module) higher than the speed of light c, but simply as particles having a negative square energy. It must be said that the expression (75) establishes a *duality* between the tachyonic universe and the bradyonic one. In fact, based on the correspondence

$$\mu \to \mp im,$$

[23] It is straightforward to note that, if we operate in the expression of E_B the correspondence $m \to \mp i\mu$, the (74) is obtained, and this without imposing $v > c$ for a tachyon. So it can be retained the E_T is the most general expression of the energy concerning a tachyon, independently from its speed value.

we can get into the bradyonic universe, characterized by

$$E_B = \frac{mc^2}{(1-\frac{v^2}{c^2})^{1/2}}.$$

Therefore, the correspondence

$$m \to \pm i\mu \tag{76}$$

allows to go from the bradyonic universe to the tachyonic one, while its *inverse*

$$\mu \to \mp im \tag{77}$$

lets pass through the tachyonic universe to the bradyonic one. Therefore, it is possible to transform a bradyon with mass m into a tachyon with mass μ simply by using the imaginary unit i.

Either way, it is well to emphasize that the tachyonic paradox can be resolved only if the speed of the generic tachyon is smaller than c. Therefore, thanks to the TP we can detect a new vision within the special Relativity, which is in total agreement with the principle of invariant light speed,[24] since it implies the tachyons really are particles with $v < c$ and $E^2 < 0$.

This suggests a review concerning the scientific terminology on these particles. In fact, the generic particle having $v < c$ and $E^2 < 0$ would not have to be called any more tachyon, but "iep" (acronym for *imaginary energy particle*), while the generic particle having $v < c$ and $E^2 > 0$ would have to be called "rep" (acronym for *real energy particle*). It goes without saying that the tachyonic universe would have to be named "IEP universe," while the bradyonic one would have to be named "REP universe."

From what we saw, it can be understood that in special Relativity a *dual* way for describing the tachyons exists: the first one through a real energy with positive square and $v > c$, the second one through an imaginary energy with negative square and $v < c$. This last case justifies the result found for the first and second order M_α equation for $v \ll c$ and it suggests us a deeper interpretation of the tachyonic universe, that resolves the paradox earlier enunciated. This alternative vision leads to conclude that the speed of light c is a *universal limit*, beyond which one does not approach to a universe with particles faster than c, but one goes into a universe where the energy has a different form than the one we know, in the sense that from our point of view it is expressed in an imaginary unit. As already shown, this vision does not change the SR results and it allows to elegantly resolve the tachyonic paradox, thus giving sense to the equations (56) and (57).

[24]Remember that, in the Einstein formulation of the special Relativity, the speed of light c cannot be overtaken, in fact the hypothetical existence of particles having speed faster-than-light was born from late speculations [11].

1.4 Four-dimensional form of tachyonic equations and their Lagrangian densities

In this section, we want to write the four-dimensional expressions of the (56) and (57) and also to determine their Lagrangian densities. We start by the equation

$$\hbar\frac{\partial\psi_s}{\partial t} = -\hbar c\vec{\varepsilon}\cdot\vec{\nabla}\psi_s + \mu c^2\delta\psi_s \;\Leftrightarrow\; \frac{\partial\psi_s}{\partial t} = -c\vec{\varepsilon}\cdot\vec{\nabla}\psi_s + \frac{\mu c^2}{\hbar}\delta\psi_s, \tag{78}$$

and multiply its both sides by δ/c

$$\frac{\delta}{c}\frac{\partial\psi_s}{\partial t} = -\delta\vec{\varepsilon}\cdot\vec{\nabla}\psi_s + \frac{\mu c}{\hbar}\delta^2\psi_s \;\Rightarrow\; \left(\delta\frac{\partial}{\partial(ct)} + \delta\vec{\varepsilon}\cdot\vec{\nabla}\right)\psi_s = \frac{\mu c}{\hbar}\psi_s \;\Leftrightarrow\; \left(\delta\partial_0 + \delta\varepsilon^i\partial_i\right)\psi_s = \frac{\mu c}{\hbar}\psi_s. \tag{79}$$

At this point, if we define the (matrix) four-vector

$$\chi^\mu \equiv (\delta, \delta\vec{\varepsilon}), \tag{80}$$

one quickly sees the (79) can be written

$$\chi^\mu\partial_\mu\psi_s = \frac{\mu c}{\hbar}\psi_s, \tag{81}$$

which is equivalent to

$$\left(\chi^\mu\partial_\mu - \frac{\mu c}{\hbar}\mathbb{1}_s\right)\psi_s(x) = 0. \tag{82}$$

The (82) represents the first order M_α equation in the four-dimensional form. In order to find the second order M_α equation in the four-dimensional form, we start from the vectorial expression

$$\hbar^2\frac{\partial^2\psi_s}{\partial t^2} = \hbar^2c^2\varepsilon_i\varepsilon_k\frac{\partial^2\psi_s}{\partial x^i\partial x^k} - \hbar\mu c^3(\varepsilon_i\delta + \delta\varepsilon_i)\frac{\partial\psi_s}{\partial x^i} + \mu^2c^4\psi_s. \tag{83}$$

Now the scalar products must be consider in the Minkowski space and so we raise all the indices based on the four-dimensional notation.[25]Making it, the (83) can be put in the following equivalent

[25]Remember that the scalar product of two vectors $\vec{A}$ and $\vec{B}$ in the Euclidean space $\mathbb{R}^3$ is defined as $\vec{A}\cdot\vec{B} \equiv A_iB_i$, while in the Minkowski space is defined as $\vec{A}\cdot\vec{B} \equiv A^iB^i$ within the scalar product of two four-vectors A^μ and B^μ, where $i \in \{1,2,3\}$ in both cases.

expressions

$$
\begin{aligned}
&\left[-\hbar^2\mathbb{1}_s\frac{\partial^2}{\partial t^2}+\hbar^2c^2\varepsilon^i\varepsilon^k\nabla^i\nabla^k-\hbar\mu c^3(\varepsilon^i\delta+\delta\varepsilon^i)\nabla^i+\mu^2c^4\mathbb{1}_s\right]\psi_s(\vec{x},t)=0\Rightarrow\\
&\left[-\frac{\delta^2}{c^2}\frac{\partial^2}{\partial t^2}+\varepsilon^i\varepsilon^k\nabla^i\nabla^k-\frac{\mu c}{\hbar}(\varepsilon^i\delta+\delta\varepsilon^i)\nabla^i+\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(\vec{x},t)=0\Rightarrow\\
&\left[-\frac{\delta^2}{c^2}\frac{\partial^2}{\partial t^2}+\varepsilon^i\varepsilon^k\nabla^i\nabla^k-\frac{1}{\hbar}(\varepsilon^i\delta+\delta\varepsilon^i)p_0\nabla^i+\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(\vec{x},t)=0\Rightarrow\\
&\left[\frac{\delta^2}{c^2}\frac{\partial^2}{\partial t^2}-\varepsilon^i\varepsilon^k\nabla^i\nabla^k+\frac{1}{\hbar}(\varepsilon^i\delta+\delta\varepsilon^i)p_0\nabla^i-\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(\vec{x},t)=0\Rightarrow\\
&\left[\frac{\delta^2}{c^2}\frac{\partial^2}{\partial t^2}-\varepsilon^i\varepsilon^k\nabla^i\nabla^k+i(\varepsilon^i\delta+\delta\varepsilon^i)\partial_0\nabla^i-\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(\vec{x},t)=0\Rightarrow\\
&\left[\delta^2\partial_0\partial_0-\varepsilon^i\varepsilon^k\partial_i\partial_k+i(\varepsilon^i\delta+\delta\varepsilon^i)\partial_0\partial_i-\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(\vec{x},t)=0,
\end{aligned}
\tag{84}
$$

where we have used the relations [26]

$$
\partial^\mu=\left(\frac{1}{c}\frac{\partial}{\partial t},-\vec{\nabla}\right),\quad p_0=\mu c,\quad p_0\to i\frac{\hbar}{c}\frac{\partial}{\partial t}=i\hbar\partial_0. \tag{85}
$$

Now, by defining the (matrix) four-vector

$$
\xi^\mu\equiv(\delta,i\vec{\varepsilon}), \tag{86}
$$

the (84) can immediately be written

$$
\left[(\xi^\mu)^2\partial_\mu\partial_\mu+\sum_{\substack{\mu>\nu\\ \mu\neq\nu}}(\xi^\mu\xi^\nu+\xi^\nu\xi^\mu)\partial_\mu\partial_\nu-\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right]\psi_s(x)=0, \tag{87}
$$

that, naturally, is equivalent to

$$
\left(\xi^\mu\xi^\nu\partial_\mu\partial_\nu-\frac{\mu^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x)=0. \tag{88}
$$

[26] It could be objected that, having the tachyons an imaginary energy, the right relation to replace would be $p_0=i\mu c$ and, therefore, the obtained result is wrong. But ultimately, if we make such a reasoning, we must also notice that the correspondence $(p_0)_{tachyon}=\frac{i}{c}E^{bradyon}\to-\frac{\hbar}{c}\frac{\partial}{\partial t}$ is valid and so our result does not change.

The (88) represents the second order M_α equation in the four-dimensional form. Therefore, we have found that the (56) and (57) in the four-dimensional form can be written

$$\left(\chi^\mu \partial_\mu - \frac{\mu c}{\hbar} \mathbb{1}_s\right) \psi_s(x) = 0 \tag{89}$$

$$\left(\xi^\mu \xi^\nu \partial_\mu \partial_\nu - \frac{\mu^2 c^2}{\hbar^2} \mathbb{1}_s\right) \psi_s(x) = 0. \tag{90}$$

These equations, as we already saw, describe particles with imaginary energy (ieps) or, using the old terminology, they describe tachyons. Let us notice χ^μ and ξ^μ satisfy the relations [27]

$$\chi_\mu^\dagger = \chi_0 \chi_\mu \chi_0, \quad (\chi^\mu)^\dagger = \chi^0 \chi^\mu \chi^0, \quad \xi_\mu^\dagger = \xi^\mu, \quad (\xi_\nu \xi_\mu)^\dagger = \xi^\mu \xi^\nu. \tag{91}$$

Now it is useful to make the following observations:

- The first and second order M_α equation, whether in vectorial form or four-dimensional one, are not two equivalent ways to describe tachyons, since the first order M_α equation describes tachyons with asymmetric energetic states, while the second order M_α equation describes tachyons with symmetric energetic states.
- While the second order M_α equation in vectorial form follows from the first order one (always in vectorial form), only if we isolate the spatial derivative and the term not risen from the time one, with the request to take the square norm of the operators (42) rather than their simple square, it is impossible to derive the second order M_α equation in the four-dimensional form by the first order M_α equation in four-dimensional form, and this because in the space-time formalism the time derivative and the spatial one are merged in the four-dimensional derivative ∂_μ.[28]

The first point tells us that only one between the equations (89) and (90) can correctly describe the "tachyons" (more precisely we would say the "ieps") and this depending on such particles have asymmetric or symmetric energetic states. How can we do then to establish what is the correct equation for the ieps? The only way would be developing both formalisms and seeing what of the two is better adapted to the experimental data. Nevertheless, since a tachyon physics does not exist, we have to study both equations. The second point tells us the reason for which the first order M_α in the four-dimensional form is expressed through the matrices χ^μ, while the second order M_α equation in the four-dimensional form is expressed through the matrices ξ^μ, *i.e.* why the (89) and (90) are not expressed through the same matrix four-vectors.

At this point, having reached this stage, we could ask about usefulness of the found equations,

[27] Note that χ^μ and ξ^μ coincide for $s = 0$ only.

[28] The M_α equation to the second order in four-dimensional form can be derived to the first order one for $s = 0$ only, since in this case χ^μ and ξ^μ coincide.

since they do not describe reps but ieps. In the next pages, we will see that the equations (89) and (90) are fundamental for writing the equations concerning the bradyonic particles with arbitrary spin.

Now let us continue the tachyonic universe review, by studying the Lagrangian densities characterizing the equations (89) and (90). For such a purpose, if we define

$$\psi_s^\dagger \chi^0 \equiv \bar{\psi}_s, \tag{92}$$

it is straightforward to see the (89) and (90) are obtained by the Lagrangian densities

$$\mathcal{L}_{M1} = \bar{\psi}_s \chi^\mu \partial_\mu \psi_s - \frac{\mu c}{\hbar} \bar{\psi}_s \psi_s \tag{93}$$

$$\mathcal{L}_{M2} = (\partial_\mu \psi_s^\dagger) \xi^\mu \xi^\nu (\partial_\nu \psi_s) + \frac{\mu^2 c^2}{\hbar^2} \psi_s^\dagger \psi_s. \tag{94}$$

For construction, the Lagrangian density $\mathcal{L}_{M1}$ describes the tachyonic universe (IEP universe) when the energetic spectrum of its particles is asymmetric, while $\mathcal{L}_{M2}$ describes the IEP universe when the energetic spectrum of its constituent particles is symmetric. Thus, it comes that, through $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$, we can identify two quantum field theories by which studying conserved amounts, symmetries, *etc.* It is simple to understand not being tachyons (or better ieps) particles with real energy, namely particles with positive square energy, such studies would be useless, above all if inserted in the context of a ground-breaking work like this. What we will study, instead, through $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$, is the spontaneous symmetry breaking of the tachyonic universe and we will make it because this discussion will incredibly put us in contact with our universe, that is with the universe of the particles having positive square energy (REP universe).[29]

[29] It is important to underline that, if we chose $A^2\psi = B^2\psi$ for finding the M_α equation to the second order, one should obtain bradyonic equations to the first and second order only (such equations will be written out in the section 2.3), without ever knowing the fundamental idea on the great spontaneous symmetry breaking explained in the next chapter.

2 The Big-Break Genesis and New Equations for Elementary Particles

Having elaborated the theory of tachyonic universe, now it is interesting to verify if this admits spontaneous symmetry breaking (SSB). This chapter starts with the analysis of the SSB on the theories $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$. However, our study will not be useless, because it will take us to unexpected cosmological considerations. In fact, it will be possible to assume that for the instability of tachyonic universe, due to its negative square energy (referring to the particles that constitute it), there was a great phase transition, which brought the tachyonic universe *to condense* into a universe with positive square energy (bradyonic universe). Therefore, the theory we have elaborated puts in relation the tachyonic universe with our universe, leaving to mean that really this last one is nothing more than the tachyonic universe condensed as result of a *big* process of spontaneous symmetry breaking. Moreover, we will see the transition from tachyonic to bradyonic universe concurs to transform the tachyonic equations in bradyonic ones, *i.e.* able to describe the elementary particles with arbitrary spin. The first order partial differential equation will take to asymmetric quantum states, while the second order one will take to symmetric quantum states. This could induce an examination on the asymmetry between matter and anti-matter, which is a characteristic of our universe (REP universe). Lastly, about the proposed equations, we will treat their probability densities, general solutions and Lorentz covariance.

2.1 Spontaneous symmetry breaking of the theories $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$

We start by describing the process of spontaneous symmetry breaking of the tachyonic theory characterized by the Lagrangian density (93). From the literature, we know the field connected to the SSB is a scalar field ϕ^4, namely a Klein-Gordon field to which we add a potential term [30]

$$-\lambda|\phi|^4, \tag{95}$$

and this means that, in order to study the SSB of the theory characterized by $\mathcal{L}_{M1}$, we must put us in the case $s=0$ and add in $\mathcal{L}_{M1}$ the potential term $-\lambda(\bar{\psi}_{s=0}\psi_{s=0})^2$, *i.e.*

$$\mathcal{L}_{M1}^{SSB} = \bar{\psi}_{s=0}\chi^{\mu}_{s=0}\partial_\mu\psi_{s=0} - \frac{\mu c}{\hbar}\bar{\psi}_{s=0}\psi_{s=0} - \lambda(\bar{\psi}_{s=0}\psi_{s=0})^2, \tag{96}$$

and, making clear the scalar products, we obtain the two Lagrangian densities (both right) [31]

[30] In this case $\lambda \in \mathbb{R}$.

[31] For simplicity, we put $\psi_{s=0} = \psi$ and $\psi^*_{s=0} = \psi^*$.

$$\mathcal{L}_{M1}^{SSB,1} = \frac{1}{c}\psi^* \frac{\partial \psi}{\partial t} + \frac{\mu c}{\hbar}\psi^*\psi - \lambda(\psi^*\psi)^2 \tag{97}$$

$$\mathcal{L}_{M2}^{SSB,2} = \frac{1}{c}\psi^* \frac{\partial \psi}{\partial t} - \frac{\mu c}{\hbar}\psi^*\psi - \lambda(\psi^*\psi)^2. \tag{98}$$

As it is known, for studying the SSB of our system we have to analyze the potentials of the Lagrangian densities (97) and (98). They are, respectively, given by

$$\begin{cases} V_1(\psi^*\psi) = \lambda(\psi^*\psi)^2 - \frac{\mu c}{\hbar}\psi^*\psi \\ V_2(\psi^*\psi) = \lambda(\psi^*\psi)^2 + \frac{\mu c}{\hbar}\psi^*\psi. \end{cases} \tag{99}$$

Naturally, the study of their sign has to be at the same time, since the (97) and (98) must be both right. Making it, we easily see the system admits SSB, *i.e.* the theory have a maximum in 0 and two relative minimums in $\mu c/2\hbar\lambda$ and $-\mu c/2\hbar\lambda$. As the careful reader already noticed, in contrast to the scalar fields which one studies in literature ($-\lambda|\phi|^4$ + Klein-Gordon field), it is not necessary distinguishing the two cases $\mu > 0$ and $\mu < 0$, since SSB happens for both and this tells us the studied system surely admits SSB and so it is an "unstable system." For having a clearer physical situation as soon as mathematically described, we consider the diagram of the potential V (in this case $V = \lambda|\psi|^4 \mp \frac{\mu c}{\hbar}|\psi|^2$) in the $V - |\psi|^2$ plan drawn in fig. 1. The plot of potential $V(|\psi|^2)$ shows a *depression*, which existing makes the system unstable. As it is straightforward seen, this depression can be put in relation with the typical negative square energy of the IEP universe.[32] In fact, indicated with

$$-\mu^2 c^4 = E_T^2 \tag{100}$$

the square energy of a "tachyonic" particle (varying μ), it is immediate to write the points of $V(|\psi|^2)$ correspondent to 0 and $-\mu^2c^2/4\hbar^2\lambda$, according to this negative energy. We have

$$0 = -\frac{(\mu = 0)^2 c^4}{4\hbar^2 c^2 \lambda} = \frac{(E_T^2)_0}{4\hbar^2 c^2 \lambda}; \quad -\frac{\mu^2 c^2}{4\hbar^2 \lambda} = -\frac{\mu^2 c^4}{4\hbar^2 c^2 \lambda} = \frac{E_T^2}{4\hbar^2 c^2 \lambda}, \tag{101}$$

i.e. the *depression* of $V(|\psi|^2)$, which corresponds to the instability of the system, occurs for those potential values such that

$$V(|\psi|^2) \propto E_T^2. \tag{102}$$

[32] In particular, it is the rest energy of a "tachyon."

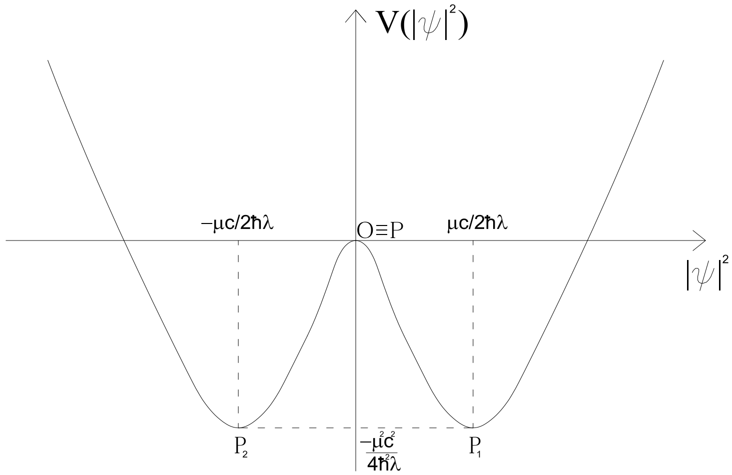

Figure 1: The potential of $\mathcal{L}_{M1}^{SSB}$ shows a typical form to "sombrero," which characterizes the unstable systems.

Thanks to this result, we can suppose the SSB is not only due to the fact the vacuum of the system is into a state different from zero, but also to the circumstance that the system is characterized by a negative square energy. Going beyond, one can think the system is into a vacuum state different from zero just because the system has a negative square energy. We will see this vision will lead to important results which will reveal the depth of the SSB process, that should not be interpreted as a property of the some physical systems only, but rather as a "universal characteristic" of the elementary particle physics, *i.e.* of our universe.

In order to try further indications, we want now to study the SSB of the theory characterized by the Lagrangian density

$$\mathcal{L}_{M2} = (\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu (\partial_\nu \psi_s) + \frac{\mu^2 c^2}{\hbar^2}\psi_s^\dagger \psi_s. \tag{103}$$

As we have previously seen, for obtaining a coherent SSB model, we must put us in the case $s = 0$ and add to the Lagrangian density, thus obtained, the term $-\lambda(\psi^*\psi)^2$. Making it, we have

$$\mathcal{L}_{M2}^{SSB} = (\partial_\mu \psi^*)\xi^\mu \xi^\nu (\partial_\nu \psi) + \frac{\mu^2 c^2}{\hbar^2}(\psi^*\psi) - \lambda(\psi^*\psi)^2, \tag{104}$$

by which, making clear all the terms, we get

$$\mathcal{L}_{M2}^{SSB} = \frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + \frac{\mu^2 c^2}{\hbar^2}(\psi^*\psi) - \lambda(\psi^*\psi)^2. \tag{105}$$

We note that, unlike the (96), the (105) is only one, and, therefore, we have the only potential

$$V(\psi^*\psi) = -\frac{\mu^2 c^2}{\hbar^2}(\psi^*\psi) + \lambda(\psi^*\psi)^2. \tag{106}$$

By studying the sign of this potential, we can distinguish the two cases

1. $\mu^2 > 0$:
   The system has maximum in 0 and minimum in $\mu^2 c^2/2\hbar^2\lambda$ and so it shows SSB.

2. $\mu^2 < 0$:
   The system has maximum in $\mu^2 c^2/2\hbar^2\lambda$ and minimum in 0 and so it does not show SSB.

If we trace the diagram of the potential (106) into $V(|\psi|) - |\psi|$ plan, we have the plot of fig. 2.

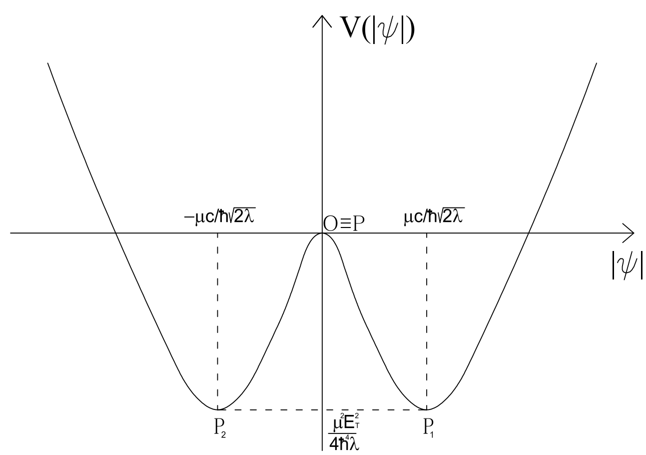

Figure 2: The potential of $\mathcal{L}_{M2}^{SSB}$ in the case $\mu^2 > 0$, for which the system is unstable and admits spontaneous symmetry breaking (SSB).

Therefore, also for the theory characterized by the (103), we have a *depression* of potential V in correspondence of the energetic interval

$$\left[E_T^2, 0\right], \tag{107}$$

that is for negative square energies ("tachyonic" energies). Therefore, also in such a case we can believe the instability of our system depends on the (negative) square energy of the particles constituting it. So, we found for our theory an already famous result, consisting in the fact that negative square energy of a system causes instability. By the diagrams of potentials, we can see indicatively one arrives to a stability condition (minimum in 0 and disappearance of *depression*), passing from negative square energies to positive ones

$$E_T^2 \rightarrow E_B^2, \tag{108}$$

where with E_B^2 we indicated the positive square energy of the particles, which we call bradyons (they are the particles of our universe, before already called reps). Saying μ the mass of a generic tachyon and m the mass of a generic bradyon,[33] we can thus rewrite the above correspondence [34]

$$-\mu^2 c^4 \rightarrow m^2 c^4, \tag{109}$$

which is equivalent to

$$-\mu^2 \rightarrow m^2, \tag{110}$$

i.e., extracting the square root [35]

$$\pm i\mu \rightarrow \pm m, \tag{111}$$

we have the only correspondence [36]

$$i\mu \rightarrow m, \tag{112}$$

that can also be written

[33] From what we earlier said, we do not mean with tachyon a superluminal particle and with bradyon a sub-luminal particle, but a particle with negative square energy (iep) and a particle with positive square energy (rep), respectively. This is their difference, because both have speeds lower or like c within our theory.

[34] Of course $\mu, m \in \mathbb{R}$.

[35] The $+$ sign is for particles, while the $-$ sign is for anti-particles.

[36] In fact, if we consider particles or anti-particles, the result does not change.

$$\mu \to -im. \tag{113}$$

The (113) is the correspondence which has to be done on our systems (the one described by $\mathcal{L}_{M1}^{SSB}$ and the one described by $\mathcal{L}_{M2}^{SSB}$) for making them stable, namely for eliminating the SSB. Naturally, such an argument is provisional and serves only to understand what happens at the system when it places itself in a stability condition. Formally, we know that the system, in order to put itself in a stability condition, will have to shift in a state whose minimum is zero. Therefore, also for the theories described by $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$, the SSB can be studied in the same way and with analogous results to those well known. In particular, it can be demonstrated that a translation of the system to a vacuum state equal to zero materializes massive and massless fields, thus as the Goldstone theorem prescribes [41, 42]. Moreover, rendering the (gauge) global symmetry a local one, *i.e.* making the "minimal coupling" in $\mathcal{L}_{M1}^{SSB}$ and $\mathcal{L}_{M2}^{SSB}$, it can be observed the Goldstone fields are coupled to the gauge fields becoming massive, thus arising the famous Higgs mechanism, fundamental for the Glashow-Weinberg-Salam electroweak theory. What is the massive fields sign? It can be demonstrated the massive fields have opposite sign towards the one of the particle fields before the transition to a state of minimum (vacuum) equal to zero (notice that the number of such fields will always depend on the group of symmetry we consider).

At this point we ask ourselves two questions

- Does the opposite sign of the massive fields, deriving from the translation of the system to a state of minimum (vacuum) equal to zero, depend effectively on a passage at positive square energy for our particles?
- What happens to the theories described from $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$ when we make the substitution $\mu \to -im$?

If, as supposed, the second question gives us the answer: "*we obtain theories describing exclusively particles with positive square energy (namely bradyons or reps)*," we will have that a relationship between SSB and bradyonic universe exists, *i.e.* the SSB can be seen like a fundamental principle for the creation of our universe, beginning from an unstable universe as the tachyonic one. Therefore, based on this reasoning, the spontaneous symmetry breaking (SSB) could not have been a phenomenon regarding the scalar fields ($s = 0$) only, but also all the other fields (arbitrary spin) staying in the tachyonic universe.

2.2 From tachyonic (IEP) to bradyonic (REP) universe

If we apply the correspondence $\mu \to -im$ in the equations of motion characterizing the tachyonic universe, *i.e.* into first and second order M_α equation given by [37]

[37]Remember that each of these equations identifies a specific tachyonic theory: one with asymmetric energetic states and other one with symmetric energetic states, for every s.

$$\left(\chi^\mu \partial_\mu - \frac{\mu c}{\hbar}\mathbb{1}_s\right)\psi_s(x) = 0 \tag{114}$$

$$\left(\xi^\mu \xi^\nu \partial_\mu \partial_\nu - \frac{\mu^2 c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0, \tag{115}$$

we obtain the equations [38]

$$\left(i\hbar\chi^\mu \partial_\mu - mc\mathbb{1}_s\right)\psi_s(x) = 0 \tag{116}$$

$$\left(\xi^\mu \xi^\nu \partial_\mu \partial_\nu + \frac{m^2 c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0, \tag{117}$$

from which we note

- The equation (116) is identical to the four-dimensional Dirac equation, apart from the substitutions [39]

$$\begin{aligned} \psi_s(x) &\to \psi_{s=3/2}(x) \\ \chi^\mu &\to \gamma^\mu \\ \mathbb{1}_s &\to \mathbb{1}_{s=3/2}. \end{aligned}$$

- The equation (117) is identical to the four-dimensional Klein-Gordon equation, apart from the substitutions

$$\begin{aligned} \psi_s(x) &\to \psi_{s=0}(x) \\ \xi^\mu \xi^\nu \partial_\mu \partial_\nu &\to \partial_\mu \partial^\mu \equiv \Box \\ \mathbb{1}_s &\to 1. \end{aligned}$$

[38] Such equations, for $s = 0$, become

$$\left(i\hbar\frac{\partial}{\partial t} \mp mc^2\right)\psi_{s=0}(x) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + \frac{m^2 c^4}{\hbar^2}\right)\psi_{s=0}(x) = 0,$$

where, for both, we have $\delta_0 = \pm 1$.

[39] Note that we have a four-dimensional field $\psi(x)$ and a (4×4) unit matrix $\mathbb{1}$, if we consider the tensor product of two fields with spin $1/2$ too. However, we will treat this argument in the next chapter.

With the promise of making further considerations in the next pages, we want now to estimate the energetic spectrum concerning the equations (116) and (117), respectively. For this purpose, by replacing in our equations the generic plane-wave solution

$$\psi_s(x) = e^{-ik\cdot x} u_s(x), \tag{118}$$

and putting us in the classical limit, we obtain, for the (116), the relation

$$E\mathbb{1}_s = mc^2\delta, \tag{119}$$

from which, by taking the representation $\delta_{\pm 1}$, we have the following energetic spectrum

for $s = 0$ $$E = \pm mc^2 \Rightarrow E_1^{s=0} = mc^2,\ \ E_2^{s=0} = -mc^2. \tag{120}$$

for $s = \frac{1}{2}$ $$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \Rightarrow E_1^{s=1/2} = mc^2,\ \ E_2^{s=1/2} = -mc^2. \tag{121}$$

for $s = 1$ $$\begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix} = \begin{pmatrix} mc^2 & 0 & 0 \\ 0 & -mc^2 & 0 \\ 0 & 0 & mc^2 \end{pmatrix} \Rightarrow E_1^{s=1} = mc^2,\ \ E_2^{s=1} = -mc^2,\ \ E_3^{s=1} = mc^2. \tag{122}$$

.... sequence of mc^2 according to the matrices of $\delta_{\pm 1}$.

Instead, for the representation $\delta_{\mp 1}$, we have

for $s = 0$ $$E_1^{s=0} = -mc^2,\ \ E_2^{s=0} = mc^2. \tag{123}$$

for $s = \frac{1}{2}$ $$E_1^{s=1/2} = -mc^2, \;\; E_2^{s=1/2} = mc^2. \tag{124}$$

for $s = 1$ $$E_1^{s=1} = -mc^2, \;\; E_2^{s=1} = mc^2, \;\; E_3^{s=1} = -mc^2. \tag{125}$$

....sequence of mc^2 according to the matrices of $\delta_{\mp 1}$.

With the same argumentation, referring now to the equation (117) in the classical limit, we obtain the relation [40]

$$E^2 \mathbb{1}_s = m^2 c^4 \mathbb{1}_s, \tag{126}$$

from which, it follows the energetic spectrum

for $s = 0$ $$mc^2, -mc^2. \tag{127}$$

for $s = \frac{1}{2}$ $$mc^2, -mc^2, mc^2, -mc^2. \tag{128}$$

[40] In such a case, it is not important if we are in the representation $\delta_{\pm 1}$ or $\delta_{\mp 1}$.

for $s = 1$ $\qquad mc^2, -mc^2, mc^2, -mc^2, mc^2, -mc^2.$

$$ \tag{129} $$

$\ldots\ldots\ldots\ldots (2s+1)$ solutions with positive energy and $(2s+1)$ solutions with negative energy.

Based on the found result, we can assert the equation obtained from the (114) through the correspondence $\mu \to -im$ describes particles with real energy (positive and negative) only, *i.e.* describes bradyons (or reps). It is important to remark that the energetic states with $s = 0$ and s half-integer are symmetric, while those with s integer are asymmetric, displaced from the part of positive and negative energy depending on the choice of representation $\delta_{\pm 1}$ or $\delta_{\mp 1}$. Instead, regarding the equation obtained from the (115), always thanks to correspondence $\mu \to -im$, it also describes bradyons with positive and negative real energy, but, unlike the (116), the energetic states of such particles are perfectly symmetric for any value of s.

Therefore, we proved that the equations deriving from the (114) and (115), through the correspondence $\mu \to -im$, describe particles with positive square energy, that is the particles of our universe. The important thing which we must underline is that the substitution $\mu \to -im$ has not been operated in an arbitrary way, but it is gushed from a reasoning about the instability of the tachyonic system earlier to the SSB process. In particular, we are now in a position to respond to the second question we made in the former pages, with regard to the result which should have to produce the already cited correspondence on the tachyonic theories described by $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$. Well, as we just predicted, the answer is that we get theories exclusively describing particles with positive square energy. Then, having ascertained the correspondence $\mu \to -im$ changes the sign of the mass term of equations (114) and (115) and having proved the change of such a sign generates a passage from particles with negative square energy to particles with positive square energy, we can also answer the first question and assert that the translation of the system to a state of minimum (vacuum) equal to zero, involving a change of sign in the mass term, leads to particles with positive square energy. It goes without saying that this observation implies the subsistence of a relationship between tachyonic SSB and bradyonic universe, in the sense that is right to think the two bradyonic equations (116) and (117) really derive from the two tachyonic ones (114) and (115), respectively (obviously, as we will explain later, only one of these equations could be the good equation for our elementary particles), through a *big* process of spontaneous symmetry breaking. If this is correct, the correspondence $\mu \to -im$ practically expresses the "reconversion" or the "tachyon condensation" [11, 12, 13], expression of the "phase transition" of the system from a condition of instability (IEP) to a stability one (REP). This drives us to guess the cosmological scenario sketched in fig. 3. According to which, after Big-Bang (the initial singularity), there was a distribution of matter characterized by the negative value of the square energy of any constituent particle (tachyonic or IEP universe). That is why this distribution of matter, revealing itself extremely unstable, received a great phase transition (SSB + tachyonic condensation), which we can call "Big-Break," that took it to a stable condition, characterized by a positive square energy for any constituent particle (bradyonic or REP universe). The just described picture is extremely fascinating and represents an exhaustive logical-deductive background which addressing the experimental and theoretical researches to. However, for now it is only a hypothesis, based on what emerges from the theory exposed in these pages. In particular, we understand that, in order to confirm such a conjecture from the merely theoretical point of view, it is absolutely necessary to generalize the SSB concept. In fact, while on the basis of the current model the spontaneous symmetry breaking is a peculiarity of the scalar fields with potential $|\phi|^4$ only, the vision proposed in these pages predicts a *great* spontaneous symmetry breaking (Big-Break), that touched all the tachyonic particles, which condensed in bradyonic particles (otherwise the correspondence $\mu \to -im$ would not lead to equations with positive energy solutions). At present, a Big-Break model, in the terms that we described, does not exist, but if one of the two proposed equations will be revealed valid for the explanation of experimental data, it will be necessary to construct such a model. Of course, joined to the definition of a wider group of gauge transformations, it could lead to the effective unification of the four physics fundamental forces (gravitational, electromagnetic, strong, weak), going complete the work begun by Glashow-Weinberg-Salam.

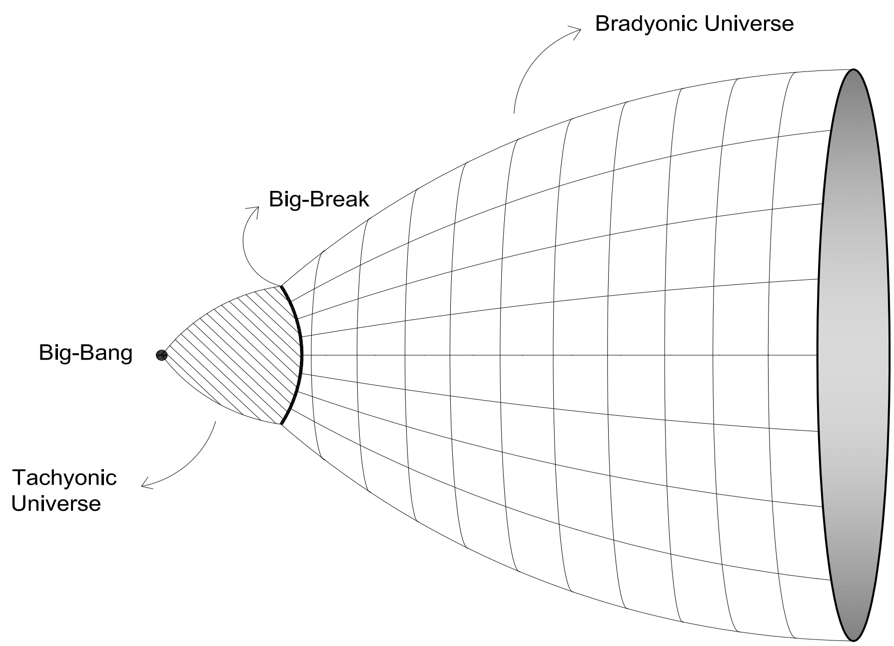

Figure 3: The α-Theory scenario. The Big-Break event is fundamental for the birth of the bradyonic universe.

So just to finalize, we can see that the correspondence

$$\mu \to -im,$$

which previously has been deducted through considerations concerning the transition from a negative square energy to a positive one of a system subject to SSB, can be considered in SR too. In fact, as we have seen in the section 1.3, thanks to the TP there are two equivalent ways for describing the energy of the bradyons and tachyons, depending on whether it is assumed these last have speed v (in module) smaller or higher than the speed of light c. They are [41]

1. $E_B = \frac{mc^2}{(1-\frac{v^2}{c^2})^{1/2}}$; $E_T = \frac{\mu c^2}{(\frac{v^2}{c^2}-1)^{1/2}}$, if $v_t > c$.

2. $E_B = \frac{mc^2}{(1-\frac{v^2}{c^2})^{1/2}}$; $E_T = \frac{i\mu c^2}{(1-\frac{v^2}{c^2})^{1/2}}$, if $v_t < c$.

where, with E_B, we indicated the energy of the generic bradyon and, with E_T, the energy of the generic tachyon. It is straightforward to see that the substitution

$$m \to i\mu$$

and its *inverse* [42]

$$\mu \to -im,$$

allow in 2 – under the hypothesis that the tachyonic speed is smaller in module than c – to pass from E_B to E_T and vice-versa, *i.e.* of being able to pass from the bradyonic universe to the tachyonic one and from the tachyonic universe to the bradyonic one without problems, letting mean these two universes are interchangeable. Therefore, the tachyonic paradox, arisen from the theory exposed in this work, allowed to give us an alternative vision on the tachyonic universe within the SR. Such a viewpoint is in line with the principle of invariant light speed, since it considers the tachyons not as superluminal particles, but like those having speed smaller than c and $E^2 < 0$ (ieps). However, according to this new framework on the special Relativity, the tachyonic universe (IEP universe) and the bradyonic one (REP universe) are interchangeable, because through the substitutions $m \to i\mu$ and $\mu \to -im$, one can go by one to the other, and vice-versa. Hence, the imaginary energy, within the SR, has not a physical meaning, but it seems to be a mathematical characteristic having the only goal to avoid that the particles of the (absolute) elsewhere are able to have $v > c$. Instead, the theory exposed in this work shows that this characteristic leads to instability, and so it has a specific physical implication. This allows to find a well-known result, consisting to think the tachyons are not particles with $v > c$, but strongly unstable particles with imaginary energy $[13, 34, 35, 41 - 45]$.

But there is more because – joined to the hypothesis that the SSB was a global physical phenomenon for tachyonic particles – our theory gives to the correspondence $\mu \to -im$ a primary meaning, establishing practically an *arrow of conversion* between the tachyonic universe and the

[41] Note that v_t is the velocity of the generic tachyon.

[42] Remember that such correspondences are with double sign in SR. We choose here only one, for confronting them with the correspondence $\mu \to -im$ arisen from our model.

bradyonic one, allowing the first one to transform itself in the second, but the second one to not transform itself in the first, namely it establishes the irreversibility of the correspondence $\mu \to -im$ ($m \nrightarrow i\mu$).[43] Therefore, into the SR the imaginary energy is a fictitious effect, while for our theory it is a real effect, caused by the instability of the tachyonic universe (IEP universe). It is because of this instability that such a universe condensed in a stable universe having real energy (REP universe). This suggests us that the theoretical model we are constructing exceeds and improves the special Relativity background.

2.3 Quantum states and asymmetric universe

In the last section, we proved that the equations

$$(i\hbar\chi^{\mu}\partial_{\mu} - mc\mathbb{1}_s)\,\psi_s(x) = 0 \tag{130}$$

$$\left(\xi^{\mu}\xi^{\nu}\partial_{\mu}\partial_{\nu} + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0 \tag{131}$$

can be applied to our universe, and they give us, respectively, solutions with positive and negative energy distributed in asymmetric and symmetric way. The energetic spectrum of such solutions has been computed in the classical limit, namely in the inertial frame in which the particles taken under investigation are practically at rest (for this it is usually called "rest frame"). We understand this condition describes the elementary states of a quantum particle, *i.e.* describes the stationary quantum states. We can, therefore, claim that the energetic analysis made on equations (130) and (131) established that the first one admits symmetric quantum states for particles with spin 0 or half-integer, and asymmetric quantum states for particles with integer spin,[44] while the second one admits symmetric quantum states for any spin value (democratic equation). We also know our universe is asymmetric, because the matter composing it is overall higher than the anti-matter. Does a relationship exist between the quantum states of equations (130) and (131) and the asymmetry of our universe? It could apparently be asserted such an asymmetry is due to the fact that the elementary particles are described just from equation (130) in the representation $\delta_{\pm 1}$, *i.e.* the asymmetry of the universe is due to the fact that the equation describing elementary particles admits an asymmetry in the fundamental quantum states. This explanation, although completely practicable, does not distinguish between micro and macro, simply thinking the cosmological effects are a direct result of the infinitesimally small world processes. But the opposite could be true, and so the asymmetry of the universe could be the product of collective events on large-scale, such as the Big-Break, whose formal description must be done as soon as possible. Then, we understand that, in order to answer these questions, and for finding many others, it is necessary to establish what

[43] It is well to underline that our theory, unlike the SR, fixes the sign on the correspondence $\mu \to -im$ too, *i.e.* there is not the possibility of double sign existing in the special Relativity.

[44] These states are shifted from the part of the solutions of positive or negative energy, depending on whether one chooses the representation $\delta_{\pm 1}$ or $\delta_{\mp 1}$: obviously such an option depends from the Nature.

between equations (130) and (131) correctly describes the elementary particles. Therefore, we have to understand what between the two found equations has a better predictive power from theoretical and experimental point of view. For this reason, in the next, we will limit ourselves in describing the properties of both equations, with the aim to facilitate later studies on their physical character. One of the things to be well emphasized is that, while the equation

$$(i\hbar\chi^{\mu}\partial_{\mu} - mc\mathbb{1}_s)\,\psi_s(x) = 0,$$

in principle, can also be obtained without calling for the tachyon condensation, *i.e.* making less of correspondence $\mu \to -im$, the equation

$$\left(\xi^{\mu}\xi^{\nu}\partial_{\mu}\partial_{\nu} + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0$$

does not exist without the substitution $\mu \to -im$, since it derives, necessarily, from a global process of spontaneous symmetry breaking (so good for any s), that leads to the tachyon condensation. In fact, it is possible to demonstrate that, considering the equation (30) (and not the (31)), where there is the square of the differential operators A and B, one arrives to the following vectorial equations[45]

$$i\hbar\frac{\partial\psi_s}{\partial t} + i\hbar c\varepsilon_i\partial_i\psi_s = mc^2\delta\psi_s(\vec{x},t) \tag{132}$$

$$\hbar^2\frac{\partial^2\psi_s}{\partial t^2} = \hbar^2c^2\varepsilon_i\varepsilon_k\frac{\partial^2\psi_s}{\partial x^i\partial x^k} + i\hbar mc^3(\varepsilon_i\delta + \delta\varepsilon_i)\frac{\partial\psi_s}{\partial x^i} - m^2c^4\psi_s(\vec{x},t), \tag{133}$$

of which the (132) can be put in the four-dimensional form and it coincides with the (130), while the (133) cannot be put in four-dimensional form and it is, however, different from the (131) in vectorial form which is

$$\hbar^2\frac{\partial^2\psi_s}{\partial t^2} = \hbar^2c^2\varepsilon_i\varepsilon_k\frac{\partial^2\psi_s}{\partial x^i\partial x^k} - \hbar mc^3(\varepsilon_i\delta + \delta\varepsilon_i)\frac{\partial\psi_s}{\partial x^i} - m^2c^4\psi_s(\vec{x},t). \tag{134}$$

This makes us understand that, if we will prove the equation (130) is the more appropriate for describing the elementary particles, we will have the *big* process of spontaneous symmetry breaking – which gives birth to the tachyon condensation – could not find full justification, while, if the (131) will attest of being the right equation for the elementary particles with arbitrary spin, this will be a sure test our universe was born from an universe with negative square energy after a Big-Break. Therefore, it will be fundamental, in future, to understand if and what between the (130) and the (131) is able to describe the quantum physics.

Before continuing with the study of these two equations, we call the (130) "Asymmetric α-

[45]In this case, we simply place $\tilde{\delta} = \delta$, without specifying if it is $\delta_{\pm 1}$ or $\delta_{\mp 1}$.

Equation" (AαE) and the (131) "Symmetric α-Equation" (SαE). Since these equations are characterized by different wave functions, one prefers in the next to indicate, with ψ_s, the wave function of the AαE and, with ψ_s, the wave function of the SαE.

2.4 Probability densities of the AαE and SαE

In this section, we want to calculate the probability densities of the asymmetric and symmetric α-equation. We begin from the first of them, that in vector components can be written

$$i\hbar\frac{\partial\psi_s}{\partial t} = -i\hbar c\varepsilon_i\partial_i\psi_s + mc^2\delta\psi_s. \tag{135}$$

The hermitian conjugate of such an equation is

$$-i\hbar\frac{\partial\psi_s^\dagger}{\partial t} = i\hbar c\partial_i\psi_s^\dagger\varepsilon_i + mc^2\psi_s^\dagger\delta, \tag{136}$$

where it has been taken account that $\varepsilon_i^\dagger = \varepsilon_i$, $\delta^\dagger = \delta$. Now, if we multiply the (135) on the left-side by $\psi_s^\dagger$ and the (136) to right-side by ψ_s, we obtain the two expressions

$$i\hbar\psi_s^\dagger\frac{\partial\psi_s}{\partial t} = -i\hbar c\psi_s^\dagger\varepsilon_i\partial_i\psi_s + mc^2\psi_s^\dagger\delta\psi_s \tag{137}$$

$$-i\hbar\frac{\partial\psi_s^\dagger}{\partial t}\psi_s = i\hbar c\partial_i\psi_s^\dagger\varepsilon_i\psi_s + mc^2\psi_s^\dagger\delta\psi_s, \tag{138}$$

from which, by considering (137) – (138), we have

$$(\psi_s^\dagger\frac{\partial\psi_s}{\partial t} + \frac{\partial\psi_s^\dagger}{\partial t}\psi_s) = -c(\psi_s^\dagger\varepsilon_i\partial_i\psi_s + \partial_i\psi_s^\dagger\varepsilon_i\psi_s), \tag{139}$$

which is not other but a continuity equation of the type

$$\frac{\partial\rho_{A\alpha}}{\partial t} + \vec{\nabla}\cdot\vec{j}_{A\alpha} = 0, \tag{140}$$

where it is placed

$$\begin{cases} \rho_{A\alpha}(\vec{x},t) \equiv \psi_s^\dagger \psi_s \equiv |\psi_s|^2 \\ \vec{j}_{A\alpha}(\vec{x},t) \equiv c\psi_s^\dagger \vec{\varepsilon} \psi_s. \end{cases} \tag{141}$$

If $\psi_s(\vec{x},t)$ is the wave function associated to a generic particle with spin s, we can identify $\rho_{A\alpha}(\vec{x},t)$ and $\vec{j}_{A\alpha}(\vec{x},t)$, respectively, like the probability density and the probability current density associated to AαE. From this, it is immediate to realize that the $\rho_{A\alpha}(\vec{x},t)$ is positive-definite just like it happens for the Dirac equation. On reflection, the $\rho_{A\alpha}(\vec{x},t)$ and $\vec{j}_{A\alpha}(\vec{x},t)$ are identical to those of the Dirac equation, on condition the Dirac wave function is replaced with $\psi_s(\vec{x},t)$ and the matrices ε_i with the matrices α_i. This is an ulterior proof of the fact the AαE can be seen like the generalization of the Dirac equation.

By proceeding to the same way for the symmetric α-equation, we obtain the following continuity equation

$$\frac{\partial \rho_{S\alpha}}{\partial t} + \vec{\nabla} \cdot \vec{j}_{S\alpha} = 0, \tag{142}$$

where we have

$$\begin{cases} \rho_{S\alpha}(\vec{x},t) \equiv (\psi_s^\dagger \dot{\psi}_s - \dot{\psi}_s^\dagger \psi_s) \\ \vec{j}_{S\alpha}(\vec{x},t) \equiv c^2 \left(\frac{\partial \psi_s^\dagger}{\partial x^k} \varepsilon_k \vec{\varepsilon} \psi_s - \psi_s^\dagger \vec{\varepsilon} \varepsilon_k \frac{\partial \psi_s}{\partial x^k} \right) + \frac{mc^3}{\hbar} \psi_s^\dagger \left(\delta \varepsilon_i + \varepsilon_i \delta \right) \psi_s. \end{cases} \tag{143}$$

This means – like the expression of $\rho_{S\alpha}(\vec{x},t)$ suggests – that the SαE has a not positive-definite probability density. Therefore, as the Klein-Gordon equation, whose SαE can be considered like a generalization, also the symmetric α-equation suffers of a not positive-definite probability density. At first sight, this could let us discard such an equation to advantage of AαE. Really, for the SαE and AαE, the same considerations made for the Klein-Gordon and Dirac equations are worth, *i.e.* the densities $\rho_{A\alpha}(\vec{x},t)$ and $\rho_{S\alpha}(\vec{x},t)$ have not to be interpreted like probability densities, but rather as charge densities. This can be made only if AαE and SαE do not represent equations for single-particle, but field equations. This interpretation is further justified, for both the AαE and SαE, by the study of the spectrum solutions made in section 2.2, already showing these equations, for any s, introduce more solutions, and therefore, necessarily, $\psi_s(\vec{x},t)$ and $\psi_s(\vec{x},t)$ cannot be seen like single-particle wave functions, but, more properly, as "field functions."

2.5 Lorentz covariance of the AαE and SαE

We want now to prove the relativistic covariance of the AαE and SαE. To be precise, we want to establish the conditions which such equations must satisfy for having the same form in any inertial frame of reference and that, based on the principle of relativity, are joined to the validity of the following two requests

- A law must exist according to which, given the $\psi_s(x)$ (or $\psi_s(x)$) in an inertial reference frame S, an observer placed into another inertial frame S' must be in a position to calculate the function $\psi'_s(x')$ (or $\psi'_s(x')$), which describes in S' the physical state correspondent to the $\psi_s(x)$ (or $\psi_s(x)$) in S.

- In agreement with the principle of relativity, the $\psi'_s(x')$ (or $\psi'_s(x')$), namely the particle field we want to study in S', must be solution of an equation which in the inertial frame S' has the same form of AαE (or SαE), *i.e.*

$$\left(i\hbar\tilde{\chi}^{\mu}\partial'_{\mu} - mc\mathbb{1}_s\right)\psi'_s(x') = 0$$

$$\left[\text{ or } \left(\tilde{\xi}^{\mu}\tilde{\xi}^{\nu}\partial'_{\mu}\partial'_{\nu} + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi'_s(x') = 0 \right],$$

  where, in general terms, the matrices $\tilde{\chi}^{\mu}$ (or $\tilde{\xi}^{\mu}$) are not said to be identical to the matrices χ^{μ} (or ξ^{μ}) defined in S, but they must have the same properties, *i.e.* must be

$$(\tilde{\chi}^{\mu})^{\dagger} = \tilde{\chi}^{0}\tilde{\chi}^{\mu}\tilde{\chi}^{0} \quad \left(\text{or} \quad \tilde{\xi}^{\dagger}_{\mu} = \tilde{\xi}^{\mu}\right). \tag{144}$$

For this reason, we understand that, in order to find the conditions for which the AαE and SαE are covariant quantities, we must search those conditions for which the (144) are equivalent to the (91). First of all, we notice that, being the matrices χ^{μ} and ξ^{μ} independent from $x \in M$ (where M is the Minkowski space), we are free to think they do not change from an inertial frame to another, if not for a unitary transformation U operated between S and S' so that

$$\tilde{\chi}^{\mu} = U\chi^{\mu}U^{-1} = U\chi^{\mu}U^{\dagger} \tag{145}$$

$$\tilde{\xi}^{\mu} = U\xi^{\mu}U^{-1} = U\xi^{\mu}U^{\dagger}. \tag{146}$$

Therefore, apart from a unitary transformation, the sets of matrices $\tilde{\chi}^{\mu}$ and $\tilde{\xi}^{\mu}$ are equal to the sets of matrices χ^{μ} and ξ^{μ} and, therefore, the above equations can be so written

$$\left(i\hbar\chi^{\mu}\partial'_{\mu} - mc\mathbb{1}_s\right)\psi'_s(x') = 0 \tag{147}$$

$$\left(\xi^{\mu}\xi^{\nu}\partial'_{\mu}\partial'_{\nu} + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi'_s(x') = 0. \tag{148}$$

In order to find the conditions allowing such equations to coincide with those in S, so that AαE and SαE are relativistically invariant quantities, one must proceed in the same way made with the Dirac equation. Hence, it is useless to repeat all the considerations and calculations we can find in any physics textbook. For this reason, we directly write these conditions, which for the (147) are

$$\begin{cases} \psi'_s(x') = S(\Lambda)\psi_s(x) \\ S(\Lambda^{-1}) = S^{-1}(\Lambda) \\ \Lambda^\mu_\nu \chi^\nu = S^{-1}(\Lambda)\chi^\mu S(\Lambda), \end{cases} \tag{149}$$

while for the (148) are [46]

$$\begin{cases} \psi'_s(x') = \tilde{S}(\Lambda)\psi_s(x) \\ \tilde{S}(\Lambda^{-1}) = \tilde{S}^{-1}(\Lambda) \\ \eta^{\mu\nu}\Lambda^\alpha_\mu \Lambda^\beta_\nu = \tilde{S}^{-1}(\Lambda)\eta^{\alpha\beta}\tilde{S}(\Lambda), \end{cases} \tag{150}$$

where, with $S(\Lambda)$ and $\tilde{S}(\Lambda)$, we indicated two generic representations of the Lorentz group $L^\uparrow_+$. From the Group Theory, it is known such a Lie group is not compact, and this makes the study of its representations problematic. However, this problem resolves by the fact that the Lie algebra of the group $L^\uparrow_+$ is isomorphic to the Lie algebra of the compact group $SO(4)$, and, therefore, the representations of $L^\uparrow_+$ are equivalent to the representations of $SO(4)$. The representations of $SO(4)$ are characterized by the couple of indices (l, m), each of which characterizes the representations of the group $SU(2)$, on strength of the isomorphism

$$LieSO(4) \sim Lie\left[SU(2) \otimes SU(2)\right].$$

The generic element of the representation of $SO(4)$ can be indicated with [47]

$$D_{l,m}(R),$$

from that, as soon as asserted, it is understood that $D_{l,m}(R)$ characterizes a matrix of dimension $(2l+1)(2m+1)$. Thence, it quickly follows that the representation $(0,0)$ gives a scalar, the inequivalent representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ give two-dimensional matrices, the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ gives a four-dimensional matrix and so on. Since the representations of $L^\uparrow_+$ and $SO(4)$ are equivalent, we have [48]

[46] We place $\eta^{\sigma\rho} \equiv \xi^\sigma \xi^\rho$.

[47] $R \in SO(4)$.

[48] $\Lambda \in L^\uparrow_+$.

$$S(\Lambda) = D_{l,m}(R) = D_{l,m}(\Lambda) \tag{151}$$

$$\tilde{S}(\Lambda) = \tilde{D}_{l,m}(R) = \tilde{D}_{l,m}(\Lambda), \tag{152}$$

where, in the last step, we have replaced R with Λ for obvious reasons. We notice, in particular, bearing in mind what has been said so far, that the transformed fields concerning the equations (147) and (148) are

$$\psi'_s(x') = S(\Lambda)\psi_s(x) = D_{l,m}(\Lambda)\psi_s(x) \tag{153}$$

$$\psi'_s(x') = \tilde{S}(\Lambda)\psi_s(x) = \tilde{D}_{l,m}(\Lambda)\psi_s(x). \tag{154}$$

For the respect of the matrix product, $D_{l,m}(\Lambda)$ and $\tilde{D}_{l,m}(\Lambda)$ have to be square matrices of dimension

$$(2l+1)(2m+1) = 2s+1.$$

It can be demonstrated the explicit form of $S(\Lambda)$ and $\tilde{S}(\Lambda)$ is

$$S(\Lambda) = D_{l,m}(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}(\mathcal{M}^{\mu\nu})^{l,m}} \tag{155}$$

$$\tilde{S}(\Lambda) = \tilde{D}_{l,m}(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{l,m}}, \tag{156}$$

where the $\omega_{\mu\nu}$ are anti-symmetric coefficients, while $(\mathcal{M}^{\mu\nu})^{l,m}$ and $(\mathcal{J}^{\mu\nu})^{l,m}$ are matrices of dimension $(2l+1)(2m+1)$, which represent the generators of $L_+^\uparrow$. The Lie algebra of these generators is

$$\left[(\mathcal{M}^{\lambda\tau})^{l,m}, (\mathcal{M}^{\rho\sigma})^{l,m}\right] = -i\left[g^{\lambda\sigma}(\mathcal{M}^{\tau\rho})^{l,m} + g^{\tau\rho}(\mathcal{M}^{\lambda\sigma})^{l,m} - g^{\lambda\rho}(\mathcal{M}^{\tau\sigma})^{l,m} - g^{\tau\sigma}(\mathcal{M}^{\lambda\rho})^{l,m}\right] \tag{157}$$

$$\left[(\mathcal{J}^{\mu\nu})^{l,m}, (\mathcal{J}^{\rho\sigma})^{l,m}\right] = -i\left[g^{\mu\sigma}(\mathcal{J}^{\nu\rho})^{l,m} + g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^{l,m} - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{l,m} - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{l,m}\right]. \tag{158}$$

It is easy to prove that the commutation relations between the generators $(\mathcal{M}^{\mu\nu})^{l,m}$ and $(\mathcal{J}^{\mu\nu})^{l,m}$ and the matrices χ^μ and ξ^μ are given by

$$\left[(\mathcal{M}^{\alpha\beta})^{l,m}, \chi^\nu\right] = -2i\left(g^{\nu\alpha}\chi^\beta - g^{\nu\beta}\chi^\alpha\right) \tag{159}$$

$$\left[(\mathcal{J}^{\mu\nu})^{l,m}, \eta^{\alpha\beta}\right] = -i\left(g^{\nu\alpha}\eta^{\mu\beta} + g^{\nu\beta}\eta^{\alpha\mu} - g^{\mu\alpha}\eta^{\nu\beta} - g^{\mu\beta}\eta^{\alpha\nu}\right). \tag{160}$$

The commutation rules as soon as written say as the generators $(\mathcal{M}^{\mu\nu})^{l,m}$ and $(\mathcal{J}^{\mu\nu})^{l,m}$ must be made so that the AαE and SαE are covariant quantities. It is clear that, due to the condition

$$(2l+1)(2m+1) = 2s+1,$$

the generators $(\mathcal{M}^{\mu\nu})^{l,m}$ and $(\mathcal{J}^{\mu\nu})^{l,m}$ have to be of the same dimension of the matrices χ^μ and ξ^μ for any fixed s. More generally they must respect the matrix product towards χ^μ and ξ^μ.

In order to conclude the search of the conditions for which the AαE and SαE are relativistically invariant quantities, it is fundamental to ask that they give rise to two four-currents which are transformed, moving from S to S', like four-vectors. Namely, by considering the four-currents [49]

$$\mathsf{j}_s^\nu = c\bar{\Psi}_s \chi^\nu \Psi_s \tag{161}$$

$$j_s^\nu = i\left[(\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu \psi_s - \psi_s^\dagger \xi^\nu \xi^\mu (\partial_\mu \psi_s)\right], \tag{162}$$

one must find the conditions for which is

$$\mathsf{j}'^\mu_s = \Lambda^\mu_\nu \mathsf{j}^\nu_s \tag{163}$$

$$j'^\mu_s = \Lambda^\mu_\nu j^\nu_s. \tag{164}$$

It can be demonstrated this happens if the generators $(\mathcal{M}^{\mu\nu})^{l,m}$ and $(\mathcal{J}^{\mu\nu})^{l,m}$ satisfy the following conditions

$$\chi^0 \left[(\mathcal{M}^{\mu\nu})^{l,m}\right]^\dagger \chi^0 = (\mathcal{M}^{\mu\nu})^{l,m} \tag{165}$$

$$\left[(\mathcal{J}^{\mu\nu})^{l,m}\right]^\dagger = (\mathcal{J}^{\mu\nu})^{l,m}. \tag{166}$$

This is the end of the research for the conditions about which the AαE and SαE are invariant quantities regarding $L_+^\uparrow$. To sum up, we can therefore assert the asymmetric α-equation is covariant when the following conditions are verified [50]

[49] These expressions can be deduced by (141) and (143).

[50] $S(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}(\mathcal{M}^{\mu\nu})^{l,m}}$.

$$\begin{cases} \psi_s'(x') = S(\Lambda)\psi_s(x) \\ S(\Lambda^{-1}) = S^{-1}(\Lambda) \\ \Lambda^\mu_\nu \chi^\nu = S^{-1}(\Lambda)\chi^\mu S(\Lambda) \\ \left[(\mathcal{M}^{\alpha\beta})^{l,m}, \chi^\nu\right] = -2i\left(g^{\nu\alpha}\chi^\beta - g^{\nu\beta}\chi^\alpha\right) \\ \chi^0\left[(\mathcal{M}^{\mu\nu})^{l,m}\right]^\dagger \chi^0 = (\mathcal{M}^{\mu\nu})^{l,m}, \end{cases} \tag{167}$$

while the symmetric α-equation is covariant when the following conditions are verified [51]

$$\begin{cases} \psi_s'(x') = \tilde{S}(\Lambda)\psi_s(x) \\ \tilde{S}(\Lambda^{-1}) = \tilde{S}^{-1}(\Lambda) \\ \eta^{\mu\nu}\Lambda^\alpha_\mu \Lambda^\beta_\nu = \tilde{S}^{-1}(\Lambda)\eta^{\alpha\beta}\tilde{S}(\Lambda) \\ \left[(\mathcal{J}^{\mu\nu})^{l,m}, \eta^{\alpha\beta}\right] = -i\left(g^{\nu\alpha}\eta^{\mu\beta} + g^{\nu\beta}\eta^{\alpha\mu} - g^{\mu\alpha}\eta^{\nu\beta} - g^{\mu\beta}\eta^{\alpha\nu}\right) \\ \left[(\mathcal{J}^{\mu\nu})^{l,m}\right]^\dagger = (\mathcal{J}^{\mu\nu})^{l,m}. \end{cases} \tag{168}$$

Hence, the covariance of our equations, like the Dirac equation, is not automatic (as it happens for the Klein-Gordon equation), but it is bounded by the existence of a special type of transformations of the Lorentz group $L_+^\uparrow$.

2.6 Solutions of the AαE and SαE for a free particle

In this section, we want to find the more general solutions of the equations

$$(i\hbar\chi^\mu\partial_\mu - mc\mathbb{1}_s)\,\psi_s(x) = 0 \tag{169}$$

$$\left(\xi^\mu\xi^\nu\partial_\mu\partial_\nu + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0. \tag{170}$$

For such a purpose, we notice that a solution of these equations is given by the plane-wave [52]

$$e^{-ik\cdot x}u_s(k), \tag{171}$$

[51] $\tilde{S}(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{l,m}}$, $\eta^{\sigma\rho} \equiv \xi^\sigma\xi^\rho$.

[52] $k^\mu \equiv (\frac{\omega}{c}, \vec{k})$.

where $u_s(k)$ is a column vector depending on k only. But it is not the only plane-wave solution of the AαE and SαE, since there is also

$$e^{ik\cdot x} v_s(k), \tag{172}$$

which, instead, gives rise to a spectrum of opposite sign for the (169) and to a spectrum equal to that obtained from the (171) for the equation (170).[53] Therefore, we can assert that the two (particular) solutions of the asymmetric and symmetric α-equation are

$$e^{-ik\cdot x} u_s(k) \text{ and } e^{ik\cdot x} v_s(k). \tag{173}$$

In order to study the free particle solutions of the AαE and SαE, one must replace within them the relations (173). Making it and starting from the AαE, we get

$$\begin{cases} (\hbar\chi^\mu k_\mu - mc\mathbb{1}_s)\, u_s(k) = 0 \\ (\hbar\chi^\mu k_\mu + mc\mathbb{1}_s)\, v_s(k) = 0 \end{cases} \tag{174}$$

and, for simplifying the calculations, we put ourselves in the "rest frame"

$$\begin{cases} (\hbar\chi^\mu k_\mu - mc\mathbb{1}_s)\, u_s(\frac{\omega}{c},\vec{0}) = 0 \\ (\hbar\chi^\mu k_\mu + mc\mathbb{1}_s)\, v_s(\frac{\omega}{c},\vec{0}) = 0, \end{cases} \tag{175}$$

from which, making clear the scalar products, by multiplying both sides by c and remembering that $E = mc^2$ and $\chi^0 = \delta_{\pm 1}$, we obtain

$$\begin{cases} \delta_{\pm 1} u_s(\frac{\omega}{c},\vec{0}) = \mathbb{1}_s u_s(\frac{\omega}{c},\vec{0}) \\ \delta_{\pm 1} v_s(\frac{\omega}{c},\vec{0}) = -\mathbb{1}_s v_s(\frac{\omega}{c},\vec{0}). \end{cases} \tag{176}$$

Now we must distinguish several cases [54]

$$u_0\left(\frac{\omega}{c},\vec{0}\right) = 1, \quad v_0\left(\frac{\omega}{c},\vec{0}\right) = 1 \qquad \text{for } s = 0 \tag{177}$$

[53]Really, the plane-waves (171) and (172) allow to pass from the representation $\delta_{\pm 1}$ to the representation $\delta_{\mp 1}$ and vice-versa in the equation (169).

[54]These solutions are all the independent ones. In the specific cases of $s = 0$ and $s = 1/2$, the first one identifies a state with positive energy, while the second one identifies a state with negative energy. Naturally, if we choose the representation $\delta_{\mp 1}$ they have to be the opposite, in the sense that the u_s solutions must be exchanged with the v_s solutions.

$$u^{(1)}_{1/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}1\\0\end{pmatrix},\quad v^{(1)}_{1/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\1\end{pmatrix}\qquad\text{for } s=\tfrac{1}{2}\tag{178}$$

$$u^{(1)}_{1}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad u^{(2)}_{1}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad v^{(1)}_{1}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad\text{for } s=1\tag{179}$$

of which, the first two identify states with positive energy, while the third one identifies a state with negative energy.

$$\begin{aligned}&u^{(1)}_{3/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad u^{(2)}_{3/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\&v^{(1)}_{3/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad v^{(2)}_{3/2}\left(\frac{\omega}{c},\vec{0}\right)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\qquad\text{for } s=\tfrac{3}{2}\end{aligned}\tag{180}$$

of which, the first two identify states with positive energy, while the third and fourth ones identify states with negative energy.

. . . . *etc* . . . . . . *etc* . . . .

Therefore, the $u_s(k)$ are linked to the states with positive energy, while the $v_s(k)$ are linked to those with negative energy. In particular, as just seen, it emerges that the asymmetric α-equation has, for $s=0$, a solution with positive energy and a solution with negative energy, for s half-integer, $(2s+1)/2$ solutions with positive energy and $(2s+1)/2$ solutions with negative energy, while for s integer, $(s+1)$ solutions with positive energy and s solutions with negative energy (for $\delta_{\mp 1}$ the situation is reversed with s solutions having positive energy and $(s+1)$ solutions having negative energy). This confirms existing, for s integer, an asymmetry in the set of the solutions with positive and negative energy for the AαE. Thanks to these considerations, we can say that in the rest frame one has

$$\text{for } s=0: \quad \begin{cases}\psi^{(+)}_{s=0}(x^0,\vec{0})=e^{-iEt/\hbar}\\ \psi^{(-)}_{s=0}(x^0,\vec{0})=e^{iEt/\hbar}\end{cases}\tag{181}$$

$$\text{for } s \text{ half-integer :} \quad \begin{cases} \psi_{s,+}^{(\alpha)}(x^0,\vec{0}) = e^{-iEt/\hbar}u_s^{(\alpha)}\left(\frac{\omega}{c},\vec{0}\right) \\ \psi_{s,-}^{(\alpha)}(x^0,\vec{0}) = e^{iEt/\hbar}v_s^{(\alpha)}\left(\frac{\omega}{c},\vec{0}\right) \end{cases} \tag{182}$$

with $\alpha \in \tilde{K} = \left\{1,\ldots,\frac{2s+1}{2}\right\}$.

$$\text{for } s \text{ integer :} \quad \begin{cases} \psi_{s,+}^{(\alpha)}(x^0,\vec{0}) = e^{-iEt/\hbar}u_s^{(\alpha)}\left(\frac{\omega}{c},\vec{0}\right) \\ \psi_{s,-}^{(\beta)}(x^0,\vec{0}) = e^{iEt/\hbar}v_s^{(\beta)}\left(\frac{\omega}{c},\vec{0}\right) \end{cases} \tag{183}$$

with $\alpha \in M = \{1,\ldots,s+1\}$ and $\beta \in N = \{1,\ldots,s\}$.

The expressions obtained for the particular solutions with positive and negative energy of the AαE are those of the rest frame. In order to obtain the generic solutions, we have to consider [55]

$$\begin{cases} u_s^{(\gamma)}(k) = S(\Lambda)u_s^{(\gamma)}\left(\frac{\omega}{c},\vec{0}\right) \\ v_s^{(\delta)}(k) = S(\Lambda)v_s^{(\delta)}\left(\frac{\omega}{c},\vec{0}\right), \end{cases} \tag{184}$$

from which easily follows [56]

$$\begin{cases} \psi_{s,+}^{(\gamma)}(x) = e^{-ik\cdot x}u_s^{(\gamma)}(k) \\ \psi_{s,-}^{(\delta)}(x) = e^{ik\cdot x}v_s^{(\delta)}(k). \end{cases} \tag{185}$$

It is not difficult to demonstrate that the particular solutions previously studied satisfy the following "orthonormality relations"

$$\text{for } s=0 : \quad \begin{cases} \bar{\psi}_{s=0}^{(+)}(x)\psi_{s=0}^{(+)}(x) = 1 \\ \bar{\psi}_{s=0}^{(-)}(x)\psi_{s=0}^{(-)}(x) = -1 \\ \bar{\psi}_{s=0}^{(+)}(x)\psi_{s=0}^{(-)}(x) = 0 = \bar{\psi}_{s=0}^{(-)}(x)\psi_{s=0}^{(+)}(x) \end{cases} \tag{186}$$

[55] $S(\Lambda) \in L_+^{\uparrow}$.
[56] Naturally, the indices γ, δ run on the same sets before defined for the rest frame solutions.

$$
\text{for } s \text{ half-integer :} \quad \begin{cases} \bar{\psi}^{(\alpha)}_{s,+}(x)\psi^{(\beta)}_{s,+}(x) = \delta_{\alpha\beta} \\ \bar{\psi}^{(\alpha)}_{s,-}(x)\psi^{(\beta)}_{s,-}(x) = -\delta_{\alpha\beta} \\ \bar{\psi}^{(\alpha)}_{s,+}(x)\psi^{(\beta)}_{s,-}(x) = 0 = \bar{\psi}^{(\alpha)}_{s,-}(x)\psi^{(\beta)}_{s,+}(x) \end{cases} \tag{187}
$$

where $\alpha, \beta \in \tilde{K} = \left\{1, \ldots, \frac{2s+1}{2}\right\}$.

$$
\text{for } s \text{ integer :} \quad \begin{cases} \bar{\psi}^{(\alpha)}_{s,+}(x)\psi^{(\beta)}_{s,+}(x) = \delta_{\alpha\beta} \\ \bar{\psi}^{(\gamma)}_{s,-}(x)\psi^{(\varepsilon)}_{s,-}(x) = -\delta_{\gamma\varepsilon} \\ \bar{\psi}^{(\alpha)}_{s,+}(x)\psi^{(\gamma)}_{s,-}(x) = 0 = \bar{\psi}^{(\varepsilon)}_{s,-}(x)\psi^{(\beta)}_{s,+}(x) \end{cases} \tag{188}
$$

where $\alpha, \beta \in M = \{1, \ldots, s+1\}$ and $\gamma, \varepsilon \in N = \{1, \ldots, s\}$.

Moreover, the particular solutions with positive and negative energy of the AαE respect also the following "completeness relations"

$$
\frac{1}{2}\left[\psi^{(+)}_{s=0}(x)\bar{\psi}^{(+)}_{s=0} - \psi^{(-)}_{s=0}(x)\bar{\psi}^{(-)}_{s=0}\right] = 1, \ \text{ for } s = 0 \tag{189}
$$

$$
\sum_{\alpha \in M, N, \tilde{K}} \left[\psi^{(\alpha)}_{s,+}(x)\bar{\psi}^{(\alpha)}_{s,+} - \psi^{(\alpha)}_{s,-}(x)\bar{\psi}^{(\alpha)}_{s,-}\right] = \mathbb{1}_s, \ \forall s \in \mathbb{N}/2 - \{0\}. \tag{190}
$$

After the description on form and properties of the (particular) solutions with positive and negative energy of the AαE, we are now in a position to write the general solution of this equation. First of all, we remember that from literature it is known the general solution of a differential equation is given by the sum of all the particular (independent) solutions of such an equation. Therefore, in our case, grouping in $\psi^{(+)}_s(x)$ all the particular solutions concerning the states with positive energy, and in $\psi^{(-)}_s(x)$ those with negative energy, we have that the general solution of the asymmetric α-equation is

$$
\psi_s(x) = \psi^{(+)}_s(x) + \psi^{(-)}_s(x). \tag{191}
$$

In order to resolve our problem, we must, therefore, make clear $\psi_s^{(+)}(x)$ and $\psi_s^{(-)}(x)$. It can be demonstrated that

$$\psi_s^{(+)}(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\sum_\alpha e^{-ik\cdot x}b_\alpha(k)u_s^{(\alpha)}(k) \tag{192}$$

$$\psi_s^{(-)}(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\sum_\beta e^{ik\cdot x}d_\beta^*(k)v_s^{(\beta)}(k), \tag{193}$$

where, with $b_\alpha(k)$ and $d_\beta^*(k)$, we have indicated the Fourier coefficients of the integral decomposition.

We still notice that the obtained expressions for $\psi_s^{(+)}(x)$ and $\psi_s^{(-)}(x)$ are Lorentz-invariant quantities. In fact, $u_s^{(\alpha)}(k)$ and $v_s^{(\beta)}(k)$ are so for construction, and also the measure $d^3k/2|\lambda|\tilde{\omega}_k$ is a Lorentz-invariant quantity, since it derives by d^4k, $\delta[\lambda(k^2 - m^2c^2/\hbar^2)]$ and $\theta(k_0)$, which are all Lorentz-invariant quantities. In view of this, we have that the general solution of the AαE is

$$\psi_s(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\left[\sum_\alpha e^{-ik\cdot x}b_\alpha(k)u_s^{(\alpha)}(k) + \sum_\beta e^{ik\cdot x}d_\beta^*(k)v_s^{(\beta)}(k)\right], \tag{194}$$

which, obviously, is invariant under the Lorentz group.

It is straightforward to see that, due to the equivalence $\bar{\psi}_s(x) = \psi_s^\dagger(x)\chi^0$, we have

$$\bar{\psi}_s(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\left[\sum_\alpha b_\alpha^*(k)\bar{u}_s^{(\alpha)}(k)e^{ik\cdot x} + \sum_\beta d_\beta(k)\bar{v}_s^{(\beta)}(k)e^{-ik\cdot x}\right]. \tag{195}$$

Let us specialize now the above expressions to the several labels s:

$$\text{for } s=0: \quad \begin{cases} \psi_{s=0}(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\left[b(k)e^{-ik\cdot x} + d^*(k)e^{ik\cdot x}\right] \\ \psi_{s=0}^\dagger(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\tilde{\omega}_k}\left[d(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right]. \end{cases} \tag{196}$$

The careful reader cannot avoid to see the expressions (196) are identical with the general solution of the complex Klein-Gordon equation and its conjugate. In fact, the only difference is into invariant measure, but it can be seen that the form of this measure depends on the constant λ with which the condition of mass-shell can be generalized in the following way

$$\lambda k^2 = \lambda \left(\frac{mc}{\hbar}\right)^2 . \tag{197}$$

Hence, if we take $\lambda = 2\pi$ and put ourselves in natural units, one obtains [57]

$$\psi_{s=0}(x) = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3 2\omega_k} \left[b(k)e^{-ik\cdot x} + d^*(k)e^{ik\cdot x}\right] \tag{198}$$

$$\psi^\dagger_{s=0}(x) = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3 2\omega_k} \left[d(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right] , \tag{199}$$

that just are the solution of the complex Klein-Gordon equation and its conjugate.

for s half-integer : [58]

$$\begin{cases} \psi_s(x) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\tilde{\omega}_k} \sum_{\alpha\in\tilde{K}} \left[b_\alpha(k)u_s^{(\alpha)}(k)e^{-ik\cdot x} + d_\alpha^*(k)v_s^{(\alpha)}(k)e^{ik\cdot x}\right] \\ \bar{\psi}_s(x) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\tilde{\omega}_k} \sum_{\alpha\in\tilde{K}} \left[d_\alpha(k)\bar{v}_s^{(\alpha)}(k)e^{-ik\cdot x} + b_\alpha^*(k)\bar{u}_s^{(\alpha)}(k)e^{ik\cdot x}\right] , \end{cases} \tag{200}$$

for s integer : [59]

$$\begin{cases} \psi_s(x) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\tilde{\omega}_k} \left[\sum_{\alpha\in M} b_\alpha(k)u_s^{(\alpha)}(k)e^{-ik\cdot x} + \sum_{\beta\in N} d_\beta^*(k)v_s^{(\beta)}(k)e^{ik\cdot x}\right] \\ \bar{\psi}_s(x) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\tilde{\omega}_k} \left[\sum_{\beta\in N} d_\beta(k)\bar{v}_s^{(\beta)}(k)e^{-ik\cdot x} + \sum_{\alpha\in M} b_\alpha^*(k)\bar{u}_s^{(\alpha)}(k)e^{ik\cdot x}\right] . \end{cases} \tag{201}$$

At this point, having characterized $\psi_s(x)$ and $\bar{\psi}_s(x)$ for any $s \in \mathbb{N}/2$, we have concluded the introductory discussion on the general solution of the AαE. The last thing we want to point out regards the coefficients $b_\alpha(k)$ and $d_\beta(k)$ and their complex conjugates $b_\alpha^*(k)$ and $d_\beta^*(k)$. Within a classical theory, they are not other but the amplitudes of the (integral) Fourier decomposition ("normal modes" of the decomposition), while in a quantum theory they represent the operators of the particles constituting the field.[60] From this point of view, they must satisfy the following canonical commutation or anti-commutation relations (CCR or CAR), depending on whether the studied field is made of bosons or fermions:

[57] In natural units $\omega_k = E/c = \tilde{\omega}_k$.

[58] $\tilde{K} = \left\{1, \ldots, \frac{2s+1}{2}\right\}$.

[59] $M = \{1, \ldots, s+1\}$ and $N = \{1, \ldots, s\}$. If $\delta = \delta_{\mp 1}$ these sets are reversed.

[60] In particular, $b_\alpha^*(k)$ and $d_\beta^*(k)$, indicated with $b_\alpha^\dagger(k)$ and $d_\alpha^\dagger(k)$, are the creation operators, while $b_\alpha(k)$ and $d_\alpha(k)$ are the annihilation ones.

$$\text{bosons}: \quad \begin{cases} [b_\alpha(k), b^\dagger_\beta(k')] = [d_\alpha(k), d^\dagger_\beta(k')] = 2|\lambda|\tilde{\omega}_k(2\pi\hbar)^2\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ [b_\alpha(k), b_\beta(k')] = [b^\dagger_\alpha(k), b^\dagger_\beta(k')] = 0 \\ [d_\alpha(k), d_\beta(k')] = [d^\dagger_\alpha(k), d^\dagger_\beta(k')] = 0 \end{cases} \tag{202}$$

$$\text{fermions}: \quad \begin{cases} \{b_\alpha(k), b^\dagger_\beta(k')\} = \{d_\alpha(k), d^\dagger_\beta(k')\} = 2|\lambda|\tilde{\omega}_k(2\pi\hbar)^2\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ \{b_\alpha(k), b_\beta(k')\} = \{b^\dagger_\alpha(k), b^\dagger_\beta(k')\} = 0 \\ \{d_\alpha(k), d_\beta(k')\} = \{d^\dagger_\alpha(k), d^\dagger_\beta(k')\} = 0. \end{cases} \tag{203}$$

The usage of such commutation or anti-commutation relations, thus as the use of $b_\alpha(k)$ and $d_\beta(k)$ like creation and annihilation operators in the limits of the theory studied in this work, will be the matter of the second quantization of our systems, which will be dealt ahead.

In order to complete our study, we want now to find the general solution of the symmetric α-equation

$$\left(\xi^\mu\xi^\nu\partial_\mu\partial_\nu + \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x) = 0. \tag{204}$$

The procedure we will utilize is similar to that previously seen and so we will try, as far as possible, to avoid arguments and calculations already met. Also for the SαE we want the particular plane-wave solutions which are given by the (173), however, unlike the AαE, if we replace these solutions in the (204), we obtain the equations

$$\begin{cases} \left(\xi^\mu\xi^\nu k_\mu k_\nu - \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)u_s(k) = 0 \\ \left(\xi^\mu\xi^\nu k_\mu k_\nu - \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)v_s(k) = 0, \end{cases} \tag{205}$$

which, substantially, say that $u_s(k)$ and $v_s(k)$ are solutions of the same equation. Therefore, in order to avoid confusion, taking account that $u_s(k)$ and $v_s(k)$ are practically the same column vector, one defines

$$z_s(k) \equiv u_s(k) = v_s(k), \tag{206}$$

from which, it easily follows

$$\begin{cases} \psi_s^{(+)}(x) = e^{-ik\cdot x} z_s(k) \\ \psi_s^{(-)}(x) = e^{ik\cdot x} z_s(k). \end{cases} \tag{207}$$

This means that $\psi_s^{(+)}(x)$ and $\psi_s^{(-)}(x)$ do not describe, like for the AαE, states with positive and negative energy respectively, but "progressive" (traveling from left to right) and "regressive" (traveling from right to left) waves only. Naturally, $z_s(k)$ satisfies the equation

$$\left(\xi^\mu \xi^\nu k_\mu k_\nu - \frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right) z_s(k) = 0. \tag{208}$$

We want now to study the form of $z_s(k)$, putting us – for reasons of simplicity – in the rest frame. By developing the calculations, we obtain

$$\left(\hbar^2\omega^2\mathbb{1}_s - m^2c^4\mathbb{1}_s\right) z_s\left(\frac{\omega}{c}, \vec{0}\right) = 0 \tag{209}$$

and, remembering that $E^2 = \hbar^2\omega^2$, $E^2 = m^2c^4$ (square energy of the rest frame), we have the identity

$$\mathbb{1}_s z_s\left(\frac{\omega}{c}, \vec{0}\right) = \mathbb{1}_s z_s\left(\frac{\omega}{c}, \vec{0}\right), \tag{210}$$

that gives

$$z_0\left(\frac{\omega}{c}, \vec{0}\right) = 1 \qquad \text{for } s = 0 \tag{211}$$

$$z_{1/2}^{(1)}\left(\frac{\omega}{c}, \vec{0}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad z_{1/2}^{(2)}\left(\frac{\omega}{c}, \vec{0}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{for } s = \tfrac{1}{2} \tag{212}$$

$$z_1^{(1)}\left(\frac{\omega}{c}, \vec{0}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_1^{(2)}\left(\frac{\omega}{c}, \vec{0}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_1^{(3)}\left(\frac{\omega}{c}, \vec{0}\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{for } s = 1 \tag{213}$$

. . . . *etc* . . . . . . *etc* . . . . . .

namely there are $(2s+1)$ "polarizations" $z_s\left(\frac{\omega}{c},\vec{0}\right)$ for any $s \in \mathbb{N}/2$. Hence, for the symmetric α-equation there is not a clean distinction between states with positive and negative energy, like it happens for the asymmetric α-equation, and this depends on the fact that, containing the SαE square energies, the polarization states *read* such energies only, and they are not able to distinguish between states with positive and negative energy. This must not worry, because the same structure of the SαE demands it, *i.e.* it follows why every solution contains within itself states with positive and negative energy, without showing them explicitly. We want now to establish the normalization conditions for these solutions. Being $\psi_s^\dagger \psi_s$ a Lorentz-invariant quantity specifically, it is natural to consider the normalization concerning this product. We estimate, for this purpose, the amount

$$z_s^{\dagger(\alpha)}\left(\frac{\omega}{c},\vec{0}\right) z_s^{(\beta)}\left(\frac{\omega}{c},\vec{0}\right), \tag{214}$$

that banally is equal to $\delta_{\alpha\beta}$ for any $s \in \mathbb{N}/2$, where $\alpha, \beta \in \{1, ..., 2s+1\}$.

For calculating $\psi_s^\dagger \psi_s$, it is important to remember that

$$\psi_s(x) = e^{\pm ik\cdot x} z_s(k), \tag{215}$$

from which follows

$$\psi_s^\dagger(x)\psi_s(x) = z_s^\dagger(k) z_s(k). \tag{216}$$

From the (216), it is clear we need $z_s(k)$ in order to estimate the normalization condition of the field $\psi_s(x)$. Since [61]

$$z_s^{(\alpha)}(k) = S(\Lambda) z_s^{(\alpha)}\left(\frac{\omega}{c},\vec{0}\right), \tag{217}$$

it follows

$$z_s^{\dagger(\alpha)}(k) z_s^{(\beta)}(k) = \delta_{\alpha\beta} \tag{218}$$

and therefore

$$\psi_s^\dagger(x)\psi_s(x) = \delta_{\alpha\beta}, \tag{219}$$

[61] Of course, $S(\Lambda) \in L_+^\uparrow$. It is underlined we consider the unitary representations of Lorentz group $L_+^\uparrow$, based on the argumentations of section 2.5.

which is the normalization condition concerning a particular solution (of single particle: for any $s \in \mathbb{N}/2$) for the SαE. It is also straightforward to try that is worth

$$\sum_{\alpha=1}^{2s+1} \left[\psi_s^{(\alpha)}(x)\psi_s^{\dagger(\alpha)}(x)\right] = \mathbb{1}_s, \tag{220}$$

representing the completeness relation of the SαE. To this point, we can proceed with the writing of the general solution of the symmetric α-equation. Being no difference between the solutions with positive and negative energy, the wanted solution will be simply given by the following Fourier integral

$$\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^4} d^4k \sum_{\alpha=1}^{2s+1} a_\alpha(k) z_s^{(\alpha)}(k) e^{\pm ik\cdot x}, \tag{221}$$

where $a_\alpha(k)$ represent the Fourier coefficients of the integral decomposition. Through some considerations, regarding: the use of the generalized mass-shell condition previously seen, the introduction of the function $\theta(k_0)$,[62] the substitution of $a_\alpha^*(k)$ with the new Fourier coefficients $b_\alpha(k)$,[63] it may be demonstrated that the (221) can be placed in the following form

$$\psi_s(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3} \frac{d^3k}{2\tilde{\omega}_k|\lambda|} \sum_{\alpha=1}^{2s+1} z_s^{(\alpha)}(k)\left[a_\alpha(k)e^{\pm ik\cdot x} + b_\alpha(k)e^{\mp ik\cdot x}\right], \tag{222}$$

which represents the (covariant) solution of the symmetric α-equation. From that, it is immediate to find the conjugate solution (adjoint field)

$$\psi_s^\dagger(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3} \frac{d^3k}{2\tilde{\omega}_k|\lambda|} \sum_{\alpha=1}^{2s+1} z_s^{\dagger(\alpha)}(k)\left[a_\alpha^*(k)e^{\mp ik\cdot x} + b_\alpha^*(k)e^{\pm ik\cdot x}\right]. \tag{223}$$

In order to confront the general solution of the SαE with that one found for AαE, we choose as exponential signs the sequence $-\ +$ and, rather than $b_\alpha(k)$, we take $b_\alpha^*(k)$. With such expedients the general solution of the SαE and its conjugate become

$$\begin{cases} \psi_s(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3} \frac{d^3k}{2\tilde{\omega}_k|\lambda|} \sum_{\alpha=1}^{2s+1} z_s^{(\alpha)}(k)\left[a_\alpha(k)e^{-ik\cdot x} + b_\alpha^*(k)e^{ik\cdot x}\right] \\ \psi_s^\dagger(x) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{K}^3} \frac{d^3k}{2\tilde{\omega}_k|\lambda|} \sum_{\alpha=1}^{2s+1} z_s^{\dagger(\alpha)}(k)\left[a_\alpha^*(k)e^{ik\cdot x} + b_\alpha(k)e^{-ik\cdot x}\right]. \end{cases} \tag{224}$$

[62]In this case, unlike the AαE, it has not to be imposed $k_0 > 0$, because it is not possible to distinguish between solutions with positive and negative energy.

[63]In fact, we have not $a_\alpha(-k) \neq a_\alpha^*(k)$, since the general solution of the SαE is not real.

We promptly notice that, unlike the solution of the AαE, the above $\psi_s(x)$ has an ultra-compact form, in the sense that is right for any $s \in \mathbb{N}/2$. Hence, the general solution of the SαE does not suffer of the *destructuring* of the general solution of the AαE, which is symptom of the asymmetry of the particular solutions of such an equation. Therefore, the expression of $\psi_s(x)$ is flexible to be dealt with and it is beautiful from the theoretical point of view. The analogy between the general solution of the SαE and the general solution of the Maxwell equations in covariant form will not certainly escape to the scrupulous reader. From this point of view, the $z_s^{(\alpha)}(k)$ play the role of polarization vectors for the quantum particles with arbitrary spin. The number of the polarization states, in this case, grows with the spin s, and, in general terms, it is equal to $(2s+1)$. In particular, for $s = 0$, we have

$$\psi_{s=0}(x) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2\tilde{\omega}_k|\lambda|} \left[a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right], \tag{225}$$

i.e. the particles with $s = 0$ have only a direction of polarization. We see that, if in the expression as soon as written we choose $\lambda = 2\pi$ + natural units, we have

$$\psi_{s=0}(x) = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right], \tag{226}$$

which is just the general solution of the complex Klein-Gordon equation.

In conclusion, we want also to point out here the role of the Fourier coefficients $a_\alpha(k)$ and $b_\alpha(k)$ and their complex conjugates $a^*_\alpha(k)$ and $b^*_\alpha(k)$, that, based on the quantum description (second quantization), become operators.[64] They can satisfy the following commutation or anti-commutation relations, depending on whether the described particles are bosons or fermions

$$\text{bosons :} \qquad \begin{cases} [a_\alpha(k), a^\dagger_\beta(k')] = [b_\alpha(k), b^\dagger_\beta(k')] = 2|\lambda|\tilde{\omega}_k(2\pi\hbar)^2\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ [a_\alpha(k), a_\beta(k')] = [a^\dagger_\alpha(k), a^\dagger_\beta(k')] = 0 \\ [b_\alpha(k), b_\beta(k')] = [b^\dagger_\alpha(k), b^\dagger_\beta(k')] = 0 \end{cases} \tag{227}$$

$$\text{fermions :} \qquad \begin{cases} \{a_\alpha(k), a^\dagger_\beta(k')\} = \{b_\alpha(k), b^\dagger_\beta(k')\} = 2|\lambda|\tilde{\omega}_k(2\pi\hbar)^2\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ \{a_\alpha(k), a_\beta(k')\} = \{a^\dagger_\alpha(k), a^\dagger_\beta(k')\} = 0 \\ \{b_\alpha(k), b_\beta(k')\} = \{b^\dagger_\alpha(k), b^\dagger_\beta(k')\} = 0. \end{cases} \tag{228}$$

[64]In particular, $a^*_\alpha(k)$ and $b^*_\alpha(k)$, indicated with $a^\dagger_\alpha(k)$ and $b^\dagger_\alpha(k)$, are the creation operators, while $a_\alpha(k)$ and $b_\alpha(k)$ are the annihilation ones.

3 Equations for Left- and Right-Handed Particles

Dynamics of left- and right-handed particles, which into Standard Model are identified with the neutrinos and anti-neutrinos respectively, represents a fundamental argument, because it has not been still possible to construct a theory able to expect all the properties of these escaping particles, such as the parity violation, the nonzero mass and resultant oscillations. In particular, the Weyl and Majorana theories, that currently describe the left- and right-handed fields, cannot explain these characteristics in a useful way. It is very likely this is caused by the inadequacy of the Dirac theory from which Weyl and Majorana models come out [14, 52]. Therefore, it is natural to try of developing a theory for left- and right-handed fields through the spin four-vector s_μ, defined in previous chapters. We will see this method will lead to some issues and so we will understand that dynamics of left- and right-handed particles can be alternatively obtained just from the AαE and SαE, only through mathematical considerations and without *ad hoc* hypothesis. Unexpectedly, we will derivate new equations for left- and right-handed particles, able to explain the neutrinos and anti-neutrinos properties and more.

3.1 Weyl theory revisited with the spin four-vector s_μ

The so far developed formalism, and, particularly, the introduction of the (matrix) four-vector of spin s^μ, which for bradyonic particles is given by $(\delta, -\varepsilon^i)$,[65] easily allows to find the equations of the motion for the right- and left-handed fields, being those which transform according to the inequivalent representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the group $L_+^\uparrow$, respectively. What we will practically make is to rewrite the Weyl equations (left- and right-handed fields with $m = 0$), in order to obtain a more rigorous instrument for the study of the particles following the dynamics of the left- and right-handed fields, like the neutrinos and anti-neutrinos. We start, therefore, from the definition of the field $\phi_R(x)$ and $\phi_L(x)$, that are transformed in the following way [48]

$$\begin{cases} \phi'_R(x') = D_R \phi_R(x) \\ \phi'_L(x') = D_L \phi_L(x), \end{cases} \tag{229}$$

where

$$\begin{cases} D_R = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} - i\vec{\phi})} \\ D_L = e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\phi})}. \end{cases} \tag{230}$$

[65] Really, it is $s^\mu \equiv (\tilde{\delta}, -\varepsilon^i) = (i\delta, -\varepsilon^i)$, but, in this case, we will obtain anti-hermitian quantities in our equations. So we place $\tilde{\delta} = \delta$, for having hermitian quantities only. Naturally, as it will be seen, this position will not change the general result of this chapter.

As it is well-known the Weyl equations are obtained from the Dirac equation using the chiral representation of the matrices γ^μ and putting $m = 0$. Nevertheless, the Weyl equations can be generated thanks to Lorentz-invariant quantities too. This method is the most suitable for constructing the equations of motion of the right- and left-handed fields through the spin four-vector s_μ without the Dirac equation and so this is the line we will follow in the next pages. For such a purpose, we construct now the two bilinear covariants

$$\phi_R^\dagger(\text{Lorentz-invariant})\phi_R; \quad \phi_L^\dagger(\text{Lorentz-invariant})\phi_L, \tag{231}$$

that must be, respectively, invariant regarding the transformations D_R and D_L of the group $L_+^\uparrow$. For the Weyl theory it can be chosen as Lorentz-invariant quantity concerning the field ϕ_R the scalar product $\sigma^\mu p_\mu$ and as Lorentz-invariant quantity concerning the field ϕ_L the scalar product $\tilde{\sigma}^\mu p_\mu$, where [66]

$$\sigma^\mu \equiv (\mathbb{1}_2, \vec{\sigma}), \quad \tilde{\sigma}^\mu \equiv (\mathbb{1}_2, -\vec{\sigma}). \tag{232}$$

Basically, for obtaining the Weyl framework, one can construct a kind of spin four-vector for defining some Lorentz-invariant quantities able to give the equations of motion of right- and left-handed fields. As we have seen in this work, then, it seems more correct to replace σ^μ and $\tilde{\sigma}^\mu$ with just the spin four-vectors concerning the fields with spin 1/2 (ϕ_R and ϕ_L are exactly of this type), *i.e.* with the four-vectors

$$\tilde{s}^\mu_{1/2} = (\delta_{1/2}, \vec{\varepsilon}_{1/2}), \quad s^\mu_{1/2} = (\delta_{1/2}, -\vec{\varepsilon}_{1/2}) \tag{233}$$

namely, remembering $\vec{\varepsilon}_{1/2} = 1/2\vec{\sigma}$, we have [67]

$$\tilde{s}^\mu_{1/2} = \left(\delta_{1/2}, \frac{1}{2}\vec{\sigma}\right), \quad s^\mu_{1/2} = \left(\delta_{1/2}, -\frac{1}{2}\vec{\sigma}\right). \tag{234}$$

Therefore, as Lorentz-invariant quantities we can take

$$\tilde{s}^\mu_{1/2} p_\mu, \quad s^\mu_{1/2} p_\mu \tag{235}$$

from which to define the bilinear forms

[66] Note that in this case with subindex 2 we indicate the dimension of the unit matrix and not the spin value.

[67] We are into representation $\delta = \delta_{\pm 1}$, even if the representation $\delta = \delta_{\mp 1}$ gives equivalent results.

$$\begin{cases} \phi_R^\dagger(\tilde{s}^\mu_{1/2}p_\mu)\phi_R = \phi_R^\dagger\left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_R \\ \phi_L^\dagger(s^\mu_{1/2}p_\mu)\phi_L = \phi_L^\dagger\left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_L. \end{cases} \tag{236}$$

It can be promptly observed the previous quantities are hermitian. Then, it can be proved that under D_R and D_L the (236) are transformed in the following way

$$\phi_R'^\dagger(\tilde{s}^\mu_{1/2}p_\mu)'\phi_R' = \phi_R^\dagger(\tilde{s}^\mu_{1/2}p_\mu)\phi_R \tag{237}$$

$$\phi_L'^\dagger(s^\mu_{1/2}p_\mu)'\phi_L' = \phi_L^\dagger(s^\mu_{1/2}p_\mu)\phi_L, \tag{238}$$

i.e. they are unchanged under the inequivalent representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of the group $L_+^\uparrow$. At this point, in order to find the equations of the motion for the fields ϕ_R and ϕ_R, it is enough to define the Lagrangian densities $\mathcal{L}_R$ and $\mathcal{L}_L$ through the Lorentz-invariant quantities (235), operating in them the substitution [68]

$$p_\mu \to i\partial_\mu, \tag{239}$$

thanks to which, we can thus define the following Lagrangian densities

$$\begin{cases} \mathcal{L}_R \equiv \phi_R^\dagger(\tilde{s}^\mu_{1/2}p_\mu)\phi_R = i\phi_R^\dagger(\tilde{s}^\mu_{1/2}\partial_\mu)\phi_R \\ \mathcal{L}_L \equiv \phi_L^\dagger(s^\mu_{1/2}p_\mu)\phi_L = i\phi_L^\dagger(s^\mu_{1/2}\partial_\mu)\phi_L. \end{cases} \tag{240}$$

As smoothly it is observed, $\mathcal{L}_R$ and $\mathcal{L}_L$ are not hermitian. In order to obviate this problem, it is sufficient to take [69]

$$\begin{cases} \mathcal{L}_R \equiv i\phi_R^\dagger(\tilde{s}^\mu_{1/2}\overleftrightarrow{\partial_\mu})\phi_R = \frac{i}{2}\left[\phi_R^\dagger(\tilde{s}^\mu_{1/2}\partial_\mu)\phi_R - (\partial_\mu\phi_R^\dagger)\tilde{s}^\mu_{1/2}\phi_R\right] \\ \mathcal{L}_L \equiv i\phi_L^\dagger(s^\mu_{1/2}\overleftrightarrow{\partial_\mu})\phi_L = \frac{i}{2}\left[\phi_L^\dagger(s^\mu_{1/2}\partial_\mu)\phi_L - (\partial_\mu\phi_L^\dagger)s^\mu_{1/2}\phi_L\right], \end{cases} \tag{241}$$

[68] Such a substitution is important for the construction of the kinetic term: for simplicity, we put us in natural units.

[69] In general, we have

$$A\overleftrightarrow{\partial_\mu}B \equiv \frac{1}{2}\Big[A(\partial_\mu B) - (\partial_\mu A)B\Big],$$

where, in our case, we place $A = i\phi_R^\dagger\tilde{s}^\mu_{1/2}$, $B = \phi_R$ and $A = i\phi_L^\dagger s^\mu_{1/2}$, $B = \phi_L$, respectively.

which, instead, as it is banal to verify, are hermitian. We now calculate the equations of the motion for the fields ϕ_R and ϕ_L. The Euler-Lagrange equations for such fields give

$$i\tilde{s}^{\mu}_{1/2}\partial_{\mu}\phi_R = 0 \tag{242}$$

$$is^{\mu}_{1/2}\partial_{\mu}\phi_L = 0. \tag{243}$$

Therefore, the equations of the motion for the fields ϕ_R and ϕ_L, written in the explicit form, are

$$\left(\delta_{1/2}\partial_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)\phi_R(\vec{x},t) = 0 \tag{244}$$

$$\left(\delta_{1/2}\partial_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)\phi_L(\vec{x},t) = 0. \tag{245}$$

In the momentum space, the (242) and (243) become instead

$$\tilde{s}^{\mu}_{1/2}p_{\mu}\phi_R = 0 \tag{246}$$

$$s^{\mu}_{1/2}p_{\mu}\phi_L = 0, \tag{247}$$

from which, it immediately follows

$$\left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_R(\vec{x},t) = 0 \tag{248}$$

$$\left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_L(\vec{x},t) = 0. \tag{249}$$

Matrix equations (248) and (249) can be put in the following form [70]

$$\begin{cases} \frac{1}{2}\left(\frac{\vec{\sigma}\cdot\vec{p}}{p_0}\right)\phi_R(\vec{x},t) = \delta_{1/2}\phi_R(\vec{x},t) \\ \frac{1}{2}\left(\frac{\vec{\sigma}\cdot\vec{p}}{p_0}\right)\phi_L(\vec{x},t) = -\delta_{1/2}\phi_L(\vec{x},t). \end{cases} \tag{250}$$

We now define the "generalized helicity operator" in the following way

$$\lambda_s \equiv \frac{\vec{s}\cdot\vec{p}}{p_0}. \tag{251}$$

[70] Note that $\delta_{1/2} = \sigma_3$.

In the case of $s = 1/2$, it becomes

$$\lambda_{1/2} = \frac{\vec{s}_{1/2} \cdot \vec{p}}{p_0} = \frac{1}{2}\left(\frac{\vec{\sigma} \cdot \vec{p}}{p_0}\right), \tag{252}$$

from which, the equations (250) can be so written

$$\begin{cases} \lambda_{1/2}\phi_R(\vec{x}, t) = \delta_{1/2}\phi_R(\vec{x}, t) \\ \lambda_{1/2}\phi_L(\vec{x}, t) = -\delta_{1/2}\phi_L(\vec{x}, t), \end{cases} \tag{253}$$

i.e. $\phi_R(x)$ is the eigenstate of the scalar operator $\lambda_{1/2}$ with eigenvalue $\delta_{1/2}$, while $\phi_L(x)$ is the eigenstate of the scalar operator $\lambda_{1/2}$ with eigenvalue $-\delta_{1/2}$. It is clear that as soon as asserted is not rigorous, since $\delta_{1/2}$ is not a scalar, but a two-dimensional matrix. However, if we are well thinking, the same misuse is made in the ordinary Weyl theory, when one asserts that ϕ_R and ϕ_L are eigenstates of the operator $\vec{\sigma} \cdot \vec{p}/p_0$, with eigenvalues 1 and -1, respectively. In fact, the operator $\vec{\sigma} \cdot \vec{p}/p_0$ is represented by a two-dimensional matrix and so the right-hand sides of the Weyl equations are not really 1 and -1, but rather $\mathbb{1}_2$ and $-\mathbb{1}_2$. But this is not the only problem of the Weyl equations, since the operator $\vec{\sigma} \cdot \vec{p}/p_0$ present in them, is not the helicity, that, like we know, is defined as the projection of the spin on the direction of the motion, because $\vec{\sigma}$ is not the spin vector, but the vector built with the three Pauli matrices. Instead, in the previous equations, there is the correct helicity for the particles of spin 1/2, which is the projection in the direction of the motion of the spin $\vec{\sigma}/2$ (in natural units). Moreover, the obtained equations, like the Weyl ones, are not invariant under the action of the parity operator. All of this takes to understand the Weyl method – applied not to the (matrix) four-vectors σ^μ and $\tilde{\sigma}^\mu$, but to $\tilde{s}^\mu_{1/2}$ and $s^\mu_{1/2}$ – gives equations that are similar and perhaps better than those commonly known in literature for the study of dynamics of right- and left-handed particles (massless).

Let us now derivate the equations for the right- and left-handed fields with mass $m \neq 0$. The road to follow is to define a Lagrangian density having for kinetic terms the sum of $\mathcal{L}_R$ and $\mathcal{L}_L$ and for potential term the amount (it is said "Dirac mass term": note its invariance under $D_R, D_L \subset L^\uparrow_+$)

$$m(\phi^\dagger_R\phi_L + \phi^\dagger_L\phi_R). \tag{254}$$

Therefore, the Lagrangian density of the theory we are constructing is given by

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_L - m(\phi^\dagger_R\phi_L + \phi^\dagger_L\phi_R) = i\left[\phi^\dagger_R(\tilde{s}^\mu_{1/2}\overleftrightarrow{\partial_\mu})\phi_R + \phi^\dagger_L(s^\mu_{1/2}\overleftrightarrow{\partial_\mu})\phi_L\right] - m(\phi^\dagger_R\phi_L + \phi^\dagger_L\phi_R), \tag{255}$$

from which, the equations of the motion for the fields ϕ_R and ϕ_L are

$$i(\tilde{s}^{\mu}_{1/2}\partial_\mu)\phi_R = m\phi_L \;\Leftrightarrow\; i\left(\delta_{1/2}\partial_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)\phi_R = m\phi_L \tag{256}$$

$$i(s^{\mu}_{1/2}\partial_\mu)\phi_L = m\phi_R \;\Leftrightarrow\; i\left(\delta_{1/2}\partial_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)\phi_L = m\phi_R. \tag{257}$$

The equations (256) and (257) mix right- and left-handed fields. In order to obtain the equations for the single fields ϕ_R and ϕ_L (that is for no coupled fields), it is sufficient to multiply the first by $i\left(\delta_{1/2}\partial_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)$ and the second one by $i\left(\delta_{1/2}\partial_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{\nabla}\right)$, both members on the left-hand sides. At the end, it is not difficult to get

$$\left[\mathbb{1}_2\partial_0^2 + \delta_{1/2}(\sigma_1\nabla_1 + \sigma_2\nabla_2)\partial_0 - \frac{\mathbb{1}_2}{2}\nabla^2\right]\phi_R = -m^2\phi_R \tag{258}$$

$$\left[\mathbb{1}_2\partial_0^2 - \delta_{1/2}(\sigma_1\nabla_1 + \sigma_2\nabla_2)\partial_0 - \frac{\mathbb{1}_2}{2}\nabla^2\right]\phi_L = -m^2\phi_L. \tag{259}$$

The expressions (258) and (259) represent the equations of the right- and left-handed fields in the case $m \neq 0$. They are more complex than those of the ordinary Dirac theory in the chiral representation, which says the fields ϕ_R and ϕ_L satisfy both the two-dimensional Klein-Gordon equation, *i.e.* component for component. In the momentum space, we have

$$\begin{cases} (\tilde{s}^{\mu}_{1/2}p_\mu)\phi_R = m\phi_L \;\Leftrightarrow\; \left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_R = m\phi_L \\ (s^{\mu}_{1/2}p_\mu)\phi_L = m\phi_R \;\Leftrightarrow\; \left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_L = m\phi_R. \end{cases} \tag{260}$$

From which, proceeding as before, it is not difficult to find the no coupled equations for the right- and left-handed fields

$$\begin{cases} \left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_R = m\left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_L = m^2\phi_R \\ \left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\left(\delta_{1/2}p_0 + \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_L = m\left(\delta_{1/2}p_0 - \frac{1}{2}\vec{\sigma}\cdot\vec{p}\right)\phi_R = m^2\phi_L, \end{cases} \tag{261}$$

that, in the explicit form, are

$$\begin{cases} \left[\mathbb{1}_2 p_0^2 - p_0\delta_{1/2}(\sigma_1 p_1 + \sigma_2 p_2) - \frac{\mathbb{1}_2}{2}|\vec{p}|^2\right]\phi_R = m^2\phi_R \\ \left[\mathbb{1}_2 p_0^2 + p_0\delta_{1/2}(\sigma_1 p_1 + \sigma_2 p_2) - \frac{\mathbb{1}_2}{2}|\vec{p}|^2\right]\phi_L = m^2\phi_L. \end{cases} \tag{262}$$

3.2 The equations for the right- and left-handed fields via AαE and SαE

It is immediate to verify that the equations (262), thus like the (250), are not invariant under parity,[71] and so they can be used in all the theories violating the parity. It also must be emphasized that the equations of the motion for the fields ϕ_R and ϕ_L, with $m = 0$ or $m \neq 0$, earlier achieved, are based on the completely arbitrary choice of the Lorentz-invariant quantities $\tilde{s}^{\mu}_{1/2} p_{\mu}$ and $s^{\mu}_{1/2} p_{\mu}$ inserted between the fields ϕ_R and ϕ_L and their conjugates. Obviously, it is possible to choose other invariant quantities as

$$\tilde{s}^{\mu}_{1/2} \tilde{s}^{\nu}_{1/2} p_{\mu} p_{\nu} \text{ and } s^{\mu}_{1/2} s^{\nu}_{1/2} p_{\mu} p_{\nu}, \tag{263}$$

that should generate other equations of the motion for the right- and left-handed fields with $m = 0$ or $m \neq 0$. The question is, then, what are the right Lorentz-invariant quantities to choose, *i.e.* what are the good equations for the right- and left-handed fields. The problem is really wrong at the beginning, because the previously found equations are nothing but an alternative version of the Weyl equations, based practically on the substitutions

$$\sigma^{\mu} = (\mathbb{1}_2, \vec{\sigma}) \rightarrow \tilde{s}^{\mu}_{1/2} = \left(\delta_{1/2}, \frac{1}{2}\vec{\sigma}\right) = \left(\sigma_3, \frac{1}{2}\vec{\sigma}\right) \tag{264}$$

$$\tilde{\sigma}^{\mu} = (\mathbb{1}_2, -\vec{\sigma}) \rightarrow s^{\mu}_{1/2} = \left(\delta_{1/2}, -\frac{1}{2}\vec{\sigma}\right) = \left(\sigma_3, -\frac{1}{2}\vec{\sigma}\right). \tag{265}$$

Therefore, the problems of our equations can be synthesized in the following points:

- Arbitrariness of the definition of bilinear $\phi^{\dagger}_{R/L}$(Lorentz scalar)$\phi_{R/L}$ like kinetic term of the theory.
- Arbitrariness in coupling the four-vector $\tilde{s}^{\mu}_{1/2}$ (or σ^{μ}) to the right-handed field and the four-vector $s^{\mu}_{1/2}$ (or $\tilde{\sigma}^{\mu}$) to the left-handed field.

From this reason, it is understood a more rigorous discussion of the problem is necessary, by proving to achieve the equations for the right- and left-handed fields by a general theory, which should be able to make visible the dynamics of such fields, without using *ad hoc* hypothesis. It is easy to see that the theory dealt in this work, based on the asymmetric and symmetric α-equation, describing in wide terms the particles with arbitrary spin, is the natural candidate for the extrapolation of the equations for the right- and left-handed fields. Hence, what now we want to make is to find the equations of the motion for the right- and left-handed fields concerning the AαE and SαE, which, for simplicity, we write in natural units

[71] In particular, such equations are inverted under parity, in the sense that the first is transformed in the second and vice-versa.

$$(i\chi^{\mu}\partial_{\mu} - m\mathbb{1}_s)\,\psi_s(x) = 0 \tag{266}$$

$$\left(\xi^{\mu}\xi^{\nu}\partial_{\mu}\partial_{\nu} + m^2\mathbb{1}_s\right)\psi_s(x) = 0. \tag{267}$$

How can we do it? How, at the same way, can we conform the (266) and (267) to the inequivalent representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group $L_+^{\uparrow}$? We intuitively understand that, being the right- and left-handed fields two-dimensional, one must work with four-dimensional representations of the (266) and (267), with the aim to obtain a two-dimensional decomposition for the right- and left-handed fields, respectively. However, we know the (266) and (267) both admit a four-dimensional representation for $s = 3/2$, which contrasts with the fact that the right- and left-handed fields describe particles having spin 1/2. This lets us conclude it has not to shrink the (266) and (267) to $s = 3/2$, but to the tensor product (also called "Kronecker product") of two fields with spin 1/2. Therefore, defined

$$\psi_{\otimes}(x) \equiv \varphi_{1/2} \otimes \zeta_{1/2} \equiv \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \otimes \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \equiv \begin{pmatrix} \tilde{x}_1\tilde{y}_1 \\ \tilde{x}_1\tilde{y}_2 \\ \tilde{x}_2\tilde{y}_1 \\ \tilde{x}_2\tilde{y}_2 \end{pmatrix} \equiv \begin{pmatrix} \psi_R^1 \\ \psi_R^2 \\ \psi_L^1 \\ \psi_L^2 \end{pmatrix} \equiv \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{268}$$

$$\psi_{\otimes}(x) \equiv \varphi_{1/2} \otimes \zeta_{1/2} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} \equiv \begin{pmatrix} \psi_R^1 \\ \psi_R^2 \\ \psi_L^1 \\ \psi_L^2 \end{pmatrix} \equiv \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{269}$$

we can write the AαE and SαE in the following way [72]

$$(i\chi_{\otimes}^{\mu}\partial_{\mu} - m\mathbb{1}_4)\,\psi_{\otimes}(x) = 0 \tag{270}$$

$$\left(\xi_{\otimes}^{\mu}\xi_{\otimes}^{\nu}\partial_{\mu}\partial_{\nu} + m^2\mathbb{1}_4\right)\psi_{\otimes}(x) = 0. \tag{271}$$

In order to calculate the equations of the motion for the right- and left-handed fields, the above expressions must be made clear. For this reason, we must calculate the matrices $\chi_{\otimes}^{\mu}$ and $\xi_{\otimes}^{\mu}$, that, in such a case, are given by the tensor product of the respective matrices for $s = 1/2$, *i.e.*

$$\chi_{\otimes}^0 = \chi_{1/2}^0 \otimes \chi_{1/2}^0 = \begin{pmatrix} \delta_{1/2} & \mathbb{O}_2 \\ \mathbb{O}_2 & -\delta_{1/2} \end{pmatrix} \tag{272}$$

[72] In such a case, the subindex of $\mathbb{1}$ is the dimension of this matrix and not the spin value.

$$\chi^1_\otimes = \chi^1_{1/2} \otimes \chi^1_{1/2} = \frac{1}{4}\begin{pmatrix} \mathbb{O}_2 & i\sigma_2 \\ -i\sigma_2 & \mathbb{O}_2 \end{pmatrix} \tag{273}$$

$$\chi^2_\otimes = \chi^2_{1/2} \otimes \chi^2_{1/2} = \frac{1}{4}\begin{pmatrix} \mathbb{O}_2 & -\sigma_1 \\ -\sigma_1 & \mathbb{O}_2 \end{pmatrix} \tag{274}$$

$$\chi^3_\otimes = \chi^3_{1/2} \otimes \chi^3_{1/2} = \frac{1}{4}\begin{pmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \tag{275}$$

$$\xi^0_\otimes = \xi^0_{1/2} \otimes \xi^0_{1/2} = \begin{pmatrix} \delta_{1/2} & \mathbb{O}_2 \\ \mathbb{O}_2 & -\delta_{1/2} \end{pmatrix} \tag{276}$$

$$\xi^1_\otimes = \xi^1_{1/2} \otimes \xi^1_{1/2} = \frac{1}{4}\begin{pmatrix} \mathbb{O}_2 & -\sigma_1 \\ -\sigma_1 & \mathbb{O}_2 \end{pmatrix} \tag{277}$$

$$\xi^2_\otimes = \xi^2_{1/2} \otimes \xi^2_{1/2} = \frac{1}{4}\begin{pmatrix} \mathbb{O}_2 & i\sigma_2 \\ -i\sigma_2 & \mathbb{O}_2 \end{pmatrix} \tag{278}$$

$$\xi^3_\otimes = \xi^3_{1/2} \otimes \xi^3_{1/2} = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix}. \tag{279}$$

It is straightforward to verify the substitution of the matrices (272)-(275) into (270) gives the two spinorial equations

$$\begin{cases} \left(i\delta_{1/2}\frac{\partial}{\partial t} + \mathbb{1}_2\frac{i}{4}\frac{\partial}{\partial z} - m\mathbb{1}_2\right)\psi_R(\vec{x},t) = \frac{1}{4}\left(\sigma_2\frac{\partial}{\partial x} + i\sigma_1\frac{\partial}{\partial y}\right)\psi_L(\vec{x},t) \\ \left(i\delta_{1/2}\frac{\partial}{\partial t} - \mathbb{1}_2\frac{i}{4}\frac{\partial}{\partial z} + m\mathbb{1}_2\right)\psi_L(\vec{x},t) = \frac{1}{4}\left(\sigma_2\frac{\partial}{\partial x} - i\sigma_1\frac{\partial}{\partial y}\right)\psi_R(\vec{x},t). \end{cases} \tag{280}$$

For decoupling these equations, it is necessary to do some mathematical artifices. With a small amount of calculations, one can arrive to the following equations for the right- and left-handed fields [73]

$$\begin{cases} \left(\frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial x^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial y^2} - \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial z^2} - \mathbb{1}_2\frac{\partial^2}{\partial t^2} + i\frac{\sigma_1\sigma_2}{8}\frac{\partial^2}{\partial x\partial y} - \frac{\delta_{1/2}}{2}\frac{\partial^2}{\partial t\partial z} - \frac{mi}{2}\mathbb{1}_2\frac{\partial}{\partial z} - 2mi\delta_{1/2}\frac{\partial}{\partial t} + m^2\mathbb{1}_2\right)\psi_R(\vec{x},t) = 0 \\ \left(\frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial x^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial y^2} - \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial z^2} - \mathbb{1}_2\frac{\partial^2}{\partial t^2} - i\frac{\sigma_1\sigma_2}{8}\frac{\partial^2}{\partial x\partial y} + \frac{\delta_{1/2}}{2}\frac{\partial^2}{\partial t\partial z} - \frac{mi}{2}\mathbb{1}_2\frac{\partial}{\partial z} + 2mi\delta_{1/2}\frac{\partial}{\partial t} + m^2\mathbb{1}_2\right)\psi_L(\vec{x},t) = 0. \end{cases} \tag{281}$$

It is immediately noticed, passing to the momentum space, that the equations as soon as written are not invariant under parity.

Let us estimate now the equations of the motion for the right- and left-handed fields starting from

$$\left(\xi^\mu_\otimes\xi^\nu_\otimes\partial_\mu\partial_\nu + m^2\mathbb{1}_4\right)\psi_\otimes(x) = 0. \tag{282}$$

Also in such a case, by replacing the previous matrices (276)-(279) into (282), with a little of calculations one arrives to the spinorial equations

$$\left(\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial x^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial y^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial z^2} - \frac{\sigma_3}{8}\frac{\partial^2}{\partial x\partial y} - \frac{\mathbb{1}_2}{2}\frac{\partial^2}{\partial t\partial z} + \mathbb{1}_2 m^2\right)\psi_R(\vec{x},t) = \frac{1}{2}\left(\frac{1}{4}\sigma_1\frac{\partial^2}{\partial y\partial z} - \frac{i}{4}\sigma_2\frac{\partial^2}{\partial x\partial z} - \sigma_1\frac{\partial^2}{\partial t\partial y} + i\sigma_2\frac{\partial^2}{\partial t\partial x}\right)\psi_L(\vec{x},t) \tag{283}$$

$$\left(\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial x^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial y^2} + \frac{\mathbb{1}_2}{16}\frac{\partial^2}{\partial z^2} + \frac{\sigma_3}{8}\frac{\partial^2}{\partial x\partial y} - \frac{\mathbb{1}_2}{2}\frac{\partial^2}{\partial t\partial z} + \mathbb{1}_2 m^2\right)\psi_L(\vec{x},t) = \frac{1}{2}\left(\frac{1}{4}\sigma_1\frac{\partial^2}{\partial y\partial z} + \frac{i}{4}\sigma_2\frac{\partial^2}{\partial x\partial z} - \sigma_1\frac{\partial^2}{\partial t\partial y} - i\sigma_2\frac{\partial^2}{\partial t\partial x}\right)\psi_R(\vec{x},t), \tag{284}$$

which, as it can be observed, have the defect to mix right- and left-handed fields. For decoupling them, it is necessary to proceed as for the (280). However, this road is not very fruitful in order to obtain the correct equations for the fields ψ_R and ψ_L. In fact, a common insight suggests we

[73] The first step is to multiply, on the left-hand sides, both members of the equations (280) by $\frac{1}{4}\left(\sigma_2\frac{\partial}{\partial x} - i\sigma_1\frac{\partial}{\partial y}\right)$ and $\frac{1}{4}\left(\sigma_2\frac{\partial}{\partial x} + i\sigma_1\frac{\partial}{\partial y}\right)$, respectively.

will obtain partial differential equations of the fourth order, which, aside from operative difficulty that can involve during explicit calculations, have a double differential order compared to the initial equation, as it happened for the equations (281) calculated from the (270). This suggests we made a conceptual mistake, and not a calculation one. How can we find such an error?

As it often happens, the problem is simpler than it seems and can be solved with a very used *formula* in physics, consisting in the change of reference. Therefore, for obtaining by the AαE and SαE the equations for the right- and left-handed fields of their same order, we must select an opportune reference that grants our wish. To vary reference, when we work with matrices, is equivalent to operate a change of basis, that is an invertible transformation which allows to express these matrices in a more manageable form. Who chews a little only of linear algebra knows the more comfortable matrix representations are the diagonal ones. From the previous pages, we see the not diagonal matrices, concerning the studied problem, are: $\chi^1_\otimes$, $\chi^2_\otimes$, $\xi^1_\otimes$, $\xi^2_\otimes$. We can choose, therefore, to put us in the reference in which one of these matrices has a diagonal form. What matrix can we choose? We note quickly that, being

$$\chi^1_\otimes = \xi^2_\otimes; \quad \chi^2_\otimes = \xi^1_\otimes, \tag{285}$$

doing diagonal a matrix, we automatically have also another and the matrices $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$ will be all expressed in the same coordinate system. Practically, what we want to make with the change of basis, by the point of view of the representation theory (of the transformation groups), is to decompose the tensor product of two representations with dimension equal to 2 in the direct sum of the irreducible representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, which are nothing but the inequivalent representations D_R and D_L of the group $L^\uparrow_+$, thanks to which the right- and left-handed fields are transformed for definition. That effectively the change of basis (or of representation) allows to obtain

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \tag{286}$$

is based on the representation theory and it depends on circumstance that a representation can be reduced in the direct sum of irreducible representations if its block matrix (or equally if all the matrices composing the representation) can be put in a diagonal form. Hence, in our specific case, we are able to reach the direct sum of the irreducible representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, only if the generic representation – obtained from the tensor product of two-dimensional representations – which we may indicate with $D_\otimes$, can be written in the following form

$$D_\otimes = \begin{pmatrix} D_1 & \mathbb{O}_2 \\ \mathbb{O}_2 & D_2 \end{pmatrix}, \tag{287}$$

where D_1 and D_2 are two representations of dimension 2, *i.e.* they are matrices 2×2. Therefore, we must verify that the coordinate system making $\chi^1_\otimes$ or $\chi^2_\otimes$ in a diagonal form (and so also $\xi^2_\otimes$ and

$\xi^1_\otimes$, respectively) is really the one about all our matrices are diagonal. Before continuing, it is well to emphasize that we will perform the diagonalization for both $\chi^1_\otimes$ and $\chi^2_\otimes$ (and not for only one), because it is not true that the matrices $\tilde{\chi}^\mu_\otimes$ and $\tilde{\xi}^\mu_\otimes$, deriving from such diagonalizations, lead to the same equations for the right- and left-handed fields.

Firstly, we proceed to the diagonalization of $\chi^1_\otimes$. It is easy to gain, by the characteristic polynomial of this matrix, that it admits eigenvalues $\lambda_1 = -1/4$ and $\lambda_2 = 1/4$, both with algebraic multiplicity equal to 2. In order to complete the diagonalization process regarding $\chi^1_\otimes$, it is necessary to find the matrix of change of basis P and its inverse P^{-1}. But this, although from the mathematical point of view is quite simple, places some problems from the physical one. For understanding what just asserted, we characterize the eigenspaces of the eigenvalues λ_1 and λ_2. They are

$$V_{-1/4} = \Big\{(-w, z, z, w) : z, w \in \mathbb{R}\Big\} \tag{288}$$

$$V_{1/4} = \Big\{(w, -z, z, w) : z, w \in \mathbb{R}\Big\}. \tag{289}$$

Since z and w can assume every real value, there are in principle ∞^2 matrices P and P^{-1} that resolves our problem and we do not know if all these matrices produce the same equations for right- and left-handed fields (in any case this would have to be demonstrated and this is impossible). Therefore, we have run into a troublesome problem, which puts our purpose to find *good* equations for right- and left-handed fields, using AαE and SαE, at risk. How can we exceed this apparently great obstacle?
As usual, we can use the smartness and beauty of the Group Theory. In fact, according to it, we know that the representations of Lie algebras and subalgebras can be visualized through the so-called "weight diagrams." Since we want to obtain the representation

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right), \tag{290}$$

we should try to shrink the choice of z and w between the fundamental weights of such a representation. What is the form of the weight diagram about $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$? From the literature, we know this is the representation of the Lorentz group $L^\uparrow_+$ given by the direct sum of the two irreducible representations $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, which are said "spinorial representations." Since the irreducible representations of $L^\uparrow_+$ having finite dimension are equivalent to the $SU(2)_l \otimes SU(2)_m$ ones,[74] they can be characterized by the couple (l, m), where l and m are integer and half-integer numbers such that $l(l+1)$ and $m(m+1)$ are the eigenvalues of the Casimir operators belonging to the Lie algebra

[74]Remember that such groups are not isomorphic, because $L^\uparrow_+ \simeq SO(3,1)$ – unlike $SU(2)_l \otimes SU(2)_m$ – is not a compact group.

of $SU(2)_l$ and $SU(2)_m$, respectively. Thence, the $LieSU(2)_l \otimes SU(2)_m$ representations can be visualized through the weight diagram obtained from $A_1 \times A_1$,[75] having l and m as maximum weights and dimension $(2l+1) \times (2m+1)$.[76] From what said, we understand the weight diagram of $\left(\frac{1}{2}, 0\right)$ is that of fig. 4, while the weight diagram of $\left(0, \frac{1}{2}\right)$ is drawn in fig. 5. As we can see such diagrams are identical unless the change of the l-axis with m-axis and vice-versa. This is not banal, since it will allow us to resolve our problem elegantly.

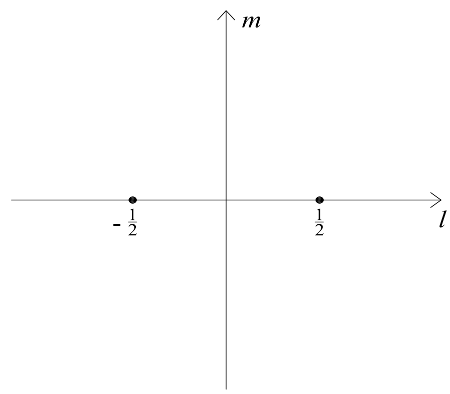

Figure 4: The weight diagram of the irreducible representation $\left(\frac{1}{2}, 0\right)$.

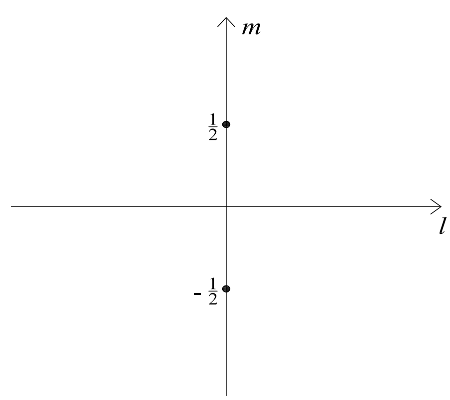

Figure 5: The weight diagram of the irreducible representation $\left(0, \frac{1}{2}\right)$.

[75] $SU(2)$ is classified with A_1 into Lie algebras theory.

[76] By and large the weights are $l, l-1, \dots, -l$ and $m, m-1, \dots, -m$.

Therefore, the weight diagram of $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ is given by the superimposition of the two previous figures, as shown in fig. 6. The weight diagram concerning the representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ resolves our problem, because, as we know, it is isomorphic to the Euclidean plane $\mathbb{R}^2$, whose axes can be chosen coinciding with z and w. This means we can take the points (z,w), which are able to resolve our problem, just equal to the weights of the representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$, according to the diagram of fig. 7, where

$$A \equiv (1/2,0),\ B \equiv (0,1/2),\ C \equiv (-1/2,0),\ D \equiv (0,-1/2). \tag{291}$$

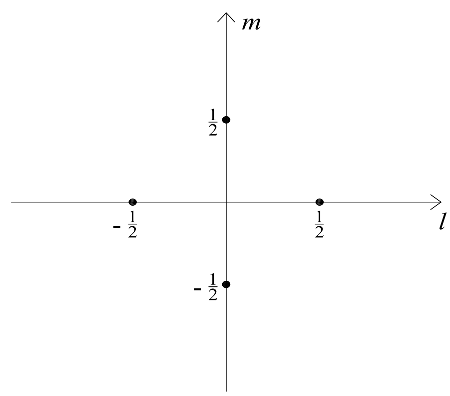

Figure 6: The weight diagram of the representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$.

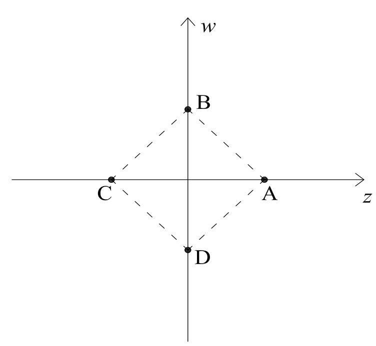

Figure 7: The square in the $z-w$ plane: each vertex coincides with the weight of the reducible representation $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$.

To restrict the choice of z and w, between the points representing the weights of $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$, allowed to transform a problem with infinite solutions to a problem having a finite number of solutions. In order to obtain the matrix of the change of basis P by the eigenspaces $V_{-1/4}$ and $V_{1/4}$, we need two different points on the diagram of fig. 7, through which it will be possible to construct a basis for such vector spaces. All the possible ordered couple of points, two by two different, which can be considered by the diagram of fig. 7, are

$$AB,\ BC,\ CD,\ DA,\ CA,\ DB,\ BD,\ BA,\ CB,\ DC,\ AD,\ AC. \tag{292}$$

Therefore, in general, there are twelve matrices of the change of basis (*i.e.* there are twelve coordinate systems), which are able to resolve our problem. Really, it is easy to prove that AC, CA, DB and BD give singular matrices P, since their determinants are null (and so, for them, it is impossible to diagonalize our system of matrices). Moreover, the matrices P deriving from AD, DA, DC, CD are of opposite sign with respect to those which are obtained from AB, BA, BC, CB and, for this, they lead to the same coordinate systems in which the matrices $\chi^{\mu}_{\otimes}$ and $\xi^{\mu}_{\otimes}$ are diagonal. For that reason, the only independent and invertible matrices of the change of basis are those which are obtained by the ordered couple of points

$$AB,\ BC,\ CB,\ BA. \tag{293}$$

This fact, from a figurative point of view, corresponds to cover the angle $\widehat{ABC}$ before counter-clockwise and then clockwise, as shown in fig. 8.[77]

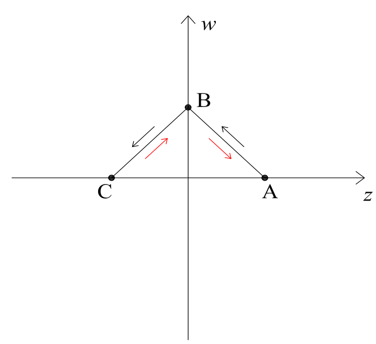

Figure 8: The magic angle: the two only independent representations for $\chi^{\mu}_{\otimes}$ and $\xi^{\mu}_{\otimes}$ can be obtained covering $\widehat{ABC}$ in the double-verse of the arrows.

[77] Otherwise, if it is preferred, before covering the triangle ABC counter-clockwise and then clockwise, where the contribution of the sides CA and AC does not count, since it gives singular matrices P.

It is straightforward to verify that, in our specific case, the sides AB, BC, CB, BA give, respectively, the following matrices of change of basis and their inverses

$$P_1 = \frac{1}{2}\begin{pmatrix} -i\sigma_2 & i\sigma_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \quad P_1^{-1} = \begin{pmatrix} i\sigma_2 & \mathbb{1}_2 \\ -i\sigma_2 & \mathbb{1}_2 \end{pmatrix} \tag{294}$$

$$P_2 = \frac{1}{2}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{1}_2 \\ -i\sigma_2 & -i\sigma_2 \end{pmatrix}, \; P_2^{-1} = \begin{pmatrix} -\mathbb{1}_2 & i\sigma_2 \\ \mathbb{1}_2 & i\sigma_2 \end{pmatrix} \tag{295}$$

$$P_3 = \frac{1}{2}\begin{pmatrix} -\sigma_3 & \sigma_3 \\ \sigma_1 & \sigma_1 \end{pmatrix}, \quad P_3^{-1} = \begin{pmatrix} -\sigma_3 & \sigma_1 \\ \sigma_3 & \sigma_1 \end{pmatrix} \tag{296}$$

$$P_4 = \frac{1}{2}\begin{pmatrix} -\sigma_1 & \sigma_1 \\ -\sigma_3 & -\sigma_3 \end{pmatrix}, \; P_4^{-1} = \begin{pmatrix} -\sigma_1 & -\sigma_3 \\ \sigma_1 & -\sigma_3 \end{pmatrix}. \tag{297}$$

With a little of calculations, it is simple to understand that (296) and (297) lead to the same coordinate systems of the (295) and (294), respectively. Hence, only the (294) and (295) get to two different representations of $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$. In the case of (294), we have [78]

$$\tilde{\chi}^0_\otimes = \begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \qquad \tilde{\xi}^0_\otimes = \begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \tag{298}$$

$$\tilde{\chi}^1_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^1_\otimes = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \tag{299}$$

$$\tilde{\chi}^2_\otimes = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \qquad \tilde{\xi}^2_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \tag{300}$$

$$\tilde{\chi}^3_\otimes = \frac{1}{4}\begin{pmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^3_\otimes = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix}, \tag{301}$$

[78] It is underlined $\tilde{\chi}^\mu_\otimes \equiv P_i^{-1}\chi^\mu_\otimes P_i$ and $\tilde{\xi}^\mu_\otimes \equiv P_i^{-1}\xi^\mu_\otimes P_i$, where $i \in \{1, 2\}$ in the studied case.

while, in the case of (295), we have [79]

$$\tilde{\chi}^0_{\otimes} = \begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \qquad \tilde{\xi}^0_{\otimes} = \begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \tag{302}$$

$$\tilde{\chi}^1_{\otimes} = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^1_{\otimes} = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \tag{303}$$

$$\tilde{\chi}^2_{\otimes} = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \qquad \tilde{\xi}^2_{\otimes} = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \tag{304}$$

$$\tilde{\chi}^3_{\otimes} = \frac{1}{4}\begin{pmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^3_{\otimes} = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix}. \tag{305}$$

As it is quickly seen, the representations obtained from (294) and (295) are the same, apart from the substitution

$$\sigma_3 \to -\sigma_3. \tag{306}$$

This is not banal, because it will affect the equations for the right- and left-handed fields.

Therefore, we obtained the diagonal representations of the matrices $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$ and, thanks to the use of the weight diagram belonging to the specific representation of the group of Lorentz $L^\uparrow_+$ with regard to which we are working, it has been possible to transform our problem with infinite solutions in a problem with a finite number of solutions. From that reason, we understand the equations for the right- and left-handed fields – obtained in the previous pages by the AαE and SαE – have not the same order of these last, because the representation of the matrices $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$, on which we worked, was not the right one, *i.e.* suitable for the representation

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right). \tag{307}$$

At this point, we re-calculate the equations for the right- and left-handed fields from the (270) and (271), by using the new representations. With the representation obtained from the (294), we have

[79] If one controls the characters of the representations obtained from (294) and (295), it is observed these are equal to the characters of the representations $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$ and so they are equivalent.

$$\left(i\tilde{\chi}^{\mu}_{\otimes}\partial_{\mu} - m\mathbb{1}_4\right)\Psi_{\otimes}(x) = \left[i\left(\tilde{\chi}^{0}_{\otimes}\frac{\partial}{\partial t} + \vec{\tilde{\chi}}_{\otimes}\cdot\vec{\nabla}\right) - m\mathbb{1}_4\right]\Psi_{\otimes}(x) = 0, \tag{308}$$

from which, the spinorial equations for the no coupled right- and left-handed fields are

$$\begin{cases} i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) + \sigma_3\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_R(\vec{x},t) = -m\Psi_R(\vec{x},t) \\ i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) - \sigma_3\left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_L(\vec{x},t) = m\Psi_L(\vec{x},t). \end{cases} \tag{309}$$

We, thus, obtained two distinct equations for the right- and left-handed fields of the same order of the AαE. Such equations, like the Weyl ones, are not invariant under the parity operator, but they include a nonzero mass. The equations deduced from the (309), by placing $m = 0$, are always distinct and not invariant under parity.[80]

Regarding the SαE, by using the representation $\tilde{\xi}^{\mu}_{\otimes}$ derived from the (294), we have

$$\left(\tilde{\xi}^{\mu}_{\otimes}\tilde{\xi}^{\nu}_{\otimes}\partial_{\mu}\partial_{\nu} + m^2\mathbb{1}_4\right)\psi_{\otimes}(x) = 0, \tag{310}$$

from which, making out all the matrix sums and products, the following two equations are obtained

$$\Bigg\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z} - \sigma_3\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z} - \frac{\partial^2}{\partial t\partial x}\right) - \sigma_3\frac{\partial^2}{\partial t\partial y}\right]\Bigg\}\psi_R(\vec{x},t) = -m^2\psi_R(\vec{x},t) \tag{311}$$

$$\Bigg\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z} + \sigma_3\left(\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z} + \frac{\partial^2}{\partial t\partial x}\right) + \sigma_3\frac{\partial^2}{\partial t\partial y}\right]\Bigg\}\psi_L(\vec{x},t) = -m^2\psi_L(\vec{x},t), \tag{312}$$

[80] It must be noticed the equations (309) have an imaginary coefficient. They become homogeneous differential equations with real coefficients for $m = 0$ only. For eliminating the imaginary unit, we simply can multiply the first equation by $i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) + \sigma_3\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial}{\partial y}\right)\right]$ and the second one by $i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) - \sigma_3\left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial}{\partial y}\right)\right]$, both members on the left-hand sides. In such a manner we obtain equations without imaginary unit, but having a different order compared to the AαE. For that reason, we prefer using the expressions (309). It is better to note that if we used the tachyonic equation (114), in order to obtain equations for right- and left-handed fields, we would have achieved the same equations (309), but without the imaginary unit i. Shall we give a physical sense to these equations?

which are of the same order of the SαE.

As it is immediate to state, they are not invariant under the parity operator and have a nonzero mass. If one places $m = 0$, they always remain distinct and not invariant under parity. What result we obtain if the representation derived from the (295) is used? By making for the AαE the same line of reasoning which led us to the equations as soon as written, we have

$$\begin{cases} i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) - \sigma_3\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_R(\vec{x},t) = -m\Psi_R(\vec{x},t) \\ i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) + \sigma_3\left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_L(\vec{x},t) = m\Psi_L(\vec{x},t) \end{cases} \tag{313}$$

and, for the SαE, we have

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z} + \sigma_3\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z} - \frac{\partial^2}{\partial t\partial x}\right) + \sigma_3\frac{\partial^2}{\partial t\partial y}\right]\right\}\psi_R(\vec{x},t) = -m^2\psi_R(\vec{x},t) \tag{314}$$

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z} - \sigma_3\left(\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z} + \frac{\partial^2}{\partial t\partial x}\right) - \sigma_3\frac{\partial^2}{\partial t\partial y}\right]\right\}\psi_L(\vec{x},t) = -m^2\psi_L(\vec{x},t). \tag{315}$$

As it could be expected, these equations are identical with those deduced by the representation deriving from the (294), unless the substitution

$$\sigma_3 \to -\sigma_3. \tag{316}$$

This means the equations for the right- and left-handed fields suffer of direction of the third spin component, depending on whether it is “up” or “down.” Therefore, matching the results of the representations deriving from the (294) and (295) through the diagonalization of $\chi^1_\otimes$, we have that the equations for the right- and left-handed fields gotten by the AαE are

$$\begin{cases} i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right) \pm \sigma_3\left(\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_R(\vec{x},t) = -m\Psi_R(\vec{x},t) \\ i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) \mp \sigma_3\left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial}{\partial y}\right)\right]\Psi_L(\vec{x},t) = m\Psi_L(\vec{x},t), \end{cases} \tag{317}$$

while the equations for the right- and left-handed fields, obtained by the SαE, are

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2}+\frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)-\frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z}\mp\sigma_3\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial z}\right)\right]-\right.$$
$$\left.\frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z}-\frac{\partial^2}{\partial t\partial x}\right)\mp\sigma_3\frac{\partial^2}{\partial t\partial y}\right]\right\}\psi_R(\vec{x},t)=-m^2\psi_R(\vec{x},t) \quad (318)$$

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2}+\frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x\partial z}\pm\sigma_3\left(\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y\partial z}\right)\right]-\right.$$
$$\left.\frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z}+\frac{\partial^2}{\partial t\partial x}\right)\pm\sigma_3\frac{\partial^2}{\partial t\partial y}\right]\right\}\psi_L(\vec{x},t)=-m^2\psi_L(\vec{x},t). \quad (319)$$

They, as already said, are not invariant under parity and have nonzero mass. Moreover, we can see that if the third spin component of the right-handed field is "down" that one of the left-handed field is always "up" and vice-versa: this means the right- and left-handed fields suffer from the direction of σ_3 concerning the particles of which they are constituted. We still notice that, if we had considered the points obtained from the fundamental weights multiplied simply by a generic $k\in\mathbb{R}$, we would have the same representations $\tilde{\chi}^\mu_\otimes$ and $\tilde{\xi}^\mu_\otimes$. This can be banally seen for the (294), which, in such a case, are transformed in

$$P_1=\frac{k}{2}\begin{pmatrix}-i\sigma_2 & i\sigma_2\\ \mathbb{1}_2 & \mathbb{1}_2\end{pmatrix},\quad P_1^{-1}=\frac{1}{k}\begin{pmatrix}i\sigma_2 & \mathbb{1}_2\\ -i\sigma_2 & \mathbb{1}_2\end{pmatrix} \quad (320)$$

which, naturally, does not change either $P_1^{-1}\chi^\mu_\otimes P_1$ nor $P_1^{-1}\xi^\mu_\otimes P_1$. This means the fundamental weights of the Lie algebra representation, that we are studying, characterize the "elementary cell" of all the geometric structures which they can generate through the multiplication by an arbitrary real constant, *i.e.* considering the frame of fig. 9, all the infinite representations, which can be constructed by the points of such a diagram, are equivalent to those which can be deduced by the elementary cell.

In order to conclude the study about the right- and left-handed fields, through the use of the AαE and SαE, we have to diagonalize the matrix $\chi^2_\otimes$. In this case, the eigenspaces, thanks to which calculating the matrices of the change of basis and their inverses, are

$$V_{-1/4}=\Big\{(w,z,z,w):z,w\in\mathbb{R}\Big\} \quad (321)$$

$$V_{1/4}=\Big\{(-w,-z,z,w):z,w\in\mathbb{R}\Big\}, \quad (322)$$

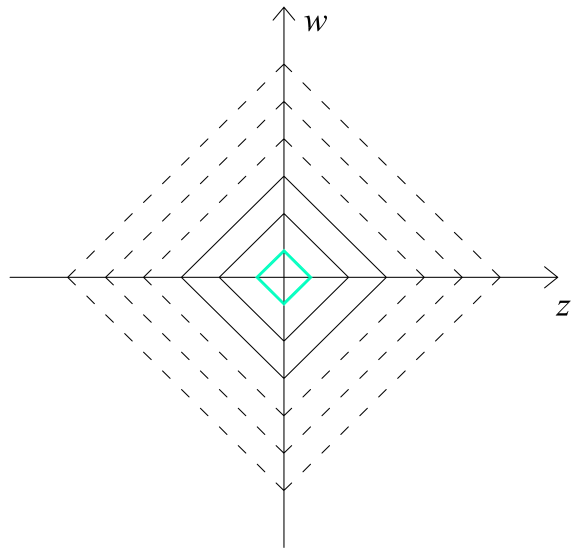

Figure 9: The infinite geometric structure generated by the “elementary cell” of $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$.

from which, making the same reasoning used for $\chi^1_{\otimes}$ and by taking the angle of fig. 8, we have

$$\tilde{P}_1=\frac{1}{2}\begin{pmatrix}\sigma_1 & -\sigma_1\\ \mathbb{1}_2 & \mathbb{1}_2\end{pmatrix},\quad \tilde{P}_1^{-1}=\begin{pmatrix}\sigma_1 & \mathbb{1}_2\\ -\sigma_1 & \mathbb{1}_2\end{pmatrix} \tag{323}$$

$$\tilde{P}_2=\frac{1}{2}\begin{pmatrix}\sigma_3 & -\sigma_3\\ -i\sigma_2 & -i\sigma_2\end{pmatrix},\ \tilde{P}_2^{-1}=\begin{pmatrix}\sigma_3 & i\sigma_2\\ -\sigma_3 & i\sigma_2\end{pmatrix} \tag{324}$$

$$\tilde{P}_3=\frac{1}{2}\begin{pmatrix}-i\sigma_2 & i\sigma_2\\ \sigma_3 & \sigma_3\end{pmatrix},\quad \tilde{P}_3^{-1}=\begin{pmatrix}i\sigma_2 & \sigma_3\\ -i\sigma_2 & \sigma_3\end{pmatrix} \tag{325}$$

$$\tilde{P}_4=\frac{1}{2}\begin{pmatrix}\mathbb{1}_2 & -\mathbb{1}_2\\ \sigma_1 & \sigma_1\end{pmatrix},\quad \tilde{P}_4^{-1}=\begin{pmatrix}\mathbb{1}_2 & \sigma_1\\ -\mathbb{1}_2 & \sigma_1\end{pmatrix}. \tag{326}$$

Also in such a case, the (325) and (326) lead to the same coordinate system of the (323) and (324), respectively. Hence, like $\chi^1_\otimes$, also $\chi^2_\otimes$ has two only representations for the $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$, which can be derived from the (323) and (324). With regards to the (323), we have

$$\tilde{\chi}^0_\otimes = \begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \qquad \tilde{\xi}^0_\otimes = \begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \tag{327}$$

$$\tilde{\chi}^1_\otimes = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \qquad \tilde{\xi}^1_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \tag{328}$$

$$\tilde{\chi}^2_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^2_\otimes = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \tag{329}$$

$$\tilde{\chi}^3_\otimes = \frac{1}{4}\begin{pmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^3_\otimes = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix}, \tag{330}$$

while, for the (324), we have [81]

$$\tilde{\chi}^0_\otimes = \begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \qquad \tilde{\xi}^0_\otimes = \begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_3 \end{pmatrix} \tag{331}$$

$$\tilde{\chi}^1_\otimes = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \qquad \tilde{\xi}^1_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \tag{332}$$

$$\tilde{\chi}^2_\otimes = \frac{1}{4}\begin{pmatrix} -\mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^2_\otimes = \frac{1}{4}\begin{pmatrix} \sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix} \tag{333}$$

$$\tilde{\chi}^3_\otimes = \frac{1}{4}\begin{pmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{1}_2 \end{pmatrix} \qquad \tilde{\xi}^3_\otimes = \frac{1}{4}\begin{pmatrix} -\sigma_3 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\sigma_3 \end{pmatrix}. \tag{334}$$

[81] Also in this case the representations obtained from (323) and (324) are equivalent to $\chi^\mu_\otimes$ and $\xi^\mu_\otimes$, since their characters coincide.

It is immediate to notice that the representations derived by the (323) and (324) are equal, apart from the substitution

$$\sigma_3 \to -\sigma_3. \tag{335}$$

Really, if we confront such representations with those obtained by the diagonalization of $\chi^1_\otimes$, we see they are exactly the same, unless the exchanges

$$\tilde{\chi}^1_\otimes \to \tilde{\chi}^2_\otimes, \quad \tilde{\xi}^1_\otimes \to \tilde{\xi}^2_\otimes. \tag{336}$$

What do such exchanges in the equations of the motion for the right- and left-handed fields entail? For seeing it, we compute such equations from the (270) and (271), joining together those obtained by the representations deriving from the (323) and (324), like already made in the case of $\chi^1_\otimes$. By making it, we have, for the AαE, the following equations for the right- and left-handed fields

$$\begin{cases} i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right) \pm \sigma_3\left(\frac{\partial}{\partial t}+\frac{1}{4}\frac{\partial}{\partial x}\right)\right]\psi_R(\vec{x},t) = -m\psi_R(\vec{x},t) \\ i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) \mp \sigma_3\left(\frac{\partial}{\partial t}-\frac{1}{4}\frac{\partial}{\partial x}\right)\right]\psi_L(\vec{x},t) = m\psi_L(\vec{x},t), \end{cases} \tag{337}$$

while, for the SαE, we have

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) - \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial y\partial z} \mp \sigma_3\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial x\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z}-\frac{\partial^2}{\partial t\partial y}\right) \mp \sigma_3\frac{\partial^2}{\partial t\partial x}\right]\right\}\psi_R(\vec{x},t) = -m^2\psi_R(\vec{x},t) \tag{338}$$

$$\left\{\mathbb{1}_2\frac{\partial^2}{\partial t^2} + \frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) + \frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial y\partial z} \pm \sigma_3\left(\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial x\partial z}\right)\right] - \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial z}+\frac{\partial^2}{\partial t\partial y}\right) \pm \sigma_3\frac{\partial^2}{\partial t\partial x}\right]\right\}\psi_L(\vec{x},t) = -m^2\psi_L(\vec{x},t). \tag{339}$$

Also in such a case, our equations have $m \neq 0$ and are not invariant under parity. Moreover, if the third spin component of the equation for the right-handed field is "down" that for the left-handed field is "up" and vice-versa. But the most evident thing is these equations are perfectly identical to the (317), (318) and (319), apart from the substitution

$$x \to y. \tag{340}$$

This means that to choose if diagonalizing $\chi^1_\otimes$ or $\chi^2_\otimes$ is equivalent to exchange the x-axis with the y-axis (or vice-versa), in the coordinate system in which we study the motion of the right- and left-handed fields. Since the equations of the motion suffer of such an exchange, we may conclude such fields are subject to a kind of "polarization."

Finally, by summarizing the results of this section, we can assert the equations for the right- and left-handed fields derived by the AαE are [82]

$$\begin{cases} i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_3}\right)\pm\sigma_3\left(\frac{\partial}{\partial t}+\frac{1}{4}\frac{\partial}{\partial x_j}\right)\right]\psi_R(\vec{x},t)=-m\psi_R(\vec{x},t)\\ i\left[\frac{\mathbb{1}_2}{4}\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial x_3}\right)\mp\sigma_3\left(\frac{\partial}{\partial t}-\frac{1}{4}\frac{\partial}{\partial x_j}\right)\right]\psi_L(\vec{x},t)=m\psi_L(\vec{x},t), \end{cases} \tag{341}$$

while the equations for the right- and left-handed fields derived from SαE are

$$\Bigg\{\mathbb{1}_2\frac{\partial^2}{\partial t^2}+\frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)-\frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x_i\partial x_3}\mp\sigma_3\left(\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial x_j\partial x_3}\right)\right]- \\ \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial x_3}-\frac{\partial^2}{\partial t\partial x_i}\right)\mp\sigma_3\frac{\partial^2}{\partial t\partial x_j}\right]\Bigg\}\psi_R(\vec{x},t)=-m^2\psi_R(\vec{x},t) \tag{342}$$

$$\Bigg\{\mathbb{1}_2\frac{\partial^2}{\partial t^2}+\frac{\mathbb{1}_2}{16}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)+\frac{1}{8}\left[\mathbb{1}_2\frac{\partial^2}{\partial x_i\partial x_3}\pm\sigma_3\left(\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial x_j\partial x_3}\right)\right]- \\ \frac{1}{2}\left[\mathbb{1}_2\left(\frac{\partial^2}{\partial t\partial x_3}+\frac{\partial^2}{\partial t\partial x_i}\right)\pm\sigma_3\frac{\partial^2}{\partial t\partial x_j}\right]\Bigg\}\psi_L(\vec{x},t)=-m^2\psi_L(\vec{x},t), \tag{343}$$

where, for all the previous equations, we have

$$\begin{cases} i=1,\ j=2 \text{ if we diagonalize } \chi^1_\otimes\\ i=2,\ j=1 \text{ if we diagonalize } \chi^2_\otimes. \end{cases} \tag{344}$$

We, thus, found a couple of distinct equations for the right- and left-handed fields of the same order of the AαE and SαE, respectively. Naturally, the equations for the right- and left-handed fields derived by the SαE are more complex than those obtained from the AαE. Nevertheless, in this last

[82] In this case, we place $\vec{x}\equiv(x_1,x_2,x_3)$.

case, the signs of the mass concerning the right- and left-handed fields are contrary and this places the found equations on a different level, in the sense that, as already explained in previous chapter, they derive by two different theories and cannot be simply seen like one the square of the other.

If we identify the left-handed particles with neutrinos and the right-handed particles with anti-neutrinos, we can assert the found equations show the following properties

1. Neutrinos and anti-neutrinos have mass $m \neq 0$.[83]

2. Neutrinos and anti-neutrinos do not conserve the parity.

3. Since neutrinos and anti-neutrinos have different equations of motion, they are different particles.

4. If the third spin component of neutrinos is "up" that one of anti-neutrinos must be "down" and vice-versa.

5. The neutrinos and anti-neutrinos equations suffer from the spin of the particles they describe, in the sense that it is fundamental to specify if the third spin component is "up" or "down": this means the sign of the third spin component enters into dynamics of such particles.

6. Except if the Nature did not choose a privileged direction for neutrinos and anti-neutrinos (which is equivalent to decide if to diagonalize $\chi^1_\otimes$ or $\chi^2_\otimes$), such particles show a *sort* of polarization, which leaves fixed the z-direction, but can change the x with the y and vice-versa.[84]

It is well to notice such properties can be deducted from the equations (270) and (271), which have been derived by the AαE and SαE, without using *ad hoc* hypothesis. What between these two couples of equations better describes neutrinos and anti-neutrinos, it depends on which between the AαE and SαE is the most adapted to describe elementary particles and this needs ulterior studies on the theory exposed in this work. It is very important to stop on the predictive power of the found equations. In fact, they describe right- and left-handed fields with nonzero mass, with distinct equations which are not invariant under parity. These, as we know, are all characteristics characterizing the neutrinos and anti-neutrinos based on current knowledges (the neutrino and anti-neutrino fields are just identified with left- and right-handed fields, respectively). And these results, as it should be well emphasized, have spontaneously emerged from the AαE and SαE, without some further physical hypothesis. Precisely, our couples of equations are always distinct, contain nonzero masses and violate the parity, all properties that the Weyl and Majorana theories do not predict at the same time. In particular, the Majorana equation predicts that particles are their own anti-particles and so neutrinos are equivalent to anti-neutrinos [14, 98]. Up till now, all the experiments on "double-beta decay" have discouraged such forecasts [15]. But, there is yet another issue, perhaps more important. In fact, as it is known the Weyl theory is nothing but a special case of the Dirac theory placed in the chiral representation and with $m = 0$. What from this follows is: the Dirac theory cannot predict the parity violation of right- and left-handed particles and they

[83] This could have caused the "flavour oscillations" experimentally observed.

[84] Depending on how our coordinate system is chosen.

have $m \neq 0$ together. Then, as the points 4, 5 and 6 show, ulterior properties of neutrino and anti-neutrino, which could be experimentally tested, arise from our equations.

At the end of this long discussion on the left- and right-handed equations concerning the theory introduced in this work, it is important to notice the association between the right-handed field and ψ_R (or ψ_R) and the left-handed field and ψ_L (or ψ_L) is a pure convention. In fact, the same results we have obtained in these pages are valid exchanging the field ψ_R (or ψ_R) with ψ_L (or ψ_L) too. Really, this have not to demoralize us, because it is not fundamental naming our fields in a precise way, but rather the quality of the equations (341), (342) and (343), which are not invariant under parity and with $m \neq 0$. After all, if we think about it, associating neutrino to the left-handed field and anti-neutrino to the right-handed field depends exclusively on the Weyl equations, which, making clear the helicity operator, conducted the physicists to do such a distinction between these two elementary particles. As an example, this does not happen into Majorana theory, since the characteristic equation of this model shows neutrino and anti-neutrino are the same particle. Instead, the (270) and (271) tell us that, by studying the equations of motion contracted to the representation $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ of the Lorentz group $L_+^\uparrow$, we get distinct equations having the above enunciated properties. Can our equations actually describe neutrinos and anti-neutrinos? Is the neutrino phenomenology more suitable to the equation of ψ_R (or ψ_R) or to that of ψ_L (or ψ_L)? Hence, we understand that the complexity of the found equations demands a deeper analysis on the physical behavior of neutrinos and anti-neutrinos. What, however, relieves us is that all the obtained results have come out simply by the mathematics of AαE and SαE and so they objectively let us reflect and invite to further inquire.

4 The α-Theory and External Gauge Fields

The AαE and SαE, developed in previous chapters, characterize two theories, which are proposed to explain the elementary particle physics and as our universe was born. Concerning what we said, only one of them could represent the new theory for elementary particles. In this chapter, the Lagrangian densities of the AαE and SαE will be obtained, and the conserved quantities arising from their invariance under the space-time translation group will be calculated, based on the Noether theorem. Successively, the coupling between these theories with an external electromagnetic field and, more in general, with a Yang-Mills field will be studied. This will concur to write the Lagrangian density of the strong interaction for both these theories (α-QCD). In addition, we will find the free propagators of the AαE and SαE, waiting for the development of a more correct perturbative model.

4.1 Lagrangian formalism of the α-Theory: invariance and conservation laws

We want now to construct the physical theory identified by the AαE and SαE. Of this theory, then, we will calculate the relative conserved amounts, deriving from the eventual invariance under space-time translations. The transition to the Lagrangian formalism is required by the fundamental circumstance that the found equations, like the Dirac and Klein-Gordon ones, do not describe single particles, but particle fields. This will allow us, as we will see, to apply the powerful formalism of the QFT to the theory developed in this work.

What name do we give to this theory? Since it conducts to AαE or SαE, which are proposed to explain the transition from the tachyonic (post Big-Bang background) to the bradyonic universe (actual background), thanks to a *big* process of spontaneous symmetry breaking that we called Big-Break, it seems natural to name it "α-Theory," *i.e.* the "beginning theory." The writer hopes someday this theory, if verified, will improve the Standard Model of elementary particles and, thus, will render it a complete theoretical model (unification of all interactions of Nature). From this point of view, the α-Theory should be really the theory beyond the Standard Model.

Another thing which one must emphasize is that the AαE and SαE identify two different theories. Therefore, we call "Asymmetric α-Theory" (AαT) the theory characterized by the AαE, while "Symmetric α-Theory" (SαT) that one characterized by the SαE. When it will be settled down if and what between these two theories is the most suitable for describing elementary particles, then we will call such a theory simply α-Theory. We indicate the Lagrangian density of the "Asymmetric α-Theory" with $\mathcal{L}_{A\alpha}$, while the one concerning the "Symmetric α-Theory" with $\mathcal{L}_{S\alpha}$. We now explicitly calculate these Lagrangian densities. For making it, we have to perform the substitution $\mu \to -im$ in the densities $\mathcal{L}_{M1}$ and $\mathcal{L}_{M2}$ written in the previous pages for characterizing the tachyonic universe, which depends on whether this universe was described by the first order M_α equation or to the second order one. We, therefore, have

$$\mathcal{L}_{M1} \overset{\mu\to -im}{\longrightarrow} \mathcal{L}_{A\alpha} = \bar{\psi}_s(x)\left(i\hbar\chi^\mu\partial_\mu - mc\mathbb{1}_s\right)\psi_s(x) \tag{345}$$

$$\mathcal{L}_{M2} \overset{\mu\to -im}{\longrightarrow} \mathcal{L}_{S\alpha} = (\partial_\mu\psi_s^\dagger)\xi^\mu\xi^\nu(\partial_\nu\psi_s) - \frac{m^2c^2}{\hbar^2}\psi_s^\dagger(x)\psi_s(x). \tag{346}$$

It promptly can be noticed that $\mathcal{L}_{A\alpha}$ has the same structure of the Dirac Lagrangian density

$$\mathcal{L}_{Dirac} = \bar{\psi}(x)\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi(x), \tag{347}$$

while $\mathcal{L}_{S\alpha}$ *remembers* the Klein-Gordon Lagrangian density for a n-dimensional complex field [85]

$$\mathcal{L}_{K-G} = (\partial_\mu\psi^\dagger)(\partial^\mu\psi) - \frac{m^2c^2}{\hbar^2}\psi^\dagger(x)\psi(x). \tag{348}$$

This makes us understand the α-Theory, describing the elementary particles in a unified way (arbitrary spin), is really hidden in the Dirac and Klein-Gordon theories.

Now we can study the behavior of the theories characterized by $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$ under space-time translations. In order to slim the calculations, we work in natural units ($\hbar = c = 1$), in which our Lagrangian densities are written

$$\mathcal{L}_{A\alpha} = \bar{\psi}_s(x)\left(i\chi^\mu\partial_\mu - m\mathbb{1}_s\right)\psi_s(x) \tag{349}$$

$$\mathcal{L}_{S\alpha} = (\partial_\mu\psi_s^\dagger)\xi^\mu\xi^\nu(\partial_\nu\psi_s) - m^2\psi_s^\dagger(x)\psi_s(x). \tag{350}$$

We want to begin from the $\mathcal{L}_{A\alpha}$. After a space-time translation

$$x'^\mu = x^\mu + a^\mu,$$

it becomes (the χ^μ obviously remain unchanged because independent from x)

$$\mathcal{L}'_{A\alpha} = \bar{\psi}'_s(x')\left(i\chi^\mu\partial'_\mu - m\mathbb{1}_s\right)\psi'_s(x'). \tag{351}$$

[85] Note that

$$\psi(x) \equiv \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix},\ \psi_i \in \mathbb{C}\ \ \forall i \in \{1,2,3,4\};\quad \psi(x) \equiv \begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix},\ \psi_i \in \mathbb{C}\ \ \forall i \in \{1,\ldots,n\}.$$

Since $\partial'_\mu = \partial_\mu$ and due to the homogeneity of space-time, it is immediate to prove that

$$\mathcal{L}'_{A\alpha} = \mathcal{L}_{A\alpha} \Rightarrow \delta\mathcal{L}_{A\alpha} = (\mathcal{L}'_{A\alpha} - \mathcal{L}_{A\alpha}) = 0. \tag{352}$$

Thence, the system described by $\mathcal{L}_{A\alpha}$ is invariant under space-time translations. Based on the Noether theorem, this gives conserved charges and four-current of the following form [86]

$$\begin{cases} \mathsf{j}_s^{\mu\nu} = -i\bar{\psi}_s\chi^\mu\partial^\nu\psi_s \\ \mathsf{Q}_s^0 = \int_{\mathbb{R}^3} d^3x \left(i\psi_s^\dagger \frac{\partial\psi_s}{\partial t}\right), \quad \mathsf{Q}_s^i = \int_{\mathbb{R}^3} d^3x \left(i\psi_s^\dagger \frac{\partial\psi_s}{\partial x_i}\right). \end{cases} \tag{353}$$

We now try to understand what these charges represent. For making it, we calculate the Hamiltonian density concerning our theory. We have

$$\mathcal{H}_{A\alpha} = \pi_s(x)\dot{\psi}_s(x) - \mathcal{L}_{A\alpha}, \tag{354}$$

where, with $\pi_s(x)$, we indicated the (canonical) momentum of the field $\psi_s(x)$ given by

$$\pi_s(x) \equiv \frac{\partial\mathcal{L}_{A\alpha}}{\partial\dot{\psi}_s}. \tag{355}$$

From the explicit calculation of $\mathcal{H}_{A\alpha}$ and $\pi_s(x)$, we get

$$\begin{cases} \mathcal{H}_{A\alpha} = i\psi_s^\dagger \frac{\partial\psi_s}{\partial t} \\ \pi_s(x) = i\psi_s^\dagger(x) \end{cases} \tag{356}$$

and so we can thus write the conserved charges

$$\begin{cases} \mathsf{Q}_s^0 = \int_{\mathbb{R}^3} d^3x \left(i\psi_s^\dagger \frac{\partial\psi_s}{\partial t}\right) = \int_{\mathbb{R}^3} d^3x \mathcal{H}_{A\alpha} = H_{A\alpha} \\ \mathsf{Q}_s^i = \int_{\mathbb{R}^3} d^3x \left(i\psi_s^\dagger\partial^i\psi_s\right) = \int_{\mathbb{R}^3} d^3x \left(\pi_s\partial^i\psi_s\right). \end{cases} \tag{357}$$

Therefore, the charge Q_s^0 is the Hamiltonian of the system described by $\mathcal{L}_{A\alpha}$, while Q_s^i are the components of the momentum transported by the field $\psi_s(x)$. Hence, we demonstrated that the

[86] The conserved charges, thus like the tensor four-current, are really s, that is, for each field characterized by a fixed spin value, there are precise conserved charges and four-current.

Lagrangian density is invariant under space-time translations (*i.e.* under the group of the four-dimensional translations) and this invariance gives, in agreement with the Noether theorem, four conserved charges, namely the energy and the three components of the momentum transported by the field $\psi_s(x)$, respectively (of course, this happens for any fixed $s \in \mathbb{N}/2$).

We now study the behavior of the Lagrangian density $\mathcal{L}_{S\alpha}$ under space-time translations

$$x'^\mu = x^\mu + a^\mu.$$

Naturally, we have

$$\mathcal{L}'_{S\alpha} = (\partial'_\mu \psi'^\dagger_s)\xi^\mu \xi^\nu (\partial'_\nu \psi'_s) - m^2 \psi'^\dagger_s(x')\psi'_s(x'), \tag{358}$$

and by remarking that

$$\begin{cases} \partial'_\mu = \partial_\mu \\ \psi'_s(x') = \psi_s(x) \\ \psi'^\dagger_s(x') = \psi^\dagger_s(x) \end{cases} \tag{359}$$

we can write

$$\mathcal{L}'_{S\alpha} = (\partial_\mu \psi^\dagger_s)\xi^\mu \xi^\nu (\partial_\nu \psi_s) - m^2 \psi^\dagger_s(x)\psi_s(x), \tag{360}$$

from which

$$\mathcal{L}'_{S\alpha} = \mathcal{L}_{S\alpha} \;\Rightarrow\; \delta\mathcal{L}_{S\alpha} = (\mathcal{L}'_{S\alpha} - \mathcal{L}_{S\alpha}) = 0. \tag{361}$$

Therefore, also the system described by $\mathcal{L}_{S\alpha}$ is invariant under space-time translations and so, from the Noether theorem, we have the following conserved charges and four-currents

$$\begin{cases} j^{\mu\nu}_s = -T^{\mu\nu} \\ Q^0_s = \int_{\mathbb{R}^3} d^3x T^{00}, \quad Q^i_s = \int_{\mathbb{R}^3} d^3x T^{0i}, \end{cases} \tag{362}$$

where

$$\begin{cases} T^{\mu\nu} = (\partial_\alpha \psi^\dagger_s)\xi^\alpha \xi^\mu (\partial^\nu \psi_s) + (\partial^\nu \psi^\dagger_s)\xi^\mu \xi^\alpha (\partial_\alpha \psi_s) - (\partial_\alpha \psi^\dagger_s)\xi^\alpha \xi^\beta (\partial_\beta \psi_s) g^{\mu\nu} + m^2 \psi^\dagger_s \psi_s g^{\mu\nu} \\ T^{00} = (\partial_\alpha \psi^\dagger_s)\xi^\alpha \xi^0 (\partial^0 \psi_s) + (\partial^0 \psi^\dagger_s)\xi^0 \xi^\alpha (\partial_\alpha \psi_s) - (\partial_\alpha \psi^\dagger_s)\xi^\alpha \xi^\beta (\partial_\beta \psi_s) + m^2 \psi^\dagger_s \psi_s \\ T^{0i} = (\partial_\alpha \psi^\dagger_s)\xi^\alpha \xi^0 (\partial^i \psi_s) + (\partial^i \psi^\dagger_s)\xi^0 \xi^\alpha (\partial_\alpha \psi_s). \end{cases} \tag{363}$$

In order to know the physical meaning of the previous charges, we calculate the Hamiltonian density $\mathcal{H}_{S\alpha}$, whose explicit form is

$$\mathcal{H}_{S\alpha} = \pi_\psi(x)\dot{\psi}_s(x) + \dot{\psi}_s^\dagger(x)\pi_{\psi^\dagger}(x) - \mathcal{L}_{S\alpha}, \tag{364}$$

where, with $\pi_\psi(x)$ and $\pi_{\psi^\dagger}(x)$, we indicated the momentum of the field $\psi_s(x)$ and the momentum of the field $\psi_s^\dagger(x)$ given, respectively, by

$$\begin{cases} \pi_\psi(x) \equiv \frac{\partial \mathcal{L}_{S\alpha}}{\partial \dot{\psi}_s} = (\partial_\alpha \psi_s^\dagger)\xi^\alpha \xi^0 \\ \pi_{\psi^\dagger}(x) \equiv \frac{\partial \mathcal{L}_{S\alpha}}{\partial \dot{\psi}_s^\dagger} = \xi^0 \xi^\alpha (\partial_\alpha \psi_s), \end{cases} \tag{365}$$

from which immediately follows

$$\mathcal{H}_{S\alpha} = (\partial_\alpha \psi_s^\dagger)\xi^\alpha \xi^0 (\partial^0 \psi_s) + (\partial^0 \psi_s^\dagger)\xi^0 \xi^\alpha (\partial_\alpha \psi_s) - (\partial_\alpha \psi_s^\dagger)\xi^\alpha \xi^\beta (\partial_\beta \psi_s) + m^2 \psi_s^\dagger \psi_s. \tag{366}$$

By comparing the (365) and (366) with Q_s^0 and Q_s^i, we can promptly write

$$\begin{cases} Q_s^0 = \int_{\mathbb{R}^3} d^3x \mathcal{H}_{S\alpha} = H_{S\alpha} \\ Q_s^i = \int_{\mathbb{R}^3} d^3x \left[\pi_\psi \partial^i \psi_s + \partial^i \psi_s^\dagger \pi_{\psi^\dagger}\right] = \int_{\mathbb{R}^3} d^3x \pi_\psi \partial^i \psi_s + \int_{\mathbb{R}^3} d^3x \partial^i \psi_s^\dagger \pi_{\psi^\dagger}. \end{cases} \tag{367}$$

Thence, the charge Q_s^0 is not other but the Hamiltonian of the system described by $\mathcal{L}_{S\alpha}$, while Q_s^i is the sum of each one of the components concerning the momentum transported by the field $\psi_s(x)$ and the momentum transported by the field $\psi_s^\dagger(x)$ and this naturally for any $s \in \mathbb{N}/2$.

Therefore, we demonstrated the Lagrangian density $\mathcal{L}_{S\alpha}$ is invariant under any space-time translation and this invariance gives, in agreement with the Noether theorem, four conserved charges, being the energy $H_{S\alpha}$ and the sum of each one of the components concerning the momentum transported by the fields $\psi_s(x)$ and $\psi_s^\dagger(x)$, respectively.

4.2 The α-Theory and electromagnetic interaction

Now we want to study the α-Theory in presence of an external electromagnetic field $A_\mu(x)$. Obviously, this implies the discussion of the AαT and SαT both in interaction with an external electromagnetic field. It is good to underline that the approach we will follow is inverse to the one usually used for studying the Dirac or the Klein-Gordon theories coupled with an electromagnetic field, which is founded on the "Gauge Principle," *i.e.* on the acknowledge that in the transition from global phase (gauge) transformations to local ones, in order to maintain the invariance of the start Lagrangian density, it is necessary to introduce a four-dimensional field coupled to the initial matter field, more the kinetic term of this new field, just coinciding with the electromagnetic field $A_\mu(x)$. At the end of this process, it is observed, through the introduction of the covariant derivative, that the Lagrangian density of the theory, we dealt in interaction with the electromagnetic field, is nothing but the initial theory to which the kinetic term of the electromagnetic field is added and to which the ordinary four-dimensional derivative is replaced with the covariant derivative. This type of substitution is said "minimal substitution" (sometimes also called "minimal coupled"). And it is just from the minimal substitution we will start, by showing the Lagrangian densities, obtained in this way from $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$, are invariant quantities under local phase transformations. Given, therefore, the Lagrangian density

$$\mathcal{L}_{A\alpha} = \bar{\psi}_s(x)\left(i\chi^\mu\partial_\mu - m\mathbb{1}_s\right)\psi_s(x), \tag{368}$$

unchanged under global phase transformations (banally, we can verify it)

$$\begin{cases} \psi'_s(x) = e^{i\theta}\psi_s(x) \\ \bar{\psi}'_s(x) = e^{-i\theta}\bar{\psi}_s(x), \end{cases} \tag{369}$$

we want to prove that the Lagrangian density, obtained by $\mathcal{L}_{A\alpha}$ through the minimal substitution $(\partial_\mu \to D_\mu + \mathcal{L}^{cin}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu})$, and given by [87]

$$\tilde{\mathcal{L}}_{A\alpha} = \bar{\psi}_s(x)\left(i\chi^\mu D_\mu - m\mathbb{1}_s\right)\psi_s(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{370}$$

is invariant under local phase transformations

$$\begin{cases} \psi'_s(x) = e^{i\theta(x)}\psi_s(x) \\ \bar{\psi}'_s(x) = e^{-i\theta(x)}\bar{\psi}_s(x) \\ A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x). \end{cases} \tag{371}$$

We have

$$\tilde{\mathcal{L}}_{A\alpha} = \bar{\psi}_s\left[i\chi^\mu(\partial_\mu - ieA_\mu) - m\mathbb{1}_s\right]\psi_s - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \\ \bar{\psi}_s(i\chi^\mu\partial_\mu - m\mathbb{1}_s)\psi_s + e\bar{\psi}_s(\chi^\mu A_\mu)\psi_s - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{372}$$

[87] $D_\mu \equiv (\partial_\mu - ieA_\mu)$.

where the second term of the last expression represents the interaction between the particle field and electromagnetic field, with coupling constant equal to the electric charge e. By replacing the local phase transformations in the (372), it is easy to verify that it is invariant under such transformations.

Since this Lagrangian density describes the interaction of the particle field having arbitrary spin (varying s) and its conjugate with the electromagnetic field, we can assert it represents the Lagrangian density of the "Asymmetric α-QED" and so to write

$$\mathcal{L}_{A\alpha}^{QED} = \bar{\psi}_s(x)\left(i\chi^\mu D_\mu - m\mathbb{1}_s\right)\psi_s(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{373}$$

from which, it is straightforward to calculate the equations of the motion for the fields $\psi_s(x)$ and $A_\mu(x)$, through the use of the Euler-Lagrange equations about these fields. They are [88]

$$(i\chi_\mu D^\mu - m\mathbb{1}_s)\psi_s(x) = 0 \tag{374}$$

$$\partial_\mu F^{\mu\nu} = e\mathrm{j}_s^\nu. \tag{375}$$

The (374) represents the asymmetric α-equation for the field $\psi_s(x)$ coupled to the electromagnetic field $A_\mu(x)$, while the (375) gives the inhomogeneous Maxwell's equations with four-current j_s^ν.

Now we want to prove that the four-current, defined as above, is just the four-current of the field $\psi_s(x)$ associated to the invariance of $\mathcal{L}_{A\alpha}$ under global phase transformations (369). In fact, thanks to this invariance, we have

$$\delta\mathcal{L}_{A\alpha} = (\mathcal{L}'_{A\alpha} - \mathcal{L}_{A\alpha}) = 0, \tag{376}$$

i.e., based on the Noether theorem, we have the following conserved charge and four-current

$$\begin{cases} \mathrm{j}_s^\nu = \frac{\partial\mathcal{L}_{A\alpha}}{\partial(\partial_\nu\psi_s)}\bar{\delta}\psi_s + \bar{\delta}\bar{\psi}_s\frac{\partial\mathcal{L}_{A\alpha}}{\partial(\partial_\nu\bar{\psi}_s)} + \mathcal{L}_{A\alpha}\bar{\delta}x^\nu \\ \mathsf{Q}_s(t) = \int_{\mathbb{R}^3} d^3x\mathrm{j}_s^0(\vec{x},t) \end{cases} \tag{377}$$

and since

$$\begin{cases} \bar{\delta}\psi_s = i\psi_s(x) \\ \bar{\delta}\bar{\psi}_s = -i\bar{\psi}_s(x) \\ \bar{\delta}x^\nu = 0, \end{cases} \tag{378}$$

[88] $\mathrm{j}_s^\nu \equiv -(\bar{\psi}_s\chi^\nu\psi_s)$.

we can write

$$\mathsf{j}_s^\nu = -(\bar{\psi}_s \chi^\nu \psi_s), \tag{379}$$

which is just the four-current we found like constant term of the previous inhomogeneous Maxwell's equations. Regarding the conserved charge $\mathsf{Q}_s(t)$, for making it positive, we note that, based on the continuity equation

$$\partial_\nu \mathsf{j}_s^\nu = 0, \tag{380}$$

we can also take

$$\mathsf{j}_s^\nu = (\bar{\psi}_s \chi^\nu \psi_s), \tag{381}$$

from which easily follows

$$\mathsf{Q}_s(t) = \int_{\mathbb{R}^3} d^3x (\bar{\psi}_s \chi^0 \psi_s) = \int_{\mathbb{R}^3} d^3x (\psi_s^\dagger \psi_s). \tag{382}$$

Therefore, the invariance of $\mathcal{L}_{A\alpha}$ under global gauge transformations gives

$$\begin{cases} \mathsf{j}_s^\nu = (\bar{\psi}_s \chi^\nu \psi_s) \\ \mathsf{Q}_s(t) = \int_{\mathbb{R}^3} d^3x (\psi_s^\dagger \psi_s). \end{cases} \tag{383}$$

It can be noted that if we had considered the local phase transformations

$$\begin{cases} \psi_s'(x) = e^{-i\theta(x)} \psi_s(x) \\ \bar{\psi}_s'(x) = e^{i\theta(x)} \bar{\psi}_s(x) \\ A_\mu'(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x), \end{cases} \tag{384}$$

in order to obtain the invariance of $\mathcal{L}_{A\alpha}$ under such transformations, we always would have to make the minimal substitution, but the covariant derivative would have to be defined with the plus sign, namely

$$D_\mu \equiv (\partial_\mu + ieA_\mu). \tag{385}$$

The equations of the motion always would have been the (374) and (375), with the only difference that the four-current j_s^ν would have been with the plus sign, *i.e.*

$$\mathsf{j}_s^\nu = (\bar{\psi}_s \chi^\nu \psi_s). \tag{386}$$

This four-current is the right conserved current of $\mathcal{L}_{A\alpha}$ under global phase transformations

$$\begin{cases} \psi_s'(x) = e^{-i\theta}\psi_s(x) \\ \bar{\psi}_s'(x) = e^{i\theta}\bar{\psi}_s(x). \end{cases} \tag{387}$$

What we saw teaches us that the theory derived by $\mathcal{L}_{A\alpha}$, such as the Dirac theory, suffers from a duplicity in the definition of the covariant derivative and in the consequent sign of the electromagnetic four-current,[89] tied to the type of global gauge transformation that is chosen.

We now study the symmetric α-Theory in interaction with an external electromagnetic field $A_\mu(x)$. Proceeding as in the previous pages, we want to show that, given

$$\mathcal{L}_{S\alpha} = (\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu (\partial_\nu \psi_s) - m^2 \psi_s^\dagger \psi_s, \tag{388}$$

invariant under global phase transformations (we banally can verify it)

$$\begin{cases} \psi_s'(x) = e^{i\theta}\psi_s(x) \\ \psi_s'^\dagger(x) = e^{-i\theta}\psi_s^\dagger(x), \end{cases} \tag{389}$$

the Lagrangian density

$$\tilde{\mathcal{L}}_{S\alpha} = (D_\mu \psi_s)^\dagger \xi^\mu \xi^\nu (D_\nu \psi_s) - m^2 \psi_s^\dagger \psi_s - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{390}$$

obtained by $\mathcal{L}_{S\alpha}$ through the minimal substitution, with the position

$$D_\mu \equiv (\partial_\mu - ieA_\mu), \tag{391}$$

is invariant under local phase transformations

[89] However, such a sign can be adjusted thanks to the continuity equation (380).

$$
\begin{cases} \psi_s'(x) = e^{i\theta(x)}\psi_s(x) \\ \psi_s'^{\dagger}(x) = e^{-i\theta(x)}\psi_s^{\dagger}(x) \\ A_\mu'(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x). \end{cases} \tag{392}
$$

Making clear in (390) the covariant derivative and making all the products, it is easy to obtain

$$
\tilde{\mathcal{L}}_{S\alpha} = (\partial_\mu\psi_s^\dagger)\xi^\mu\xi^\nu(\partial_\nu\psi_s) - m^2\psi_s^\dagger\psi_s + ieA_\mu(\psi_s^\dagger\xi^\mu\xi^\nu\partial_\nu\psi_s - \partial^\nu\psi_s^\dagger\xi^\nu\xi^\mu\psi_s) + e^2 A_\mu A_\nu\psi_s^\dagger(\xi^\mu\xi^\nu)\psi_s - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{393}
$$

where the third and fourth term represent the interaction between the particle field (with arbitrary spin s) and the electromagnetic field, of first and second order, respectively, in the coupling constant e. With some tedious calculation, it is not difficult to prove that $\tilde{\mathcal{L}}_{S\alpha}$ is invariant under local gauge transformations (392). Such a Lagrangian density, describing the connection between the field ψ_s and the field A_μ, represents the Lagrangian density of the "Symmetric α-QED." Hence, we can write

$$
\mathcal{L}_{S\alpha}^{QED} = (D_\mu\psi_s)^\dagger\xi^\mu\xi^\nu(D_\nu\psi_s) - m^2\psi_s^\dagger\psi_s - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{394}
$$

which gives, for the fields $\psi_s(x)$ and $A_\mu(x)$, the following equations of the motion [90]

$$
(\xi^\mu\xi^\nu D_\mu D_\nu + m^2\mathbb{1}_s)\psi_s(x) = 0 \tag{395}
$$

$$
\partial_\mu F^{\mu\nu} = e\mathcal{J}_s^\nu. \tag{396}
$$

The (395) represents the symmetric α-equation of the field $\psi_s(x)$ coupled with the electromagnetic field $A_\mu(x)$, while the (396) gives the inhomogeneous Maxwell's equations with four-current $\mathcal{J}_s^\nu$.

It can be verified such a four-current is just the Noether current deriving from the invariance of $\mathcal{L}_{S\alpha}^{QED}$ under local gauge transformations (392). In fact, in virtue of this invariance one has

$$
\begin{cases} \tilde{\mathcal{J}}_s^\nu = \frac{\partial\mathcal{L}_{S\alpha}^{QED}}{\partial(\partial_\nu\psi_s)}\bar{\delta}\psi_s + \bar{\delta}\psi_s^\dagger\frac{\partial\mathcal{L}_{S\alpha}^{QED}}{\partial(\partial_\nu\psi_s^\dagger)} + \frac{\partial\mathcal{L}_{S\alpha}^{QED}}{\partial(\partial_\nu A_\mu)}\bar{\delta}A_\mu + \mathcal{L}_{S\alpha}^{QED}\bar{\delta}x^\nu \\ \tilde{Q}_s(t) = \int_{\mathbb{R}^3} d^3x\tilde{\mathcal{J}}_s^0(\vec{x},t) \end{cases} \tag{397}
$$

[90] $\mathcal{J}_s^\nu \equiv i\left[(D_\mu\psi_s)^\dagger\xi^\mu\xi^\nu\psi_s - \psi_s^\dagger\xi^\nu\xi^\mu(D_\mu\psi_s)\right]$.

and since

$$\begin{cases} \bar{\delta}\psi_s = i\psi_s(x) \\ \bar{\delta}\psi_s^\dagger = -i\psi_s^\dagger(x) \\ \bar{\delta}A_\mu = 0 \\ \bar{\delta}x^\nu = 0, \end{cases} \tag{398}$$

we can write

$$\tilde{\mathcal{I}}_s^\nu = i(\partial_\mu\psi_s^\dagger\xi^\mu\xi^\nu\psi_s - \psi_s^\dagger\xi^\nu\xi^\mu\partial_\mu\psi_s) - eA_\mu(\psi_s^\dagger\xi^\mu\xi^\nu\psi_s + \psi_s^\dagger\xi^\nu\xi^\mu\psi_s), \tag{399}$$

which just coincides with the four-current $\mathcal{I}_s^\nu$. With concern to the conserved charge $\tilde{Q}_s(t)$, we have

$$\tilde{\mathcal{I}}_s^0 = \mathcal{I}_s^0 = i(\partial_\mu\psi_s^\dagger\xi^\mu\xi^0\psi_s - \psi_s^\dagger\xi^0\xi^\mu\partial_\mu\psi_s) - eA_\mu(\psi_s^\dagger\xi^\mu\xi^0\psi_s + \psi_s^\dagger\xi^0\xi^\mu\psi_s), \tag{400}$$

from which it follows

$$\tilde{Q}_s(t) = \int_{\mathbb{R}^3} d^3x \left[i(\partial_\mu\psi_s^\dagger\xi^\mu\xi^0\psi_s - \psi_s^\dagger\xi^0\xi^\mu\partial_\mu\psi_s) - eA_\mu(\psi_s^\dagger\xi^\mu\xi^0\psi_s + \psi_s^\dagger\xi^0\xi^\mu\psi_s)\right]. \tag{401}$$

We note that being the Noether current equal to that of the equation of motion (396), this should be a result of enormous importance, because it concurs to characterize a theoretical superiority of the SαT towards AαT. In order to understand what just said, we have to calculate the Noether current of the Lagrangian density $\mathcal{L}_{A\alpha}^{QED}$ associated with the invariance under the local gauge transformations (371) and the Noether current of the Lagrangian density $\mathcal{L}_{S\alpha}$ associated with the invariance under the global gauge transformations (389) (it is banal to show $\mathcal{L}_{S\alpha}$ is unchanged under these last transformations). Calling $\tilde{\mathrm{j}}_s^\nu$ and j_s^ν such currents respectively, with some small calculations, we get

$$\begin{cases} \tilde{\mathrm{j}}_s^\nu = -(\bar{\Psi}_s\chi^\nu\Psi_s) = \mathrm{j}_s^\nu \\ j_s^\nu = i\left[(\partial_\mu\psi_s^\dagger)\xi^\mu\xi^\nu\psi_s - \psi_s^\dagger\xi^\nu\xi^\mu(\partial_\mu\psi_s)\right]. \end{cases} \tag{402}$$

This result says that, in reference to the asymmetric α-equation, in the same way that happens for the Dirac equation, the Noether current, concerning the invariance of $\mathcal{L}_{A\alpha}$ under global phase transformations, and that one, deriving from the invariance of $\mathcal{L}_{A\alpha}^{QED}$ under local phase transformations, are identical. On the contrary, the Noether current, concerning the invariance of $\mathcal{L}_{S\alpha}$ under global phase transformations, and that one, linked with the invariance of $\mathcal{L}_{S\alpha}^{QED}$ under local phase transformations, are different.

Strictly speaking, this diversity could be the indication that symmetric α-equation works better than asymmetric α-equation. In fact, the circumstance the AαT have $\tilde{\mathrm{j}}^{\nu}_{s}$ equal to j^{ν}_{s} means that:

1. While the equation of the motion for the particle field suffers from the presence of the electromagnetic field through an effective connection between ψ_s and A_μ, the equation of the motion of the electromagnetic field does not suffer from the particle field at all, being this last only present in the explicit expression of the four-current j^{ν}_{s} or $\tilde{\mathrm{j}}^{\nu}_{s}$.
2. The Gauge Principle, of which the minimal substitution is a directed consequence, is not completely respected by the theory $\mathcal{L}_{A\alpha}$. In fact, while the $\mathcal{L}^{QED}_{A\alpha}$ and the equation for ψ_s in presence of electromagnetic field are obtained by $\mathcal{L}_{A\alpha}$ and asymmetric α-equation simply replacing the ordinary derivative with the covariant derivative, the equation of the electromagnetic field does not suffer from this substitution, neither in the current term in which the field ψ_s is.

The issues 1 and 2, as it is immediately noticed, are resolved by the symmetric α-Theory (banally we can see that $\mathcal{J}^{\nu}_{s}$ can be obtained from j^{ν}_{s} simply by minimal substitution) and this seems to suggest this theory is more complete than the one characterized by $\mathcal{L}_{A\alpha}$. It must also be emphasized that problems 1 and 2 also plague the Dirac equation, but not the Klein-Gordon one. However, the fact these last equations and relative theories were constructed for different types of particles never allowed the overcoming of such an inconsistency. In short, the asymmetric α-Theory, being the generalization of the Dirac theory, inevitably suffers from the problems 1 and 2 too. Instead, the symmetric α-Theory, being the generalization of the Klein-Gordon theory, exceeds these problems, determining an effective connection between particle field and electromagnetic field, which is physically desirable in an interaction theory.

In order to conclude this section, we note that if we had considered the local phase transformations

$$\begin{cases} \psi'_s(x) = e^{-i\theta(x)}\psi_s(x) \\ \psi'^{\dagger}_s(x) = e^{i\theta(x)}\psi^{\dagger}_s(x) \\ A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x), \end{cases} \tag{403}$$

for obtaining the invariance of $\mathcal{L}_{S\alpha}$ under such transformations, we would have had to perform the minimal substitution with the covariant derivative thus defined

$$D_\mu \equiv (\partial_\mu + ieA_\mu). \tag{404}$$

The equations of motion (395) and (396) would have been the same ones, except for the four-current $\mathcal{J}^{\nu}_{s}$, which, in this case, is

$$\mathcal{J}^{\nu}_{s} \equiv -i\left[(D_\mu\psi_s)^{\dagger}\xi^{\mu}\xi^{\nu}\psi_s - \psi^{\dagger}_s\xi^{\nu}\xi^{\mu}(D_\mu\psi_s)\right] = i(\psi^{\dagger}_s\xi^{\nu}\xi^{\mu}\partial_\mu\psi_s - \partial_\mu\psi^{\dagger}_s\xi^{\mu}\xi^{\nu}\psi_s) - eA_\mu(\psi^{\dagger}_s\xi^{\mu}\xi^{\nu}\psi_s + \psi^{\dagger}_s\xi^{\nu}\xi^{\mu}\psi_s), \tag{405}$$

i.e. formally with opposite sign with regard to the four-current found with the covariant derivative having minus sign. Explicitly, it introduces a term of the particle field with changed sign and a term of connection between particle field and electromagnetic field that is unchanged. This means that the difference of sign in the definition of the covariant derivative does not change the interaction term concerning the interaction of the radiation with the field ψ_s. At level of $\mathcal{L}_{S\alpha}^{QED}$, this is translated in a difference of sign in the linear term of A_μ and a constancy of sign in the quadratic term of A_μ, as it straightforward can be seen by making clear the covariant derivative $D_\mu = (\partial_\mu + ieA_\mu)$ in the expression

$$\mathcal{L}_{S\alpha}^{QED} = (D_\mu \psi_s)^\dagger \xi^\mu \xi^\nu (D_\nu \psi_s) - m^2 \psi_s^\dagger \psi_s - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{406}$$

4.3 Free propagators and Yang-Mills theory within the α-Theory

The writing of the Lagrangian density of the α-Theory (asymmetric or symmetric), in interaction with an external electromagnetic field, concurs to use the formalism of the scattering matrix S or the Feynman integral for the study of the physical processes relative to the α-QED. However, such a study involves a more complex analysis of the perturbation theory applied to the α-Theory, that is outside the aims of the present work. Hoping, therefore, in an exhaustive review of this argument in another place, we want to deal, in this section, with the calculation of the propagators regarding the asymmetric and symmetric α-Theory, and the possibility to extend the Yang-Mills theory to the α-Theory. It should be clear this is made for pure academic spirit, because like it has been shown in the previous pages, the α-Theory, asymmetric or symmetric, was born from Big-Break. This inexorably drives to conclude that all the interactions known in Nature were born by a great process of SSB (for the time unknown), whose aspect developed into the Glashow-Weinberg-Salam model could be the simpler one. It goes without saying that the Big-Break study, most probably, will carry to a radical redefinition of the gauge theories and, hence, the Lagrangian density of the electromagnetic interaction, written in the last section, and the one concerning the Yang-Mills theory, which we will see in the next pages, are not really definitive ones and so we must handle them with great care.

We proceed now to the calculation of the free propagator of the asymmetric α-Theory. From the study of the QFT, we know the propagator of a certain physical field is given by the Green's function associated with the equation of the motion of such a field in absence of interaction. Therefore, in order to get the free propagator of the AαT, it is enough to calculate the Green's function of the equation [91]

[91] Really $0 \equiv 0_{d\times 1}$, where $d \equiv (2s+1)$. In this case, the subindex s should be indicated with d (or $d \times 1$ for ψ_s) too, but we leave it intact for a slimmer notation.

$$(i\chi^\mu\partial_\mu - m\mathbb{1}_s)\psi_s(x) = 0. \tag{407}$$

The Green equation associated with the (407) is

$$(i\chi^\mu\partial_\mu - m\mathbb{1}_s)\mathsf{G}_s(x-y) = \delta^4_{d\times 1}(x-y) \equiv \delta^4(x-y)1_{d\times 1}, \tag{408}$$

where, we placed

$$1_{d\times 1} \equiv \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}. \tag{409}$$

For resolving the Green equation (408), we use the Fourier integral transformations of $\mathsf{G}_s(x-y)$ and $\delta^4(x-y)$, which are

$$\begin{cases} \mathsf{G}_s(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\tilde{\mathsf{G}}_s(p)d^4p \\ \delta^4(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}d^4p. \end{cases} \tag{410}$$

According with them, our Green equation becomes

$$(i\chi^\mu\partial_\mu - m\mathbb{1}_s)\left[\frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\tilde{\mathsf{G}}_s(p)d^4p\right] = \frac{1}{(2\pi)^4}\left[\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}d^4p\right]1_{d\times 1} \tag{411}$$

and since

$$i\chi^\mu\partial_\mu e^{-ip\cdot(x-y)} = \chi^\mu p_\mu e^{-ip\cdot(x-y)}, \tag{412}$$

we can still write

$$\frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\left[(\chi^\mu p_\mu - m\mathbb{1}_s)\tilde{\mathsf{G}}_s(p) - 1_{d\times 1}\right]d^4p = 0, \tag{413}$$

from which, it banally follows

$$(\chi^\mu p_\mu - m\mathbb{1}_s)\tilde{\mathsf{G}}_s(p) = 1_{d\times 1} \;\Rightarrow\; \tilde{\mathsf{G}}_s(p) = \frac{1_{d\times 1}}{(\chi^\mu p_\mu - m\mathbb{1}_s)} \tag{414}$$

and we immediately have

$$\mathsf{G}_s(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\frac{1_{d\times 1}}{(\chi^\mu p_\mu - m\mathbb{1}_s)}d^4p. \tag{415}$$

This integral is divergent, because the integrand has a pole in $\chi^\mu p_\mu = m\mathbb{1}_s$. Such a problem is easily resolved using the Feynman's $i\epsilon$ prescription, *i.e.* by adding $i\epsilon$ to the denominator, namely writing

$$\mathsf{G}_s(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\left(\frac{1_{d\times 1}}{\chi^\mu p_\mu - m\mathbb{1}_s + i\epsilon}\right)d^4p, \tag{416}$$

where it is implied that $i\epsilon$ is multiplied by the matrix [92]

$$1_{d\times d} \equiv \begin{pmatrix} 1 & 1 & \dots & 1 & \dots \\ 1 & 1 & \dots & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ 1 & 1 & \dots & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \end{pmatrix}. \tag{417}$$

The Green's function as soon as calculated represents the free propagator of the asymmetric α-Theory in the position space. By indicating this propagator with $\Delta_s^{A\alpha}(x-y)$, we can write

$$\Delta_s^{A\alpha}(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{P}^4} e^{-ip\cdot(x-y)}\left(\frac{1_{d\times 1}}{\chi\cdot p - m\mathbb{1}_s + i\epsilon}\right)d^4p. \tag{418}$$

Instead, the propagator in the momentum space, which we indicate with $\Delta_s^{A\alpha}(p)$, is just the Green's function $\tilde{\mathsf{G}}_s(p)$. Hence

$$\Delta_s^{A\alpha}(p) = \frac{1_{d\times 1}}{\chi\cdot p - m\mathbb{1}_s}. \tag{419}$$

We note these propagators really are column vectors, thus as every Green equation concerning a column field, like $\psi_s(x)$, demands. This means the (419) must be read in such a way

$$\Delta_s^{A\alpha}(p) = \frac{1_{d\times 1}}{\chi\cdot p - m\mathbb{1}_s} = (\chi\cdot p - m\mathbb{1}_s)^{-1}1_{d\times 1}, \tag{420}$$

[92] Obviously this matrix has $(2s+1)$-rows and $(2s+1)$-columns.

in the total respect of the matrix multiplication. We also understand that the existence of $\Delta_s^{A\alpha}(x-y)$ and $\Delta_s^{A\alpha}(p)$ depends on the invertibility of the matrix $(\chi \cdot p - m\mathbb{1}_s)$, *i.e.* of the fact that

$$\det(\chi \cdot p - m\mathbb{1}_s) \neq 0. \tag{421}$$

If there were a $s \in \mathbb{N}/2$ for which this determinant is null for that precise s, a free propagator could not be defined and, therefore, the perturbation theory, for this s, would not be applicable. For little s it is easy to see, with some small calculations, that this does not happen unless to place $\vec{p} = \vec{0}$. Note that, for $s = 0$, we have

$$\begin{cases} \Delta_{s=0}^{A\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left(\frac{1}{\pm m - m + i\epsilon}\right) d^4p \\ \Delta_{s=0}^{A\alpha}(p) = \frac{1}{\pm m - m}, \end{cases} \tag{422}$$

i.e., by separating the case $\delta_0 = 1$ from that one $\delta_0 = -1$, we have [93]

$$\delta_0 = 1 : \begin{cases} \Delta_{\delta=1}^{A\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left(\frac{1}{i\epsilon}\right) d^4p \\ \Delta_{\delta=1}^{A\alpha}(p) = -\frac{1}{0} \equiv \infty \end{cases} \tag{423}$$

$$\delta_0 = -1 : \begin{cases} \Delta_{\delta=-1}^{A\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left(\frac{1}{-2m + i\epsilon}\right) d^4p \\ \Delta_{\delta=-1}^{A\alpha}(p) = -\frac{1}{2m}. \end{cases} \tag{424}$$

Since, for $\delta_0 = 1$, the propagator $\Delta_{\delta=1}^{A\alpha}(p)$ is infinite, we *suppose* the free propagators concerning the position and momentum spaces, when $s = 0$, are

$$\begin{cases} \Delta_{s=0}^{A\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left(\frac{1}{(\overline{\chi\cdot p})_0 - m + i\epsilon}\right) d^4p \\ \Delta_{s=0}^{A\alpha}(p) = \frac{1}{(\overline{\chi\cdot p})_0 - m}, \end{cases} \tag{425}$$

where, with $(\overline{\chi \cdot p})_0$, we indicated the average of the scalar product $\chi \cdot p$ calculated for $s = 0$ when $\delta_0 = \pm 1$, namely

$$(\overline{\chi \cdot p})_0 \equiv \frac{1}{2}\left[(\chi \cdot p)_{\delta_0=1} + (\chi \cdot p)_{\delta_0=-1}\right]. \tag{426}$$

[93]Remember that, for $s = 0$, δ can be 1 and -1.

Since $\chi^\mu = (\delta, \delta\varepsilon^i)$, for $s = 0$ we have $\chi_0^\mu = (\pm 1, 0, 0, 0)$ and so $(\chi \cdot p)_{\delta_0 = 1} = m$, $(\chi \cdot p)_{\delta_0 = -1} = -m$, that is

$$(\overline{\chi \cdot p})_0 = 0, \tag{427}$$

from which it follows

$$\begin{cases} \Delta^{A\alpha}_{s=0}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left(\frac{1}{i\epsilon - m}\right) d^4p \\ \Delta^{A\alpha}_{s=0}(p) = -\frac{1}{m} = -\frac{1}{p_0}. \end{cases} \tag{428}$$

Naturally, for $\delta_{\mp 1}$, the result does not change, because the sign sequence of scalar 1 cannot certainly alter the average scalar product previously defined.

At this point, it is worth remembering that, by the theory of the Green's functions, a general solution of the AαE can be written in the following way

$$\psi_s(x) = \Delta_s^{A\alpha}(x) + \tilde{\psi}_s(x) \quad \forall x \in M - \{0\}, \tag{429}$$

where $\tilde{\psi}_s(x)$ is an arbitrary function, fixed by the initial conditions ("initial value problem").

In order to conclude the discussion on the free propagator of the AαT, we observe an alternative form existing for the just studied propagator, which can be banally obtained multiplying $\Delta_s^{A\alpha}(p)$ up and down on the left by $(\chi^\nu p_\nu + m\mathbb{1}_s)$. It is given by

$$\Delta_s^{A\alpha}(p) = \frac{(\chi^\nu p_\nu + m\mathbb{1}_s)1_{d\times 1}}{\chi^\mu\chi^\nu p_\mu p_\nu - m^2\mathbb{1}_s}, \tag{430}$$

from which it follows

$$\Delta_s^{A\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{P}^4} e^{-ip\cdot(x-y)} \left[\frac{(\chi^\nu p_\nu + m\mathbb{1}_s)1_{d\times 1}}{\chi^\mu\chi^\nu p_\mu p_\nu - m^2\mathbb{1}_s + i\epsilon}\right] d^4p. \tag{431}$$

We now calculate the free propagator of the symmetric α-Theory.
Given the symmetric α-equation

$$\left(\xi^\mu\xi^\nu\partial_\mu\partial_\nu + m^2\mathbb{1}_s\right)\psi_s(x) = 0, \tag{432}$$

we have that its associated Green equation is

$$\left(\xi^\mu \xi^\nu \partial_\mu \partial_\nu + m^2 \mathbb{1}_s\right) G_s(x-y) = \delta^4(x-y) 1_{d\times 1}. \tag{433}$$

In order to resolve it, we use the usual integral transformations

$$\begin{cases} G_s(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{K}^4} e^{-ik\cdot(x-y)} \tilde{G}_s(k) d^4k \\ \delta^4(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{K}^4} e^{-ik\cdot(x-y)} d^4k, \end{cases} \tag{434}$$

which, replaced in the (433), with some little calculations, give

$$\left(m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu\right) \tilde{G}_s(k) = 1_{d\times 1} \Rightarrow \tilde{G}_s(k) = \frac{1_{d\times 1}}{m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu}, \tag{435}$$

from which banally

$$G_s(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{K}^4} e^{-ik\cdot(x-y)} \left(\frac{1_{d\times 1}}{m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu + i\epsilon}\right) d^4k, \tag{436}$$

where we have used the Feynman's $i\epsilon$ prescription (remember that in the studied cases $i\epsilon \equiv i\epsilon 1_{d\times d}$), since the integral (436) is divergent in the pole $\xi^\mu \xi^\nu k_\mu k_\nu = m^2 \mathbb{1}_s$. The Green's function we have calculated represents the free propagator of the symmetric α-Theory in the position space. By indicating such a propagator with $\Delta_s^{S\alpha}(x-y)$, we can write

$$\Delta_s^{S\alpha}(x-y) = \frac{1}{(2\pi)^4} \int_{\mathbb{K}^4} e^{-ik\cdot(x-y)} \left(\frac{1_{d\times 1}}{m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu + i\epsilon}\right) d^4k. \tag{437}$$

Instead, the propagator in the momentum space, that we indicate with $\Delta_s^{S\alpha}(k)$, is just the Green's function $\tilde{G}_s(k)$. Therefore

$$\Delta_s^{S\alpha}(k) = \frac{1_{d\times 1}}{m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu}. \tag{438}$$

Obviously, also for these propagators the same discussion made about AαT is true, that is their existence depends on the invertibility of the matrix $(m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu)$, *i.e.* on the condition

$$\det\left(m^2 \mathbb{1}_s - \xi^\mu \xi^\nu k_\mu k_\nu\right) \neq 0. \tag{439}$$

Naturally, if a $s \in \mathbb{N}/2$ were supposed to exist, for which such a determinant is null, for this s the perturbation theory could not be applied. For little values of s, it is easy to see that the propagators

$\Delta_s^{S\alpha}(x-y)$ and $\Delta_s^{S\alpha}(k)$ exist, unless to take $\vec{k}=\vec{0}$.

Let us now calculate the free propagator of the symmetric α-Theory for $s=0$. Since we expect such a propagator to be equal to the Klein-Gordon one, concerning a particle that moves on t-axis, we calculate before this propagator, starting from the equation [94]

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{m^2c^2}{\hbar^2}\right)\psi(x)=0. \tag{440}$$

As it is observed, we have written the Klein-Gordon equation relative to the t-axis, not in natural units. This choice, as we will see in the next page, will concur to completely confirm the hypothesis previously made about the propagator for $s=0$. The Green equation associated to the (440) is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{m^2c^2}{\hbar^2}\right)G(x-y)=\delta^4(x-y), \tag{441}$$

from which, by replacing the usual integral transformations of $G(x-y)$ and $\delta^4(x-y)$, at the end one obtains

$$\Delta_t^{K-G}(k)=\frac{1}{m^2c^2\left(\frac{1}{\hbar^2}-1\right)}. \tag{442}$$

We now calculate the propagator of the symmetric α-Theory in the momentum space for $s=0$ and confront it with $\Delta_t^{K-G}(k)$. The propagator (438) previously calculated is expressed in natural units, since it is referred to the SαE written in such units. For obtaining the propagator in explicit form, it is sufficient to consider the equation

$$\left(\xi^\mu\xi^\nu\partial_\mu\partial_\nu+\frac{m^2c^2}{\hbar^2}\mathbb{1}_s\right)\psi_s(x)=0, \tag{443}$$

from which, it is easy to have

$$\Delta_s^{S\alpha}(k)=\frac{1_{d\times 1}}{\frac{m^2c^2}{\hbar^2}\mathbb{1}_s-\xi^\mu\xi^\nu k_\mu k_\nu}. \tag{444}$$

We now estimate this propagator for $s=0$. As for the AαT, we *suppose* that

$$\begin{cases}\Delta_{s=0}^{S\alpha}(x-y)=\frac{1}{(2\pi)^4}\int_{\mathbb{K}^4}e^{-ik\cdot(x-y)}\left(\frac{1}{\frac{m^2c^2}{\hbar^2}-(\xi^\mu\xi^\nu k_\mu k_\nu)_0+i\epsilon}\right)d^4k\\ \Delta_s^{S\alpha}(k)=\frac{1}{\frac{m^2c^2}{\hbar^2}-(\xi^\mu\xi^\nu k_\mu k_\nu)_0},\end{cases} \tag{445}$$

[94]Since we stay in the bradyonic (REP) universe, it is necessary to use such an equation and not the (44) one.

where

$$(\overline{\xi^\mu \xi^\nu k_\mu k_\nu})_0 \equiv \frac{1}{2}\left[(\xi^\mu \xi^\nu k_\mu k_\nu)_{\delta_0=1} + (\xi^\mu \xi^\nu k_\mu k_\nu)_{\delta_0=-1}\right]. \tag{446}$$

Since $\xi^\mu = (\delta, i\varepsilon^i)$, for $s=0$ we have $\xi_0^\mu = (\pm 1, 0, 0, 0)$, and so

$$(\xi^\mu \xi^\nu k_\mu k_\nu)_{s=0} = m^2 c^2, \tag{447}$$

from which, it follows

$$(\overline{\xi^\mu \xi^\nu k_\mu k_\nu})_0 = m^2 c^2. \tag{448}$$

Therefore [95]

$$\Delta^{S\alpha}_{s=0}(x-y) = \frac{1}{(2\pi)^4}\int_{\mathbb{K}^4} e^{-ik\cdot(x-y)}\left(\frac{1}{\frac{m^2c^2}{\hbar^2} - m^2c^2 + i\epsilon}\right) d^4k \tag{449}$$

$$\Delta^{S\alpha}_{s=0}(k) = \frac{1}{\frac{m^2c^2}{\hbar^2} - m^2c^2} = \frac{1}{m^2c^2\left(\frac{1}{\hbar^2} - 1\right)}. \tag{450}$$

We notice (450) is equal to the free propagator in the momentum space of the Klein-Gordon equation for a particle moving on the t-axis. This means the *ansatz* to consider the average of the cases $\delta_0 = 1$ and $\delta_0 = -1$ of the products $\chi \cdot p$ and $\xi^\mu \xi^\nu k_\mu k_\nu$, when $s=0$, inside the free propagators of the AαT and SαT, is correct, because, as it should be, it allows to $\Delta^{S\alpha}_{s=0}(k)$ and $\Delta^{K-G}_t(k)$ of coinciding.[96]

In order to conclude the discussion on the free propagator of the symmetric α-Theory, we note that a general solution of the SαE is given by

$$\psi_s(x) = \Delta^{S\alpha}_s(x) + \tilde{\psi}_s(x) \quad \forall x \in M - \{0\}, \tag{451}$$

where $\tilde{\psi}_s(x)$ is an arbitrary function, fixed by the initial conditions.

Now we want to deal shortly – after having studied in the previous section the electromagnetic interaction within the α-Theory – with the Yang-Mills theory applied to the α-Theory, asymmetric or symmetric, succeeding also to write the Lagrangian density of the QCD in terms of AαT and

[95] Obviously, for the representation $\delta_{\mp 1}$, we would have had the same result.

[96] The same thing is, obviously, true for $\Delta^{S\alpha}_{s=0}(x-y)$ and $\Delta^{K-G}_t(x-y)$.

SαT. Naturally, as already explained at the top of this section, this is provisional, waiting for a global theory on the SSB. First of all, we start writing the most general Yang-Mills theory of the particle field ψ_s described by AαT and, then, we specialize it to the group $SU(3)$ of colour. For this aim, we consider the Lagrangian density

$$\mathcal{L}_{A\alpha} = \bar{\psi}_s(x)(i\chi^\mu \partial_\mu - m\mathbb{1}_s)\psi_s(x), \tag{452}$$

unchanged under global phase transformations (we can banally verify it)

$$\begin{cases} \psi'_s(x) = e^{-iT_a\theta_a}\psi_s(x) \\ \bar{\psi}'_s(x) = e^{iT_a\theta_a}\bar{\psi}_s(x), \end{cases} \tag{453}$$

where, with $e^{\pm iT_a\theta_a}$, we indicated an element of a generic non-abelian Lie group G having n complex parameters θ_a, and, with T_a, the generators of such a group in a unitary representation of dimension n, satisfying the Lie algebra

$$[T^a, T^b] = it^{abc}T^c, \tag{454}$$

where the t^{abc}, anti-symmetric in the indices, are the structure constants of the group G.

It can be easily demonstrated that the Lagrangian density $\mathcal{L}_{A\alpha}$, when the group G is locally shrunk (*i.e.* one goes from G to G_x), does not remain invariant under the non-abelian local gauge transformations

$$\begin{cases} \psi'_s(x) = e^{-iT_a\theta_a(x)}\psi_s(x) \\ \bar{\psi}'_s(x) = e^{iT_a\theta_a(x)}\bar{\psi}_s(x). \end{cases} \tag{455}$$

In order to maintain the invariance, as seen for the abelian local phase transformations, a covariant derivative must be introduced

$$D_\mu \equiv (\partial_\mu + ig\tilde{A}_\mu), \tag{456}$$

and to apply at $\mathcal{L}_{A\alpha}$ the minimal substitution. Before making this, we note that in the covariant derivative expression we indicated with $\tilde{A}_\mu$ the "gauge field" (also called "Yang-Mills field") given by

$$\tilde{A}_\mu(x) \equiv T^a\tilde{A}^a_\mu(x), \tag{457}$$

and, with g, the coupling constant of the generic theory characterized by the Lagrangian density obtained from the initial one through minimal substitution. In our case, the Lagrangian density is

$$\mathcal{L}^{Y-M}_{A\alpha} = \bar{\psi}_s(x)(i\chi^\mu D_\mu - m\mathbb{1}_s)\psi_s(x) - \frac{1}{2g^2}Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{458}$$

where the last term represents the kinetic term of the Yang-Mills field $\tilde{A}_\mu$. The tensor $\tilde{F}_{\mu\nu}$, generalization of the tensor $F_{\mu\nu}$ of the electromagnetic field, is called the "Yang-Mills tensor" and it is thus defined [97]

$$\tilde{F}_{\mu\nu} \equiv (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + ig[\tilde{A}_\mu, \tilde{A}_\nu]. \tag{459}$$

It is straightforward to try it satisfies the relation

$$[D_\mu, D_\nu] = ig\tilde{F}_{\mu\nu}. \tag{460}$$

It can be proved that the Lagrangian density (458) is invariant under the non-abelian local gauge transformations [98]

$$\begin{cases} \psi'_s(x) = e^{-iT_a\theta_a(x)}\psi_s(x) \\ \bar{\psi}'_s(x) = e^{iT_a\theta_a(x)}\bar{\psi}_s(x) \\ \tilde{A}'_\mu(x) = U(x)\tilde{A}_\mu(x)U^{-1}(x) + \frac{i}{g}(\partial_\mu U(x))U^{-1}(x), \end{cases} \tag{461}$$

where the last expression gives the transformation law of the field $\tilde{A}_\mu$.

Now we want to write the equations of the motion for the field ψ_s and for the Yang-Mills field $\tilde{A}_\mu$. From the Euler-Lagrange equations relative to these fields, one obtains [99]

$$(i\chi^\mu D_\mu - m\mathbb{1}_s)\psi_s(x) = 0 \tag{462}$$

$$\partial_\mu \tilde{F}^{\mu\nu} - ig[\tilde{A}_\mu, \tilde{F}^{\mu\nu}] = g\mathrm{j}^\nu_s. \tag{463}$$

The (462) represents the asymmetric α-equation of the field ψ_s coupled with the Yang-Mills field $\tilde{A}_\mu$, while the (463) gives the equation of the motion for the gauge field $\tilde{A}_\mu$ (or for the components $\tilde{A}^a_\mu$, if we express the (463) in terms of its components). It can be noted that the four-current j^ν_s – in contrast to the electromagnetic case ($G = U(1)$) – is not conserved, *i.e.* it is not the Noether

[97] If the covariant derivative is defined with the minus sign also the sign of g changes in the definition of $\tilde{F}_{\mu\nu}$.

[98] $U(x) \equiv e^{-iT_a\theta_a(x)} \in G_x$.

[99] $\mathrm{j}^\nu_s \equiv (\bar{\psi}_s\chi^\nu\psi_s)$.

current deriving from the invariance of $\mathcal{L}_{A\alpha}^{Y-M}$ under non-abelian local gauge transformations. The right Noether current is given, instead, by

$$\mathcal{J}_s^\nu \equiv i[\tilde{A}_\mu, \tilde{F}^{\mu\nu}] + (\bar{\psi}_s \chi^\nu \psi_s), \tag{464}$$

which is nothing but the total Noether current of $\mathcal{L}_{A\alpha}^{Y-M}$ under non-abelian local gauge transformations given by the sum of the current $i[\tilde{A}_\mu, \tilde{F}^{\mu\nu}]$ concerning the Yang-Mills field and the current $(\bar{\psi}_s \chi^\nu \psi_s)$ of the particle field ψ_s (this last four-current is the Noether current associated to the invariance of $\mathcal{L}_{A\alpha}$ under non-abelian global phase transformations). The fact the total current $\mathcal{J}_s^\nu$ is conserved means that, similarly to what happens for the Dirac theory coupled with a Yang-Mills interaction, also for the asymmetric α-Theory in interaction with a gauge field $\tilde{A}_\mu$ results that the charge g can flow from the field ψ_s to the field $\tilde{A}_\mu$ (or to its components $\tilde{A}_\mu^a$) for any $s \in \mathbb{N}/2$. Due to the Noether current $\mathcal{J}_s^\nu$, the equation (463) can be written in the following compact form

$$\partial_\mu \tilde{F}^{\mu\nu} = g \mathcal{J}_s^\nu, \tag{465}$$

which is nothing but the generalization of the inhomogeneous Maxwell's equation in covariant form.[100] Naturally, according to what was said, we can see the conserved charge of the theory is

$$\tilde{\mathsf{Q}}_s(t) = \int_{\mathbb{R}^3} d^3x \mathcal{J}_s^0 = \int_{\mathbb{R}^3} d^3x \left\{ i[\tilde{A}_\mu, \tilde{F}^{\mu 0}] + (\bar{\psi}_s \chi^0 \psi_s) \right\}. \tag{466}$$

We described, therefore, in broad outline, the Yang-Mills theory concerning the field ψ_s of the asymmetric α-Theory. As usual, we obtained results formally consistent with the Dirac field. In particular, in our case too, the strange circumstance the four-current $\mathcal{J}_s^\nu$ is composed by two terms, describing independent fields, can be noted. Since ψ_s and $\tilde{A}_\mu$ interact, it would have been more logical to expect their interaction was shown also within the structure of $\mathcal{J}_s^\nu$. We will see such an anomaly – already present in the electromagnetic case – will be neatly exceeded by the Yang-Mills theory of the field ψ_s concerning the symmetric α-Theory.

At this point, we are in a position to write the Lagrangian density of the "Asymmetric α-QCD." Practically, we can obtain it from $\mathcal{L}_{A\alpha}^{Y-M}$, specializing the generic Lie group G to $SU(3)_c$. Before making it, we must remember that the QCD is the theory explaining the strong interaction through the quarks and gluons. The quarks are the particles of spin 1/2 which compose the hadrons, while the gluons are the gauge bosons able to mediate the interaction. Inside $\mathcal{L}_{A\alpha}^{Y-M}$, the fields ψ_s and $\bar{\psi}_s$ will have, then, to be replaced by the quark and anti-quark fields. Since quarks and anti-quarks have spin 1/2, they will be represented by two-dimensional column and row vectors, respectively. This means each of them will have two components. Moreover, every quark and anti-quark will have to be characterized by a "flavour" index f and by "colour" and "anti-colour" indices, respectively

[100] If $G = U(1)$, one always obtains the equation (465), with the only difference that $\mathcal{J}_s^\nu$, $\tilde{F}^{\mu\nu}$ and g, are reduced to j_s^ν, $F^{\mu\nu}$ and e, respectively.

(we remember the colours are R, G, B and the anti-colours are $\bar{R}$, $\bar{G}$, $\bar{B}$). This means one must make to $\mathcal{L}_{A\alpha}^{Y-M}$ such substitutions

$$\begin{aligned}
\psi_s(x) &\longrightarrow \mathbf{q}_\beta^{f,j}(x),\ \beta \in \{1,2\},\ f = \text{flavour index},\ j \in \{R,G,B\} \\
\bar{\psi}_s(x) &\longrightarrow \bar{\mathbf{q}}_\alpha^{f,i}(x),\ \alpha \in \{1,2\},\ f = \text{flavour index},\ i \in \{\bar{R},\bar{G},\bar{B}\} \\
m &\longrightarrow m_f, \qquad f = \text{flavour index.}
\end{aligned}$$

With regard to gluons, in this model, they are described by the components $\tilde{A}_\mu^a$ of the gauge field concerning $SU(3)_c$. Since the generators of this group, given by [101]

$$T^a = \frac{\lambda^a}{2}, \tag{467}$$

are eight, we will have these massless bosons are eight too. It is good to underline that the generic structure constants t^{abc} must formally be replaced with the structure constants f^{abc} (always anti-symmetric in the indices) of the group $SU(3)_c$. This prerequisite concurs to immediately write the Lagrangian density of the "Asymmetric α-QCD" for a fixed flavour [102]

$$\mathcal{L}_{A\alpha}^{QCD} = \bar{\mathbf{q}}_\alpha^{f,i}(x)\left(i\chi_{1/2}^\mu D_\mu - m_f \mathbb{1}_{1/2}\right)\mathbf{q}_\beta^{f,j}(x) - \frac{1}{2g_{QCD}^2} Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{468}$$

where

$$D_\mu = (\partial_\mu + i g_{QCD}\tilde{A}_\mu); \quad \tilde{A}_\mu(x) = \frac{1}{2}\lambda^a \tilde{A}_\mu^a(x); \quad [\lambda^a, \lambda^b] = i f^{abc}\lambda^c$$

$$\tilde{F}_{\mu\nu} = (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + i g_{QCD}[\tilde{A}_\mu, \tilde{A}_\nu].$$

The expert reader will have certainly noticed that, into expression of $\mathcal{L}_{A\alpha}^{QCD}$, the terms of the ghost fields and "gauge-fixing," usually found in literature, have been omitted. Obviously, this is intentional, and it is due to the fact these terms derive by the quantization of the Yang-Mills theory and the use of the functional formalism. Since in this work, like announced, neither the perturbation theory, constructed through the matrix S, nor the Feynman integral, are dealt, and the general form of the second quantization will be studied in the next chapter, it should have been wrong to add in $\mathcal{L}_{A\alpha}^{QCD}$ terms whose right expression we do not know within the framework of the theory we built here.

At this point, we want to write the Lagrangian density of the "Symmetric α-QCD." In order

[101] λ^a = Gell-Mann matrices, $a \in \{1,2,3,\ldots,8\}$.
[102] If we want to consider all the flavours, it is needed $\sum_f$.

to make it, we must before write the Lagrangian density of the most general Yang-Mills theory concerning the field ψ_s described by $\mathcal{L}_{S\alpha}$ and, then, specialize it to the group $SU(3)_c$. It is immediate to see the Lagrangian density

$$\mathcal{L}_{S\alpha} = (\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu (\partial_\nu \psi_s) - m^2 \psi_s^\dagger \psi_s, \tag{469}$$

is unchanged under the non-abelian global phase transformations

$$\begin{cases} \psi_s'(x) = e^{-iT_a\theta_a}\psi_s(x) \\ \psi_s'^\dagger(x) = e^{iT_a\theta_a}\psi_s^\dagger(x), \end{cases} \tag{470}$$

with Noether current

$$j_s^\nu = i\left[\psi_s^\dagger \xi^\nu \xi^\mu (\partial_\mu \psi_s) - (\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu \psi_s\right]. \tag{471}$$

If one contracts G to G_x, the above $\mathcal{L}_{S\alpha}$ will not be more invariant under the non-abelian gauge transformations, which became now of local phase. In order to re-establish the invariance, we must introduce a gauge field

$$\tilde{A}_\mu(x) = T^a \tilde{A}_\mu^a, \tag{472}$$

which, under $U(x) \in G_x$, transforms in the following way

$$\tilde{A}'_\mu(x) = U(x)\tilde{A}_\mu(x)U^{-1}(x) + \frac{i}{g}(\partial_\mu U(x))U^{-1}(x), \tag{473}$$

and a covariant derivative [103]

$$D_\mu \equiv (\partial_\mu + ig\tilde{A}_\mu), \tag{474}$$

and, then, apply to $\mathcal{L}_{S\alpha}$ the principle of minimal substitution, by remarking that the kinetic term of the Yang-Mills field $\tilde{A}_\mu$ is given, like already previously seen, by

$$\mathcal{L}_{cin}^{Y-M} = -\frac{1}{2g^2}Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{475}$$

[103]It can be observed that if it is defined with the minus sign all the signs of g would change.

where

$$\tilde{F}_{\mu\nu} \equiv (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + ig[\tilde{A}_\mu, \tilde{A}_\nu]. \tag{476}$$

Hence, the Lagrangian density, obtained from $\mathcal{L}_{S\alpha}$ through the minimal substitution, and invariant under the non-abelian gauge transformations of local phase [104]

$$\begin{cases} \psi'_s(x) = e^{-iT_a\theta_a(x)}\psi_s(x) \\ \psi'^{\dagger}_s(x) = e^{iT_a\theta_a(x)}\psi^{\dagger}_s(x) \\ \tilde{A}'_\mu(x) = U(x)\tilde{A}_\mu(x)U^{-1}(x) + \frac{i}{g}(\partial_\mu U(x))U^{-1}(x), \end{cases} \tag{477}$$

is given by

$$\mathcal{L}^{Y-M}_{S\alpha} = (D_\mu\psi_s)^\dagger \xi^\mu \xi^\nu (D_\nu \psi_s) - m^2 \psi^\dagger_s \psi_s - \frac{1}{2g^2} Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{478}$$

The equations of motion for ψ_s and $\tilde{A}_\mu$, deriving from such a Lagrangian density, are [105]

$$\left(\xi^\mu \xi^\nu D_\mu D_\nu + m^2 \mathbb{1}_s\right)\psi_s(x) = 0 \tag{479}$$

$$\partial_\mu \tilde{F}^{\mu\nu} - ig[\tilde{A}_\mu, \tilde{F}^{\mu\nu}] = g \mathcal{Y}^\nu_s. \tag{480}$$

The (479) represents the symmetric α-equation of the field ψ_s coupled with the Yang-Mills field $\tilde{A}_\mu$, while the (480) gives the equation of the motion for the gauge field $\tilde{A}_\mu$ (or for the components$\tilde{A}^a_\mu$, if we express the (480) in terms of components). Also in this case, it can be demonstrated that $\mathcal{Y}^\nu_s$ is not the Noether current deriving from the invariance of $\mathcal{L}^{Y-M}_{S\alpha}$ under non-abelian gauge transformations of local phase. The right Noether current, instead, is

$$\Gamma^\nu_s \equiv i[\tilde{A}_\mu, \tilde{F}^{\mu\nu}] + i\left[\psi^\dagger_s \xi^\nu \xi^\mu (D_\mu \psi_s) - (D_\mu\psi_s)^\dagger \xi^\mu \xi^\nu \psi_s\right], \tag{481}$$

which is nothing but the total Noether current of $\mathcal{L}^{Y-M}_{S\alpha}$ under non-abelian gauge transformations of local phase, obtained by the sum of the current $i[\tilde{A}_\mu, \tilde{F}^{\mu\nu}]$ concerning the Yang-Mills field and of the current $i\left[\psi^\dagger_s \xi^\nu \xi^\mu (D_\mu \psi_s) - (D_\mu\psi_s)^\dagger \xi^\mu \xi^\nu \psi_s\right]$ of the field ψ_s. Thanks to the four-current Γ^ν_s, the equation (480) can immediately be written in the more compact form

[104] $U(x) \equiv e^{-iT_a\theta_a(x)} \in G_x$.
[105] $\mathcal{Y}^\nu_s \equiv i\left[\psi^\dagger_s \xi^\nu \xi^\mu (D_\mu \psi_s) - (D_\mu\psi_s)^\dagger \xi^\mu \xi^\nu \psi_s\right]$.

$$\partial_\mu \tilde{F}^{\mu\nu} = g\Gamma_s^\nu, \tag{482}$$

which is not more than the generalization of the inhomogeneous Maxwell's equation in covariant form.[106] Obviously, the conserved charge of the theory is

$$\mathcal{Q}_s(t) = \int_{\mathbb{R}^3} d^3x \Gamma_s^0 = \int_{\mathbb{R}^3} d^3x \left\{ i[\tilde{A}_\mu, \tilde{F}^{\mu 0}] + i\left[\psi_s^\dagger \xi^0 \xi^\mu (D_\mu \psi_s) - (D_\mu \psi_s)^\dagger \xi^\mu \xi^0 \psi_s\right]\right\}. \tag{483}$$

All that as soon as discussed represents, in broad terms, the Yang-Mills theory concerning the field ψ_s of the symmetric α-Theory. We notice from such a discussion that some characteristics, already found in the Yang-Mills theory of the field ψ_s belonging to the asymmetric α-Theory, raise, but, above all, there are many improvements. For instance, also in such a theory the form of Γ_s^ν says that the charge g can flow from the field ψ_s to the gauge field $\tilde{A}_\mu$, for any $s \in \mathbb{N}/2$. Nevertheless, the form of Γ_s^ν, with in particular the four-current of the field ψ_s, states that an interaction between the particle field ψ_s and the gauge field $\tilde{A}_\mu$ exists too, thus as logically one expects. This fact, from the theoretical point of view, places the SαT on a higher step regarding the AαT.

At this point, we are ready to write the Lagrangian density of the "Symmetric α-QCD."
Based on the substitutions

$$\begin{aligned}
\psi_s(x) &\longrightarrow q_\beta^{f,j}(x), \quad \beta \in \{1,2\},\ f = \text{flavour index},\ j \in \{R, G, B\} \\
\psi_s^\dagger(x) &\longrightarrow q_\alpha^{\dagger f,i}(x),\ \alpha \in \{1,2\},\ f = \text{flavour index},\ i \in \{\bar{R}, \bar{G}, \bar{B}\} \\
m &\longrightarrow m_f, \qquad f = \text{flavour index}
\end{aligned}$$

and by remembering the gluons are described by the components $\tilde{A}_\mu^a$ of the gauge field relative to $SU(3)_c$, having the generators given by

$$T^a = \frac{\lambda^a}{2}, \tag{484}$$

such that

$$[\lambda^a, \lambda^b] = i f^{abc} \lambda^c, \tag{485}$$

we have for a fixed flavour [107]

[106] If $G = U(1)$, one always obtains the equation (482), with the only difference that Γ_s^ν, $\tilde{F}^{\mu\nu}$ and g, are reduced to $\mathcal{J}_s^\nu$, $F^{\mu\nu}$ and e, respectively

[107] Naturally, it is needed $\sum_f$ for all the flavours.

$$\mathcal{L}_{S\alpha}^{QCD} = (D_\mu q_\alpha^{f,i})^\dagger \xi_{1/2}^\mu \xi_{1/2}^\nu (D_\nu q_\beta^{f,j}) - m_f^2 q_\alpha^{\dagger f,i} q_\beta^{f,j} - \frac{1}{2g_{QCD}^2} Tr(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{486}$$

where

$$D_\mu = (\partial_\mu + ig_{QCD}\tilde{A}_\mu); \;\; \tilde{A}_\mu(x) = \frac{1}{2}\lambda^a \tilde{A}_\mu^a(x)$$

$$\tilde{F}_{\mu\nu} = (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + ig_{QCD}[\tilde{A}_\mu, \tilde{A}_\nu].$$

Also in this case, we, intentionally, omitted the ghost fields and the "gauge-fixing" term, by waiting that the perturbation theory and functional integral will be suitably studied and developed for the α-Theory.

5 Quantization and Statistics

This chapter is concerned about the second quantization of the α-Theory and the study of statistics which the particles it describes are subject to. We will see this will take us to the important generalization of the Pauli principle, through which it will be possible to widen the concept of Dirac sea, which we could apply to all the elementary particles (except the gauge bosons), thus reaching a more general idea of vacuum. Then, the generalized Pauli principle will be in a position to characterize new types of statistics for the particles of matter, according to their spins (multi-statistics). This will allow to assume the existence of clusters (s-matter), consisting of particles with $s \neq 1/2$, which could be the true sources of the lacking mass of our universe (Dark Matter).

5.1 The second quantization of the α-Theory

In this section, we want to deal with the second quantization of the theories characterized by $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$ (as already explained the α-Theory is only one of the two: the one which will be better in agreement with the experiments). We will see this will give indications about the type of statistics that the particles described by ψ_s or ψ_s should satisfy.

In general terms, the process of the second quantization of a classical field consists of the following points

1. All the variables of the examined system become operators.
2. The field operator must be written according to creation and annihilation operators.
3. One establishes the statistics of particles constituting the field, for fixing the algebra of creation and annihilation operators.
4. One applies to the observables of the system the process of "Normal Ordering," in order to avoid divergences on their vacuum expectation value.
5. One assumes the existence of the vacuum state defined by

$$f(\alpha_n)\,|0\rangle \equiv 0$$

and such that [108]

$$|\alpha_1, \ldots, \alpha_j, \ldots\rangle \equiv f^\dagger(\alpha_1)\cdots f^\dagger(\alpha_j)\cdots|0\rangle\,,$$

[108] Such a system of vectors characterizes the Fock space of the system.

where with α_n, $f(\alpha_n)$ and $f^\dagger(\alpha_n)$ we defined the generic *nth* occupation number, annihilation and creation operators, respectively.

We promptly notice the points 1, 2, 4, can be applied without particular problems to $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$, while the points 3 and 5 need to know the elementary particles statistics, which for such theories we did not fix *a priori*. In particular, regarding point 3, we could also omit it, because, thanks to the appendix A, we established no relationship between CCR (or CAR) and statistics exists.

Now we want singularly to deal with the second quantization of $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$, just starting from the asymmetric α-Theory. Based on points 1 and 2, by putting us in natural units, we have (the hat $\wedge$ indicates the written quantities are operators) [109]

$$\begin{cases} \mathcal{L}_{A\alpha} = (\hat{\psi}_s(x), \hat{\bar{\psi}}_s(x)) = \hat{\bar{\psi}}_s(i\chi^\mu\partial_\mu - m\mathbb{1}_s)\hat{\psi}_s \\ \hat{\pi}_s(x) = i\hat{\psi}_s^\dagger(x) \\ \mathcal{H}_{A\alpha}(\hat{\pi}_s(x), \hat{\psi}_s(x)) = i\hat{\psi}_s^\dagger\frac{\partial\hat{\psi}_s}{\partial t} = \hat{\pi}_s\frac{\partial\hat{\psi}_s}{\partial t} \end{cases} \tag{487}$$

$$\begin{cases} \hat{\psi}_{s=0}(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\left[\hat{b}(k)e^{-ik\cdot x} + \hat{d}^\dagger(k)e^{ik\cdot x}\right] \\ \hat{\psi}^\dagger_{s=0}(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\left[\hat{d}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}\right] \end{cases} \tag{488}$$

$$\text{for } s \text{ half-integer}: \quad \begin{cases} \hat{\psi}_s(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha\in\tilde{K}}\left[\hat{b}_\alpha(k)\hat{u}_s^{(\alpha)}(k)e^{-ik\cdot x} + \hat{d}_\alpha^\dagger(k)\hat{v}_s^{(\alpha)}(k)e^{ik\cdot x}\right] \\ \hat{\bar{\psi}}_s(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha\in\tilde{K}}\left[\hat{d}_\alpha(k)\hat{\bar{v}}_s^{(\alpha)}(k)e^{-ik\cdot x} + \hat{b}_\alpha^\dagger(k)\hat{\bar{u}}_s^{(\alpha)}(k)e^{ik\cdot x}\right] \end{cases} \tag{489}$$

$$\text{for } s \text{ integer}: \quad \begin{cases} \hat{\psi}_s(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\left[\sum_{\alpha\in M}\hat{b}_\alpha(k)\hat{u}_s^{(\alpha)}(k)e^{-ik\cdot x} + \sum_{\beta\in N}\hat{d}_\beta^\dagger(k)\hat{v}_s^{(\beta)}(k)e^{ik\cdot x}\right] \\ \hat{\bar{\psi}}_s(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\left[\sum_{\beta\in N}\hat{d}_\beta(k)\hat{\bar{v}}_s^{(\beta)}(k)e^{-ik\cdot x} + \sum_{\alpha\in M}\hat{b}_\alpha^\dagger(k)\hat{\bar{u}}_s^{(\alpha)}(k)e^{ik\cdot x}\right]. \end{cases} \tag{490}$$

From all these expressions, we see the old coefficients d_α and b_α now have become creation and annihilation operators. In particular, the operators $\hat{b}_\alpha$, multiplying in the several $\hat{\psi}_s$ the solutions with positive energy, represent the annihilation operators, while the $\hat{d}_\alpha^\dagger$, multiplying the solutions

[109] Remember that: $\tilde{K} = \{1, \ldots, \frac{2s+1}{2}\}$; $M = \{1, \ldots, s+1\}$ and $N = \{1, \ldots, s\}$.

with negative energy, represent the creation operators. Since more types of solutions for the asymmetric α-Theory exist, we understand that making quantitative evaluations for this theory will be more boring than not for the symmetric one, which, as we know, has a single expression of ψ_s and $\psi_s^\dagger$, for any $s \in \mathbb{N}/2$. We begin our study calculating the energy $\mathcal{H}_{A\alpha}$. In particular, we write it in terms of creation and annihilation operators $\hat{d}_\alpha^\dagger$ and $\hat{b}_\alpha$. For such a purpose, we remember that (henceforth, for convenience, we omit the hat $\wedge$)

$$H_{A\alpha} = \int_{\mathbb{R}^3} d^3x \mathcal{H}_{A\alpha}, \quad \mathcal{H}_{A\alpha} = i\psi_s^\dagger \tfrac{\partial \psi_s}{\partial t}. \tag{491}$$

Since the expression of ψ_s is not equal for all $s \in \mathbb{N}/2$, we have to distinguish three cases:

1. $s = 0$
2. s half-integer
3. s integer,

from which three different expressions of $H_{A\alpha}$ will follow. We begin to estimate the energy for $s = 0$ and to this aim we calculate $i\partial\psi_{s=0}/\partial t$. We have

$$i\frac{\partial \psi_{s=0}}{\partial t} = \frac{1}{(2\pi)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\omega_k} \left[k_0 b(k) e^{-ik\cdot x} - k_0 d^\dagger(k) e^{ik\cdot x}\right]. \tag{492}$$

Now we can estimate $H_{A\alpha}^{s=0}$. By using the properties of the Dirac delta function and the mass-shell condition, it can be obtained, with some calculation, the expression [110]

$$H_{A\alpha}^{s=0} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k} \left[b^\dagger(k) b(k) - d(k) d^\dagger(k)\right], \tag{493}$$

which represents the energy of the particles having $s = 0$ described through $\mathcal{L}_{A\alpha}$. Before making any consideration on its form, we write the energy of the particles with s integer and half-integer, respectively. Proceeding like $H_{A\alpha}^{s=0}$, it can be demonstrated that

$$H_{A\alpha}^{s\ half-integer} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k} \sum_{\alpha\in\tilde{K}} \left[b_\alpha^\dagger(k) b_\alpha(k) - d_\alpha(k) d_\alpha^\dagger(k)\right] \tag{494}$$

$$H_{A\alpha}^{s\ integer} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k} \left[\sum_{\alpha\in M} b_\alpha^\dagger(k) b_\alpha(k) - \sum_{\beta\in N} d_\beta(k) d_\beta^\dagger(k)\right]. \tag{495}$$

[110] In this case, we choose $\lambda = \pi$.

How it was easy to anticipate, the energies concerning the asymmetric α-Theory have a structure similar to the Dirac theory one, of which the AαT is a particular generalization. Therefore, for our theory, we can make the same argumentation which works for the energy of the Dirac theory, that means giving to creation and annihilation operators canonical commutation or anti-commutation relations allowing to make it positive. Since

$$b^\dagger(k)b(k) \equiv |b(k)|^2; \quad b^\dagger_\alpha(k)b_\alpha(k) \equiv |b_\alpha(k)|^2, \tag{496}$$

the only way for having in the expressions of the energies, concerning several s, a positive term is to impose the following anti-commutation rules

$$d(k)d^\dagger(k) = -d^\dagger(k)d(k) + \varkappa = -|d(k)|^2 + \varkappa \tag{497}$$

$$d_\alpha(k)d^\dagger_\alpha(k) = -d^\dagger_\alpha(k)d_\alpha(k) + \varkappa = -|d_\alpha(k)|^2 + \varkappa \tag{498}$$

and so to obtain positive-definite energies of the form [111]

$$H^{s=0}_{A\alpha} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k}\left[|b(k)|^2 + |d(k)|^2\right] \tag{499}$$

$$H^{s\ half-integer}_{A\alpha} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k}\sum_{\alpha\in\tilde{K}}\left[|b_\alpha(k)|^2 + |d_\alpha(k)|^2\right] \tag{500}$$

$$H^{s\ integer}_{A\alpha} = \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3\omega_k}\left[\sum_{\alpha\in M}|b_\alpha(k)|^2 + \sum_{\beta\in N}|d_\beta(k)|^2\right]. \tag{501}$$

From these considerations, it emerges that creation and annihilation operators of the particles with arbitrary spin, described by the asymmetric α-Theory, must respect canonical anti-commutation relations (CAR), *i.e.* (we are always in natural units) [112]

[111] The constant $\varkappa$ is omitted in the Hamiltonians, since it plays the same role of "the energy of the Dirac sea" within them.

[112] In such a case $\varkappa = (2\pi)^3\omega_k\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta}$.

$$\begin{cases} \{b_\alpha(k), b^\dagger_\beta(k')\} = \{d_\alpha(k), d^\dagger_\beta(k')\} = (2\pi)^3\omega_k\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ \{b_\alpha(k), b_\beta(k')\} = \{b^\dagger_\alpha(k), b^\dagger_\beta(k')\} = 0 \\ \{d_\alpha(k), d_\beta(k')\} = \{d^\dagger_\alpha(k), d^\dagger_\beta(k')\} = 0. \end{cases} \tag{502}$$

These CAR, naturally, are valid for any $s \in \mathbb{N}/2$. It is interesting to notice that, for $s = 0$, they take to

$$\{\psi_{s=0}(x), \psi^\dagger_{s=0}(y)\} = \int_{\mathbb{K}^3} \frac{d^3k}{4\pi^3\omega_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \cos k_o(x_0 - y_0) \tag{503}$$

$$\{\psi_{s=0}(x), \psi_{s=0}(y)\} = \{\psi^\dagger_{s=0}(x), \psi^\dagger_{s=0}(y)\} = 0. \tag{504}$$

Before commenting in detail the result obtained for the AαT, it is better to quantize the symmetric α-Theory too, thus being able to make a general discussion for both our theories, by also using the results found in the appendix A.

Let us look now to quantize the symmetric α-Theory. We understood the fundamental point in order to apply the rules of the second quantization – seen at the beginning of this section – it is to find the algebra concerning the creation and annihilation operators of the system. This, like previously explained, can be just made through the study of the energy concerning the system described by $\mathcal{L}_{S\alpha}$, according to creation and annihilation operators. Unfortunately, the energy of $\mathcal{L}_{S\alpha}$ has a more complex form than $\mathcal{L}_{A\alpha}$ one and this renders our goal complicated enough. Therefore, what firstly we must do is to put $H_{S\alpha}$ in a simpler form to deal with. To this purpose, we rewrite the expression of $H_{S\alpha}$ seen in the previous chapter

$$H_{S\alpha} = \int_{\mathbb{R}^3} d^3x \mathcal{H}_{S\alpha} = \int_{\mathbb{R}^3} d^3x \Big[(\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^0(\partial_0\psi_s) + (\partial_0\psi^\dagger_s)\xi^0\xi^\alpha(\partial_\alpha\psi_s) - (\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^\beta(\partial_\beta\psi_s) + m^2\psi^\dagger_s\psi_s\Big], \tag{505}$$

and we try to place the term in the square bracket in a more manageable form. It is simple to see that

$$(\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^0(\partial_0\psi_s) + (\partial_0\psi^\dagger_s)\xi^0\xi^\alpha(\partial_\alpha\psi_s) - (\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^\beta(\partial_\beta\psi_s) + m^2\psi^\dagger_s\psi_s = \dot{\psi}^\dagger_s\dot{\psi}_s - (\partial_i\psi^\dagger_s)\xi^i\xi^j(\partial_j\psi_s) + m^2\psi^\dagger_s\psi_s, \tag{506}$$

where we have taken advantage of the fact that $\xi^0 = \delta$ and $\delta^2 = \mathbb{1}_s$. Therefore, we have

$$\mathcal{H}_{S\alpha} = |\dot{\psi}_s|^2 - (\nabla^i \psi_s^\dagger)\xi^i \xi^j (\nabla^j \psi_s) + m^2 |\psi_s|^2. \tag{507}$$

But the above expression can be further simplified by remembering that in general $\xi^i = i\varepsilon^i$, from which promptly follows [113]

$$\mathcal{H}_{S\alpha} = |\dot{\psi}_s|^2 + (\nabla_i \psi_s^\dagger)\varepsilon_i \varepsilon_j (\nabla_j \psi_s) + m^2 |\psi_s|^2, \tag{508}$$

which is nothing but the generalization of the Hamiltonian density of a n-dimensional complex Klein-Gordon field ϕ given by

$$\mathcal{H}_{K-G} = |\dot{\phi}|^2 + |\vec{\nabla}\phi|^2 + m^2 |\phi|^2 \tag{509}$$

and that can also be written

$$\mathcal{H}_{K-G} = |\dot{\phi}|^2 + (\nabla_i \phi^\dagger)\delta_{ij}(\nabla_j \phi) + m^2 |\phi|^2. \tag{510}$$

Hence, $\mathcal{H}_{S\alpha}$ and $\mathcal{H}_{K-G}$ are equal, apart from the substitutions

$$\phi(x) \to \psi_s(x)$$
$$(\nabla_i \phi^\dagger)\delta_{ij}(\nabla_j \phi) \to (\nabla_i \psi_s^\dagger)\varepsilon_i \varepsilon_j (\nabla_j \psi_s).$$

What we must make now is to express $\mathcal{H}_{S\alpha}$ in function of creation and annihilation operators of the symmetric α-Theory, thus to characterize their algebra. This happens by replacing in the (508) the fields (they are expressed always in natural units)

$$\psi_s(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\omega_k} \sum_{\alpha=1}^{2s+1} z_s^{(\alpha)}(k) \left[a_\alpha(k) e^{-ik\cdot x} + b_\alpha^\dagger(k) e^{ik\cdot x} \right] \tag{511}$$

$$\psi_s^\dagger(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{K}^3} \frac{d^3k}{2|\lambda|\omega_k} \sum_{\alpha=1}^{2s+1} z_s^{\dagger(\alpha)}(k) \left[a_\alpha^\dagger(k) e^{ik\cdot x} + b_\alpha(k) e^{-ik\cdot x} \right], \tag{512}$$

where, with $a_\alpha(k)$ and $b_\alpha^\dagger(k)$, we indicated, respectively, the annihilation and creation operators of our theory. From ψ_s and $\psi_s^\dagger$, one easily finds

[113] We put the lower indices only, since $\mathcal{H}_{S\alpha}$ is now expressed in vectorial form.

$$\dot{\psi}_s = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha=1}^{2s+1} z_s^{(\alpha)}(k)\left[-ik_0 a_\alpha(k)e^{-ik\cdot x} + ik_0 b_\alpha^\dagger(k)e^{ik\cdot x}\right] \tag{513}$$

$$\dot{\psi}_s^\dagger = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha=1}^{2s+1} z_s^{\dagger(\alpha)}(k)\left[ik_0 a_\alpha^\dagger(k)e^{ik\cdot x} - ik_0 b_\alpha(k)e^{-ik\cdot x}\right] \tag{514}$$

$$\nabla_j\psi_s = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha=1}^{2s+1} z_s^{(\alpha)}(k)\left[ik_j a_\alpha(k)e^{-ik\cdot x} - ik_j b_\alpha^\dagger(k)e^{ik\cdot x}\right] \tag{515}$$

$$\nabla_i\psi_s^\dagger = \frac{1}{(2\pi)^2}\int_{\mathbb{K}^3}\frac{d^3k}{2|\lambda|\omega_k}\sum_{\alpha=1}^{2s+1} z_s^{\dagger(\alpha)}(k)\left[-ik_i a_\alpha^\dagger(k)e^{ik\cdot x} + ik_i b_\alpha(k)e^{-ik\cdot x}\right], \tag{516}$$

from which it is possible to derivate, using the Dirac delta function and the "generalized mass-shell condition" [114]

$$k_0^2\mathbb{1}_s - k_i k_j \varepsilon_i\varepsilon_j = m^2\mathbb{1}_s, \tag{517}$$

the following expression of the energy about the theory characterized by $\mathcal{L}_{S\alpha}$[115]

$$H_{S\alpha} = \int_{\mathbb{K}^3}\frac{d^3k}{(2\pi)^3\omega_k}\omega_k\sum_{\alpha=1}^{2s+1}\left[a_\alpha^\dagger(k)a_\alpha(k) + b_\alpha(k)b_\alpha^\dagger(k)\right]. \tag{518}$$

We observe that the above expression is positive-definite only if $b_\alpha(k)b_\alpha^\dagger(k)$ can be replaced with

$$b_\alpha^\dagger(k)b_\alpha(k) \equiv |b_\alpha(k)|^2, \tag{519}$$

and this can only happen if creation and annihilation operators of the elementary particles described by $\mathcal{L}_{S\alpha}$ satisfy the commutation relations

$$\begin{cases} [a_\alpha(k), a_\beta^\dagger(k')] = [b_\alpha(k), b_\beta^\dagger(k')] = 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ [a_\alpha(k), a_\beta(k')] = [a_\alpha^\dagger(k), a_\beta^\dagger(k')] = 0 \\ [b_\alpha(k), b_\beta(k')] = [b_\alpha^\dagger(k), b_\beta^\dagger(k')] = 0, \end{cases} \tag{520}$$

[114] The (517) is nothing but a constraint derived from the relation (in natural units):

$$\xi^\mu\xi^\nu k_\mu k_\nu = k_0^2\mathbb{1}_s - k_i k_j\varepsilon_i\varepsilon_j - i(\delta\varepsilon_i + \varepsilon_i\delta)k_0 k_i = m^2\mathbb{1}_s,$$

which the imaginary part is removed to, since it obviously cannot enter in the reckoning of a bradyonic energy.

[115] In this case, we choose $\lambda = 2\pi$.

which are valid for any $s \in \mathbb{N}/2$. It is interesting to notice that based on them, for $s = 0$, one has

$$[\psi_{s=0}(x), \psi^{\dagger}_{s=0}(y)] = -i \int_{\mathbb{K}^3} \frac{d^3k}{(2\pi)^3 \omega_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sin k_o(x_0 - y_0) \tag{521}$$

$$[\psi_{s=0}(x), \psi_{s=0}(y)] = [\psi^{\dagger}_{s=0}(x), \psi^{\dagger}_{s=0}(y)] = 0. \tag{522}$$

Therefore, we demonstrated that the theory characterized by $\mathcal{L}_{A\alpha}$ admits CAR for creation and annihilation operators and for the particle fields, while the theory characterized by $\mathcal{L}_{S\alpha}$ admits CCR for creation and annihilation operators and for the particle fields. Does this mean that $\mathcal{L}_{A\alpha}$ describes fermions for any $s \in \mathbb{N}/2$, while $\mathcal{L}_{S\alpha}$ describes bosons only? It is not really thus, because a proof on the existence of a direct relationship between CCR (or CAR) and statistics does not exist, as we can see in the appendix A. This wrong conception was born from the analysis of the Dirac and Klein-Gordon theories. In such a case, starting by the hypothesis that the Dirac theory describes the fermions and Klein-Gordon one the bosons, and stating that, in order to render positive the energy of the first, CAR must be imposed, while, in order to render positive the energy of the second, CCR must be imposed, it has been arbitrarily believed that the CAR are reserved to fermions and the CCR are reserved to bosons. Instead, as it can be observed in the appendix A, there is no proof that anti-commutation rules are related to the Fermi-Dirac statistics and commutation rules to the Bose-Einstein statistics. Hence, in the case of the theories described by $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$, we could assume, without problems, that the particle fields which they describe satisfy the algebraic relations previously found (CAR for $\mathcal{L}_{A\alpha}$ and CCR for $\mathcal{L}_{S\alpha}$).

Thereupon, a problem rises: if the commutation and anti-commutation relations are not direct consequence of the statistics of particles, what is the statistics of particles described by $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$? The next section just proposes to clear this point, thus completing the speech on the second quantization of the α-Theory.

5.2 Statistics of the α-Theory: generalized Pauli principle and Dirac sea extension

We want now, thanks to the previous results and precious appendix A, to establish the statistics of the asymmetric and symmetric α-Theory. According to what we saw in the already cited appendix A, a good particle theory would have to be in a position to predict – varying the spin s – the structure of the single quantum state or, equally, the maximum occupation number of the elementary quantum state of a particle with arbitrary spin s. Of course, a theory able to make this must also predict that the elementary quantum state for $s = 1/2$ has maximum occupation number equal to 1, *i.e.* the Pauli exclusion principle must naturally emerge.

Does the asymmetric and symmetric α-Theory satisfy this request?

From the study of the energetic spectra of the AαE and SαE, made in the section (2.2), one sees, regarding the particles, the following energetic distribution [116]

Energetic Distribution		
Spin	AαE	SαE
0	mc^2	mc^2
$\frac{1}{2}$	mc^2	mc^2, mc^2
1	mc^2, mc^2	mc^2, mc^2, mc^2
$\frac{3}{2}$	mc^2, mc^2	mc^2, mc^2, mc^2, mc^2
2	mc^2, mc^2, mc^2	mc^2, mc^2, mc^2, mc^2, mc^2
...	....	....

Where, in general terms, the AαE admits $(s+1)$ particles for s integer (zero included) and $(2s+1)/2$ particles for s half-integer, while the SαE admits $(2s+1)$ particles for any $s \in \mathbb{N}/2$.[117]

What now we want to make is to extrapolate the Pauli exclusion principle from the above spectra of particles, producing a one-to-one correspondence between such spectra and the elementary quantum states. The fact that AαE and SαE describe field theories (number of particles tending to infinite), while their spectra identify, for any s, a finite number of particles (or anti-particles), already suggests a relationship existing between these spectra and the elementary quantum states, *i.e.* among number of particles and occupation numbers. From the above table, we can see the Pauli principle comes out in a natural way from the AαE (and so from the asymmetric α-Theory). In fact, if we associate the energetic distribution of our theory to the elementary quantum states, the AαE, having for $s = 1/2$ one particle with energy mc^2, satisfies the Pauli principle concerning the maximum occupation number, which must be equal to 1. This does not happen for the SαE, that, for $s = 1/2$, admits two particles having energy mc^2 and so the elementary quantum state, in such a case, has maximum occupation number equal to 2, against the Pauli principle. Nevertheless, before definitely rejecting the SαE and thus the symmetric α-Theory, we need to consider that the own properties of the quantum states and their filling (*i.e.* occupation numbers) could be more deep and complex than we think, namely the generic maximum occupation number of a quantum state could not exactly coincide with the energetic distribution, respecting clearly the Pauli principle as the AαE one. Based on this consideration, we want to understand if a simple relation between maximum occupation number and spin exists, such that also the symmetric α-Theory is able to satisfy the Pauli principle for $s = 1/2$. It is easy to verify that, defined for the AαE

$$\max \alpha_s \equiv \frac{s\theta_s + 1}{\theta_s}, \quad \theta_s \equiv \begin{cases} 1 & \text{if } s \text{ is an integer, included } s = 0 \\ 2 & \text{if } s \text{ is a half-integer.} \end{cases} \tag{523}$$

[116] Regarding the anti-particles see the next pages.

[117] On the contrary, regarding the anti-particles, the AαE admits $(s+\delta_{0s})$ anti-particles for s integer (zero included) and $(2s+1)/2$ anti-particles for s half-integer, while the SαE has – perfectly balanced with the particles – $(2s+1)$ anti-particles for any $s \in \mathbb{N}/2$.

and for the SαE [118]

$$\max\alpha_s \equiv \frac{2s+1}{\theta_s}, \quad \theta_s \equiv \begin{cases} 1 & \text{if } s \text{ is an integer, included } s=0 \\ 2 & \text{if } s \text{ is a half-integer.} \end{cases} \tag{524}$$

the Pauli principle is satisfied for both equations, *i.e.* it is valid for the asymmetric and symmetric α-Theory. But relations (523) and (524) do not only concur to find $\max\alpha_s = 1$ at $s = 1/2$ for both theories, but they establish a correspondence between occupation numbers and spin of the elementary particles. In fact, by bearing in mind

$$\alpha_s = 0, 1, 2, \ldots, \max\alpha_s \tag{525}$$

we have that, while the particles having $s = 0$ and $s = 1/2$ satisfy the Pauli principle, those with higher spin continue to respect an exclusion principle, but more general.

What has just been said can be immediately visualized by making clear the occupation numbers of our theories, based on (523) and (524), varying s:

Occupation Number Distribution		
α_s	AαT	SαT
$\alpha_{s=0}$	0, 1	0, 1
$\alpha_{s=1/2}$	0, 1	0, 1
$\alpha_{s=1}$	0, 1, 2	0, 1, 2, 3
$\alpha_{s=3/2}$	0, 1, 2	0, 1, 2
$\alpha_{s=2}$	0, 1, 2, 3	0, 1, 2, 3, 4, 5
$\alpha_{s=5/2}$	0, 1, 2, 3	0, 1, 2, 3
......	....	....

This allows to enunciate the following "generalized Pauli principle":

Depending on that the elementary particles are described by the asymmetric or symmetric α-Theory, the maximum occupation number of their fundamental quantum state is given, respectively, by

[118] This expression reduces the maximum filling of the energetic distribution on the half-integer spins, without changing the one on the integer spins.

- $\max \alpha_s \equiv \frac{s\theta_s+1}{\theta_s}$.

- $\max \alpha_s \equiv \frac{2s+1}{\theta_s}$.

The scrupulous reader can also notice the similarities between the number θ_s, defined in (523) and (524), and the gyromagnetic ratio of the electrons, which is, as is well-known, equal to 2. Can θ_s be the generalization of the gyromagnetic ratio concerning the elementary particles? If this were true, named g_s the "generalized gyromagnetic ratio," one could place

$$g_s = \theta_s \equiv \begin{cases} 1 & \text{if } s \text{ is an integer, included } s = 0. \\ 2 & \text{if } s \text{ is a half-integer.} \end{cases} \tag{526}$$

and so to have quantities with intrinsic physical meaning inside of our expressions.

The just enunciated generalized Pauli principle is able to consider the eventuality that in our universe there is another type of "matter," made by clusters having particles with spin different from 1/2. In fact, like the ordinary matter – constituted by atoms – exists because the electrons into atomic orbitals are subject to the Pauli principle, thus it can be assumed also the other particles with $s \neq 1/2$, having a different filling for their fundamental quantum state, can form a type of matter which is not the known one, *i.e.* that of which we are made. This means the matter too depends in some way on the spin s of the elementary particles, just thanks to the generalized Pauli principle. From this point of view, it would be more correct to speak about "s-matter," intended as that cluster which, depending on spin of the elementary particles constituting it, originates a certain "condensate." Therefore, we can assert that we are made of "1/2-matter" and the other matter we are not able to directly observe – but that however exists – it is nothing but a *stratification* of others s-matters raising after the Big-Break. Hence, what we called "Dark Matter" could be nothing more than such s-matters, *i.e.* that type of matter in which the particles having $s \neq 1/2$ are clustered, in virtue of the generalized Pauli principle (naturally this could include more complex form of interactions, maybe different from the *four* commonly known).

Furthermore, this principle concurs to take in consideration the extension of another key concept of the elementary particle physics too, namely the "Dirac sea." As we know, this idea was introduced by Dirac, in order to justify the solutions with negative energy, which at the dawn of the relativistic quantum theories were considered like formulation errors. For obviating this problem, Dirac thought to consider the vacuum not like the state lacking in matter, but as "sea" of particles with negative energy (according to Dirac the vacuum has, therefore, energy tending to $-\infty$). Hence, the particles with positive energy are stable, because the elementary quantum states of the sea are all occupied thanks to the validity of the Pauli exclusion principle, which does not allow to have occupation number greater than 1 for the elementary quantum states (so the particles with positive energy cannot fall in the sea, otherwise the states with occupation number equal to 2 should be). Nevertheless, one can consider cases in which a particle with negative energy, after an energetic absorption, *jumps* in a state with positive energy, thus leaving a hole in the sea. If the considered fields are constituted by electrons, the hole represents a positron, particle effectively observed. The possibility about creation of holes in the Dirac sea is called, in the case of electron fields, "electron-positron pair production." Another possibility, expected from the Dirac sea, is that a particle with positive energy *falls* in a hole, emitting radiation. This phenomenon, experimentally observed, is said "annihilation." Thanks to experimental confirmation of these theoretical predictions, the Dirac sea has had great success in the elementary particle physics. However, like already pointed out, it was constructed for electrons or – more general – for particles with $s = 1/2$, *i.e.* for those particles described by the Dirac equation only. With the birth of the α-Theory, in the light of what explained in these pages, and, in particular, thanks to the generalized Pauli principle, we can think to extend the idea of the Dirac sea to all elementary particles having arbitrary spin. What involves to introduce the concept of "generalized Dirac sea"? Firstly, it concurs to have an idea of vacuum which is homogeneous for all elementary particles, in the sense that generalized Dirac sea could just be the "vacuum state" of the s-matter. This means we can imagine this vacuum like a *stratification* of all the particles with negative energy, that interact with the correspondent particles having positive energy, due to the generalized Pauli principle. But now the question is: can the generalized Pauli principle, expressed through the (523) and (524), be applied to the anti-particles? Concerning the SαT, the answer is affirmative, since we have an equal number of particles and anti-particles for any $s \in \mathbb{N}/2$. On the contrary, for the AαT, it is not so, because we have $(s + \delta_{0s})$ anti-particles for integer spins and $(2s + 1)/2$ anti-particles for half-integer spins. This means the generalized Pauli principle, for the anti-particles described by the AαT, must be defined in such a way

$$\max \alpha_s \equiv \frac{s\theta_s + \delta_{s\theta_s,2s}}{\theta_s}. \tag{527}$$

Instead, regarding the anti-particles described from the SαT, we always have

$$\max \alpha_s \equiv \frac{2s + 1}{\theta_s}. \tag{528}$$

Naturally, this does not change the ordinary Dirac sea, since for particles as well as anti-particles with spin $s = 0,\ 1/2$, being the elementary quantum states characterized by an occupation number equal to 0 or 1, we always have phenomena of particle-antiparticle pair creation and annihilation of particles. Instead, for the elementary particles and anti-particles having higher spin, being for them an increase of s with the occupation number, not only the above-mentioned phenomena will occur, but there could be "multiple annihilations" and "particle-antiparticles multiplets" too. The generalized Dirac sea does not concur to heal the philosophical problems of the original Dirac sea only, but it allows to have a concept of vacuum more in line with the modern cosmological theories, based on the "vacuum energy" and Dark Matter. Naturally, the expressions (527) and (528) are referred to the occupation numbers concerning the quantum states of the generalized Dirac sea and so they inform as the vacuum is made. For the AαT, such a vacuum has a different generalized Pauli principle than the one of the particles, while, for the SαT, it is, as we easily could expect, perfectly symmetric. In any case, we can see that, for $s = 1/2$, the Pauli principle is always respected, and so the original Dirac idea about the "sea" does not change for $s = 1/2$.

Now we are in a position to construct the statistics of the asymmetric and symmetric α-Theory.

Based on (523), we saw in the previous pages that AαT presents an increasing occupation number with the spin. More precisely, the particles characterized by s integer and $(s+1/2)$ half-integer have equal occupation numbers and so there is a degeneracy phenomenon. The situation is analogue for the SαT but much more interesting, because it presents increasing and decreasing maximum occupation numbers with the rise of s. In particular, it is easy to see the particles with integer spin ($s = 0$ included) have maximum occupation number following the run of the odd numbers, while the particles with half-integer spin have maximum occupation number following the run of the natural numbers.[119] Therefore, like the AαT, the SαT does not present equal filling only for $s = 0$ and $s = 1/2$, but also for other spins (as an example $s = 1$ and $s = 5/2$, $s = 2$ and $s = 9/2$). Generally, it is straightforward to see that within this theory the fields with integer spin ($s = 0$ included) and fields with half-integer spin equal to [120]

$$\frac{4s^* + 1}{2} \tag{529}$$

have equal filling for the elementary quantum state.

Now we can proceed to the individuation of the statistics concerning the AαT and SαT. This can be easily made thanks to the general relation

$$\Omega_k = -\mathsf{k}T \ln \sum_{\alpha_k} \left[e^{\beta(\mu - E_k)}\right]^{\alpha_k}, \quad \begin{cases} \mathsf{k} \equiv \text{Boltzmann constant} \\ T \equiv \text{absolute temperature} \\ \mu \equiv \text{chemical potential} \\ E_k \equiv \text{energy of a single particle into a generic } k\text{-state} \\ \beta \equiv \mathsf{k}T. \end{cases} \tag{530}$$

which connects the thermodynamic potential Ω_k to the occupation numbers α_k, by taking the definition

$$\bar{\alpha}_k \equiv -\frac{\partial \Omega_k}{\partial \mu}, \tag{531}$$

where $\bar{\alpha}_k$ is the average occupation number of the generic quantum state k, that basically represents the statistical distribution function of a gas of particles characterized by elementary quantum states having, singularly, occupation number α_k.

As an example, it is easy to find for a gas of particles which satisfy the Pauli exclusion principle ($\alpha_k = 0, 1$), the following statistical distribution

$$\bar{\alpha}_k = \frac{1}{e^{\beta(E_k - \mu)} + 1}, \tag{532}$$

[119] This is a very interesting aspect, because we could give a definition of odd and natural numbers based on physics.
[120] With s^*, we intended any integer spin.

which is the well-known Fermi-Dirac distribution.

On the contrary, for a gas of particles having unlimited occupation numbers ($\alpha_k = 0, 1, 2, 3, \ldots, \to \infty$), under the hypothesis $e^{\beta(\mu - E_k)} < 1$, it is banal to have

$$\bar{\alpha}_k = \frac{1}{e^{\beta(E_k - \mu)} - 1}, \tag{533}$$

which represents the universally renowned Bose-Einstein distribution.

We promptly understand that, in order to find the statistics of the AαT and SαT, it must be considered that each field with arbitrary spin s has an own occupation number and, therefore, a general statistical distribution must be found, giving, case by case, the statistics of the field with s fixed. From this point of view, it should be more corrected to speak about "multi-statistics," because each field with fixed spin s, described by the AαT or SαT, has an own statistical distribution (except the cases of *degeneracy* previously described). For resolving this problem, we can start from the expression (530) and then to impose

$$\alpha_k = 0, 1, 2, \ldots, n - 1 \tag{534}$$

from which, we have

$$\Omega_k = -\mathsf{k}T \ln \left\{ \sum_{\alpha_k = 0, 1, \ldots,}^{n-1} \left[e^{\beta(\mu - E_k)} \right]^{\alpha_k} \right\} = -\mathsf{k}T \ln \left[\frac{1 - e^{n\beta(\mu - E_k)}}{1 - e^{\beta(\mu - E_k)}} \right] \quad \forall n \in \mathbb{N} - \{0\}, \tag{535}$$

where n is the number of the ordered terms appearing in the sum: for example $n = 1$ refers to the first term of the sum and so on. Naturally, for construction, we have

$$\max \alpha_k = n - 1 \;\Rightarrow\; n = \max \alpha_k + 1. \tag{536}$$

To sum up, we found that, regarding the AαT and SαT, the right thermodynamic potential is

$$\Omega_k = -\mathsf{k}T \ln \left[\frac{1 - e^{n\beta(\mu - E_k)}}{1 - e^{\beta(\mu - E_k)}} \right] = \mathsf{k}T \ln \left[\frac{1 - e^{\beta(\mu - E_k)}}{1 - e^{n\beta(\mu - E_k)}} \right], \tag{537}$$

from which it follows that the statistical distribution of a generic gas of particles with maximum occupation number ($\max \alpha_k$) is

$$\bar{\alpha}_k \equiv -\frac{\partial \Omega_k}{\partial \mu} = \frac{1}{e^{\beta(E_k - \mu)} - 1} - \frac{n}{e^{n\beta(E_k - \mu)} - 1}. \tag{538}$$

For testing the adequacy of the obtained result, one must verify the (538) gives as result the famous Fermi-Dirac and Bose-Einstein distributions. We begin to notice that, for $n = 1$ (*i.e.* for $\max \alpha_k = 0$), one has

$$\bar{\alpha}_k = \frac{1}{e^{\beta(E_k-\mu)} - 1} - \frac{1}{e^{\beta(E_k-\mu)} - 1} = 0, \tag{539}$$

which, correctly, is the result expected for $\alpha_k = 0$. Instead, for $n = 2$ (precisely for $\max \alpha_k = 1$, *i.e.* $\alpha_k = 0, 1$), we have

$$\bar{\alpha}_k = \frac{1}{e^{\beta(E_k-\mu)} - 1} - \frac{2}{e^{2\beta(E_k-\mu)} - 1} = \frac{1}{e^{\beta(E_k-\mu)} + 1}, \tag{540}$$

just corresponding to the Fermi-Dirac distribution.

What happens if $n \to \infty$ ($\alpha_k = 0, 1, 2, 3, \ldots, \to \infty$) and $e^{\beta(\mu - E_k)} < 1$? In this case, since $e^{n\beta(E_k-\mu)}$ approaches ∞ faster than n, we have

$$\bar{\alpha}_k = \lim_{n\to\infty} \left[\frac{1}{e^{\beta(E_k-\mu)} - 1} - \frac{n}{e^{n\beta(E_k-\mu)} - 1} \right] = \frac{1}{e^{\beta(E_k-\mu)} - 1}, \tag{541}$$

which is the Bose-Einstein distribution.

Therefore, the general expression (538) is correct, because it shows a perfect agreement with the known results of the Statistical Mechanics. It can be noted that, having to be $n = \max \alpha_k + 1$, we can also write

$$\bar{\alpha}_k = \frac{1}{e^{\beta(E_k-\mu)} - 1} - \frac{\max \alpha_k + 1}{e^{\beta(\max \alpha_k + 1)(E_k-\mu)} - 1}, \tag{542}$$

and this, inextricably, links the statistical distribution of a particle field with arbitrary spin s to the maximum occupation number of the elementary quantum state of this field. But there is more, because from the relations

$$\max \alpha_s \equiv \frac{s\theta_s + 1}{\theta_s} \qquad (\text{A}\alpha\text{T})$$

$$\max \alpha_s \equiv \frac{2s + 1}{\theta_s}, \qquad (\text{S}\alpha\text{T})$$

we can write (the use of double index k, s is obvious)

$$\bar{\alpha}_{k,s} = \frac{1}{e^{\beta(E_k-\mu)}-1} - \frac{1+\theta_s(1+s)}{\theta_s\left[e^{\frac{\beta[1+\theta_s(1+s)](E_k-\mu)}{\theta_s}}-1\right]} \tag{543}$$

$$\bar{\alpha}_{k,s} = \frac{1}{e^{\beta(E_k-\mu)}-1} - \frac{2s+\theta_s+1}{\theta_s\left[e^{\frac{\beta(2s+\theta_s+1)(E_k-\mu)}{\theta_s}}-1\right]}, \tag{544}$$

where the (543) represents the statistical distribution concerning a gas of particles with fixed spin s described by the AαT, while the (544) represents the statistical distribution concerning a gas of particles with fixed spin s described by the SαT. The wonderful thing is the fact these distributions determine an effective relation between spin of the particle fields which we want to study and the statistics which such fields are subject to. This means the two found distributions allow to establish a "spin-statistics relation," which remembers the one shown by Pauli in his theorem, even if it is completely different. In fact, while Pauli established (incorrectly) the existence of only two types of statistics for the elementary particles, *i.e.* the Bose-Einstein, for particles with integer spin (bosons), and the Fermi-Dirac, for particles with half-integer spin (fermions), the α-Theory, asymmetric or symmetric, proves each particle (field) with fixed spin s follows a type of statistics different from that concerning other particles with not equal spin (excluding the degeneration cases, that we previously described), and so the α-Theory opens the doors to the new concept of "s-matter" and "multi-statistics," which define a wholly different relation between spin and statistics, compared with the one used in QFT. As already said, this could induce to consider other "clusters" (s-matter), which could be what we called Dark Matter.

Before continuing, it is good to focus us on the nature of particles described by the α-Theory (asymmetric or symmetric). We established such particles follow different types of statistics varying the spin s (multi-statistics). The Fermi-Dirac statistics is obtained for particles having $s = 0$ or $s = 1/2$, while only for $n \to \infty$ we have the Bose-Einstein one. This induces to think the gauge bosons (such as the photons) are not really described by the AαE or SαE, but they could be somewhat a product of Big-Break, which, in some way, generated these *new* particles, that – unlike the "particles of matter," born from Big-Bang and described by the α-Theory (asymmetric or symmetric) – could have an own dynamics, which only the accurate study of Big-Break will be able to highlight. This means two classes of bosons could exist: those with integer spins born from Big-Bang and described by the α-Theory (matter bosons) and those born from Big-Break, having the task to mediate the fundamental interactions (gauge bosons). What is the spin value of such bosons? The physicists tend to assign them $s = 1$ ($s = 2$ for the hypothetical gravitons), even if in view of the theory dealt in this work it is suitable to leave this issue under discussion, before the full explanation of the Big-Break. In fact, only then we will be able to have answers about the dynamics these particles follow, their spin value and statistics which they are subjected to. It is clear, however, that their existence is direct consequence of the Big-Break and so their study cannot leave aside the α-Theory.

In order to conclude the discussion on the second quantization of the α-Theory, we must perform the point 5 of the previous section and *i.e.* to define the vacuum state $|0\rangle$, that, practically, is equivalent to characterize the Fock space of our system. Naturally, the asymmetric and symmetric α-Theory have a peculiar vacuum state (each s its own vacuum) and, therefore, distinguished Fock spaces. However, the Fock spaces of the AαT and SαT, in turn, decompose them in multi-spaces for each field with fixed spin, in virtue of the existing relationship between spin and maximum occupation number, which we largely treated in this section. Now we begin to define the vacuum state of the AαT, using the specific properties of the creation and annihilation operators of such a theory. Since, in this case, the creation and annihilation operators have three different well-defined sets, we have to distinguish many subsets. Therefore, we have

1. $s = 0$:

   Since $b(k)\,|0\rangle_1$=0, $d(k)|0\rangle_2$=0, we can define the vacuum state in this way

$$|0\rangle_{s=0} \equiv |0\rangle_1 \otimes |0\rangle_2 = |0, 0, 0, \ldots, 0, 0, \ldots\rangle. \tag{545}$$

   The generic (normalized) vector of this Fock space, having $|0\rangle_{s=0}$ as vacuum state, is given by the tensor product of the following vectors

$$|^{-\infty\leftarrow} \ldots, \alpha^b_{-k}, \ldots, \alpha^b_k, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha^b_{-k}! \cdots \alpha^b_k! \cdots}} \cdots \left(b^\dagger(-k)\right)^{\alpha^b_{-k}} \cdots \left(b^\dagger(k)\right)^{\alpha^b_k} \cdots |0\rangle_{s=0} \tag{546}$$

$$|^{-\infty\leftarrow} \ldots, \alpha^d_{-k}, \ldots, \alpha^d_k, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha^d_{-k}! \cdots \alpha^d_k! \cdots}} \cdots \left(d^\dagger(-k)\right)^{\alpha^d_{-k}} \cdots \left(d^\dagger(k)\right)^{\alpha^d_k} \cdots |0\rangle_{s=0}, \tag{547}$$

   that is

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}, \ldots, \alpha_k, \ldots^{\rightarrow+\infty}\rangle = |^{-\infty\leftarrow} \ldots, \alpha^b_{-k}, \ldots, \alpha^b_k, \ldots^{\rightarrow+\infty}\rangle \otimes |^{-\infty\leftarrow} \ldots, \alpha^d_{-k}, \ldots, \alpha^d_k, \ldots^{\rightarrow+\infty}\rangle. \tag{548}$$

   Having to be, in this case, the occupation numbers equal to 0 or 1, and, by remembering $0! = 1! = 1$, we get

$$\frac{1}{\sqrt{\cdots \alpha^b_{-k}! \cdots \alpha^b_k! \cdots}} = \frac{1}{\sqrt{\cdots \alpha^d_{-k}! \cdots \alpha^d_k! \cdots}} = 1. \tag{549}$$

2. s half-integer:

Since

$$\begin{cases} b_\gamma(k)|0\rangle_\gamma = 0 \\ d_\gamma(k)|0\rangle_{\gamma+\frac{2s+1}{2}} = 0 \quad \forall\gamma \in \tilde{K} = \left\{1, \ldots, \frac{2s+1}{2}\right\}, \end{cases} \tag{550}$$

we have $(2s+1)$ types of vacuum and, therefore, we can thus write the vacuum state of our system

$$|0\rangle_{s\ half\text{-}integer} \equiv |0\rangle_1 \otimes |0\rangle_2 \ldots \otimes |0\rangle_{2s+1} = |0,0,0,\ldots,0,0,\ldots\rangle. \tag{551}$$

The generic (normalized) vector of such a Fock space, having $|0\rangle_{s\ half\text{-}integer}$ like vacuum state, is given by the tensor product of the following vectors

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}^{b_\gamma}, \ldots, \alpha_k^{b_\gamma}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha_{-k}^{b_\gamma}! \cdots \alpha_k^{b_\gamma}! \cdots}} \cdots \left(b_\gamma{}^\dagger(-k)\right)^{\alpha_{-k}^{b_\gamma}} \cdots \left(b_\gamma{}^\dagger(k)\right)^{\alpha_k^{b_\gamma}} \cdots |0\rangle_{s\ half\text{-}integer} \tag{552}$$

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}^{d_\gamma}, \ldots, \alpha_k^{d_\gamma}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha_{-k}^{d_\gamma}! \cdots \alpha_k^{d_\gamma}! \cdots}} \cdots \left(d_\gamma{}^\dagger(-k)\right)^{\alpha_{-k}^{d_\gamma}} \cdots \left(d_\gamma{}^\dagger(k)\right)^{\alpha_k^{d_\gamma}} \cdots |0\rangle_{s\ half\text{-}integer} \tag{553}$$

namely

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}, \ldots, \alpha_k, \ldots^{\rightarrow+\infty}\rangle = |^{-\infty\leftarrow} \ldots, \alpha_{-k}^{b_\gamma}, \ldots, \alpha_k^{b_\gamma}, \ldots^{\rightarrow+\infty}\rangle \otimes |^{-\infty\leftarrow} \ldots, \alpha_{-k}^{d_\gamma}, \ldots, \alpha_k^{d_\gamma}, \ldots^{\rightarrow+\infty}\rangle. \tag{554}$$

For $s = 1/2$ ($\gamma = 1$), having to be the occupation numbers equal to 0 or 1, similarly to the previous case of $s = 0$, one has

$$\frac{1}{\sqrt{\cdots \alpha_{-k}^{b_1}! \cdots \alpha_k^{b_1}! \cdots}} = \frac{1}{\sqrt{\cdots \alpha_{-k}^{d_1}! \cdots \alpha_k^{d_1}! \cdots}} = 1. \tag{555}$$

3. s integer ($s = 0$ excluded):

Since

$$\begin{cases} b_\gamma(k)|0\rangle_\gamma = 0 \\ d_\beta(k)|0\rangle_{\beta+(s+1)} = 0 \end{cases} \quad ; \quad \begin{cases} \gamma \in M = \{1, \ldots, s+1\} \\ \beta \in N = \{1, \ldots, s\}, \end{cases} \tag{556}$$

also in such a case we have $(2s+1)$ types of vacuum. The (total) vacuum state of this system can, thus, be defined

$$|0\rangle_{s\ integer} \equiv |0\rangle_1 \otimes |0\rangle_2 \ldots \otimes |0\rangle_{2s+1} = |0,0,0,\ldots,0,0,\ldots\rangle. \tag{557}$$

The generic (normalized) vector of this Fock space, having $|0\rangle_{s\ integer}$ as vacuum state, is given by the tensor product of the following vectors

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}^{b_\gamma}, \ldots, \alpha_k^{b_\gamma}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha_{-k}^{b_\gamma}! \cdots \alpha_k^{b_\gamma}! \cdots}} \cdots \left(b_\gamma{}^\dagger(-k)\right)^{\alpha_{-k}^{b_\gamma}} \cdots \left(b_\gamma{}^\dagger(k)\right)^{\alpha_k^{b_\gamma}} \cdots |0\rangle_{s\ integer} \tag{558}$$

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}^{d_\beta}, \ldots, \alpha_k^{d_\beta}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha_{-k}^{d_\beta}! \cdots \alpha_k^{d_\beta}! \cdots}} \cdots \left(d_\beta{}^\dagger(-k)\right)^{\alpha_{-k}^{d_\beta}} \cdots \left(d_\beta{}^\dagger(k)\right)^{\alpha_k^{d_\beta}} \cdots |0\rangle_{s\ integer} \tag{559}$$

that is

$$|^{-\infty\leftarrow} \ldots, \alpha_{-k}, \ldots, \alpha_k, \ldots^{\rightarrow+\infty}\rangle = |^{-\infty\leftarrow} \ldots, \alpha_{-k}^{b_\gamma}, \ldots, \alpha_k^{b_\gamma}, \ldots^{\rightarrow+\infty}\rangle \otimes |^{-\infty\leftarrow} \ldots, \alpha_{-k}^{d_\beta}, \ldots, \alpha_k^{d_\beta}, \ldots^{\rightarrow+\infty}\rangle. \tag{560}$$

For the three just studied cases, regarding the operators $b_\epsilon(k)$ and $d_\epsilon(k)$, the following CAR are valid

$$\begin{cases} \{b_\epsilon(k), b^\dagger_\zeta(k')\} = \{d_\epsilon(k), d^\dagger_\zeta(k')\} = (2\pi)^3\omega_k\delta^3(\vec{k}-\vec{k}')\delta_{\epsilon\zeta} \\ \{b_\epsilon(k), b_\zeta(k')\} = \{b^\dagger_\epsilon(k), b^\dagger_\zeta(k')\} = 0 \\ \{d_\epsilon(k), d_\zeta(k')\} = \{d^\dagger_\epsilon(k), d^\dagger_\zeta(k')\} = 0, \end{cases} \tag{561}$$

where the subindices ϵ and ζ run on the sets already defined, according to the value of s. It is good to underline that, for all the introduced Fock spaces, we have [121]

$$\max \alpha_s \equiv \frac{s\theta_s + 1}{\theta_s}.$$

We now study the Fock space of the symmetric α-Theory. In such a case, being only a class of creation and annihilation operators for each value of $s \in \mathbb{N}/2$, our job will be less tedious than the just discussed one. In fact, by remembering that, in general terms, $a_\gamma(k)|0\rangle_\gamma = 0$ and $b_\gamma(k)|0\rangle_{\gamma+(2s+1)} = 0$ for any $\gamma \in \{1, 2, \dots, 2s+1\}$, we have, in this case, $(4s+2)$ types of vacuum. The (total) vacuum state of the SαT is

$$|0\rangle_s \equiv |0\rangle_1 \otimes |0\rangle_2 \ldots \otimes |0\rangle_{4s+2} = |0, 0, 0, \ldots, 0, 0, \ldots\rangle. \tag{562}$$

The generic (normalized) vector of this Fock space, having $|0\rangle_s$ as vacuum state, is given by the tensor product of the following vectors

$$|^{-\infty\leftarrow} \ldots, \alpha^{a_\gamma}_{-k}, \ldots, \alpha^{a_\gamma}_{k}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha^{a_\gamma}_{-k}! \cdots \alpha^{a_\gamma}_{k}! \cdots}} \cdots \left(a_\gamma{}^\dagger(-k)\right)^{\alpha^{a_\gamma}_{-k}} \cdots \left(a_\gamma{}^\dagger(k)\right)^{\alpha^{a_\gamma}_{k}} \cdots |0\rangle_s \tag{563}$$

$$|^{-\infty\leftarrow} \ldots, \alpha^{b_\gamma}_{-k}, \ldots, \alpha^{b_\gamma}_{k}, \ldots^{\rightarrow+\infty}\rangle = \frac{1}{\sqrt{\cdots \alpha^{b_\gamma}_{-k}! \cdots \alpha^{b_\gamma}_{k}! \cdots}} \cdots \left(b_\gamma{}^\dagger(-k)\right)^{\alpha^{b_\gamma}_{-k}} \cdots \left(b_\gamma{}^\dagger(k)\right)^{\alpha^{b_\gamma}_{k}} \cdots |0\rangle_s, \tag{564}$$

[121] This is true for the particles. Regarding the anti-particles, we should have

$$\max \alpha_s = \frac{s\theta_s + \delta_{s\theta_s, 2s}}{\theta_s}.$$

But this places a question: thinking the vacuum formed by anti-particles based on Dirac sea (or generalized Dirac sea), does it make sense to define a Fock space (or better several Fock spaces) for the anti-particles?

namely

$$|^{-\infty\leftarrow}\ldots,\alpha_{-k},\ldots,\alpha_k,\ldots^{\rightarrow+\infty}\rangle = |^{-\infty\leftarrow}\ldots,\alpha_{-k}^{a_\gamma},\ldots,\alpha_k^{a_\gamma},\ldots^{\rightarrow+\infty}\rangle\otimes|^{-\infty\leftarrow}\ldots,\alpha_{-k}^{b_\gamma},\ldots,\alpha_k^{b_\gamma},\ldots^{\rightarrow+\infty}\rangle, \tag{565}$$

where the operators $a_\epsilon(k)$ and $b_\epsilon(k)$ satisfy the following CCR

$$\begin{cases}[a_\epsilon(k),a_\zeta^\dagger(k')]=[b_\epsilon(k),b_\zeta^\dagger(k')]=2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{\epsilon\zeta}\\ [a_\epsilon(k),a_\zeta(k')]=[a_\epsilon^\dagger(k),a_\zeta^\dagger(k')]=0\\ [b_\epsilon(k),b_\zeta(k')]=[b_\epsilon^\dagger(k),b_\zeta^\dagger(k')]=0,\end{cases} \tag{566}$$

which are valid for any $s\in\mathbb{N}/2$.

In order to conclude, we remember that the relationship between spin and maximum occupation number in the symmetric α-Theory on the introduced Fock spaces is

$$\max\alpha_s\equiv\frac{2s+1}{\theta_s}.$$

What seen in this section ends our discussion on the second quantization of the α-Theory (AαT or SαT).

6 Miscellaneous α-Theory

In this chapter, we consider two topics rising as a result of the α-Theory development. The first one regards the gauge bosons, whose nature, based on the Big-Break model, could be more complex than that presented by the gauge theories. In particular, we will analyze the possibility that the global symmetry breaking is able to generate massive gauge bosons as mediators of the four fundamental interactions. As it will be explained, this could concur to resolve one of the most important open question of the modern mathematical physics, *i.e.* the existence of a Yang-Mills theory on $\mathbb{R}^4$ having a positive "mass gap." The second one, instead, is concerning why in our universe there are no particles with too high spin, but only those having a relatively low spin value. To such purpose, by assuming the energy of our universe distributed itself according to some group representation, we will define a simple selection rule, which will allow to connect the symmetry groups with the spin value of the elementary particles. This will contribute to obtain, by and large, the spins of the known particles, by making us to think a relationship between spin, energy and symmetry groups effectively exists.

6.1 Big-Break and mass gap. An approach to the solution of the Yang-Mills millennium prize problem

Like already pointed out in the previous section, the α-Theory, asymmetric or symmetric, if it will be proved to be the right theory for elementary particles, demands a substantial change of the interaction mechanisms, above all for what concerns the gauge theories. In fact, since the α-Theory arises by a process of "tachyon condensation," based on a phase transition from an unstable universe (IEP universe) to a stable universe (REP universe), we have that the transition from an *unstable vacuum* to a *stable vacuum*, shown in fig. 10, must be valid for the entire universe, *i.e.* all the interactions are derived by a spontaneous symmetry breaking mechanism. Such a process is not still known, but it could consist in a generalization of the "Higgs mechanism," which unified electromagnetic and weak interactions. Of course this process, that we called Big-Break, could lead to the unification of all the forces of the Nature, making of the α-Theory a GUT. The Big-Break could have generated and made massive all the gauge bosons and this could involve the existence of more types of Higgs bosons, or also of a single type, but that spans all the interactions. In particular, this global Higgs mechanism could have given mass to gluons, mediators of the strong interaction, thus explaining the short-range of the strong force, which is clear by the confinement of the quarks, and this avoiding to introduce exotic elements as the glueballs.

The possibility of a mass gap within the QCD is expected by the scientific community, that offered an high money prize for who will resolve the so-called "mass gap problem," which has been inserted in the *millennium seven problems*. However, as we can read in the specific description of the problem made by Jaffe and Witten [16], one thinks the existence of a mass gap $\Delta > 0$ bound with a quantum Yang-Mills theory, non-trivial, defined on $\mathbb{R}^4$ and that can be applied to any gauge group G, simple and compact, depends on mathematical properties inherent in the Yang-Mills theory, which have not been still sounded. It is clear that, if the α-Theory will be right, its generalization through the explanation of Big-Break could conduct to massive gauge fields, resolving, in such a way, the problem of the mass gap not from a mathematical point of view, but from a physical one. Hence, the "mass gap," supposed in QCD, could not be due to hidden mathematical properties, but to the incompleteness of modern theories on the elementary particles. For this reason, one thinks the study and verification of the α-Theory represent a fundamental stage for the high energy physics.

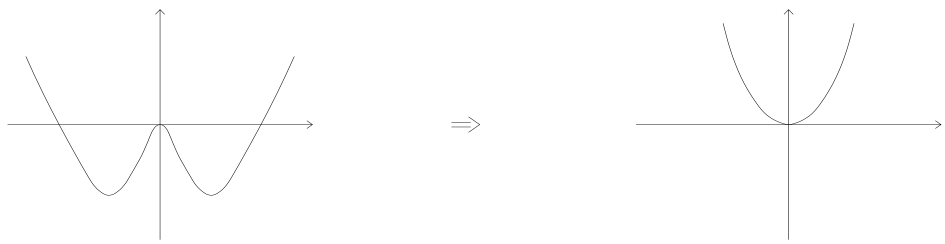

Figure 10: The transition from unstable to stable vacuum within the Big-Break framework. This process could have made massive all the gauge bosons.

6.2 Spins, gauge groups and energy. A simple selection rule

Arrived to this point, one may wonder why we observe in Nature particles with low spin values only, while, in principle, the asymmetric or symmetric α-Theory allow to estimate the physics of particles having arbitrary spin (*e.g.* also 10000001/2). We can answer this question by supposing a relationship exists between spins, gauge groups and energy, that *selections* only lower values of s. In particular, we can imagine that when our universe came into being, the great energy which generated, *distributed* itself in a uniform way, according with the representations of the groups $U(1)$, $SU(2)$, $SU(3)$, *etc.*, and so it makes sense to believe such representations have fixed the spin value of the particles subjected to the interactions characterizing these symmetry groups. Therefore, given a symmetry group $G(n)$, a selection rule R could exist such that

$$R\left[G(n)\right] = \text{finite set of } s \text{ derived by } n \in \mathbb{N}. \tag{567}$$

Naturally, if $n = \mathbb{N}$, we must have

$$R\left[G(\mathbb{N})\right] = \text{all the representations of } SU(2). \tag{568}$$

We now consider a selection rule $\tilde{R}$ such that, given the symmetry group $G(n)$, one has

$$l = \frac{n}{2}, \frac{n-1}{2}, \frac{n-2}{2}, \ldots, -\frac{n}{2} \tag{569}$$

and, by defining

$$s \equiv |l|, \tag{570}$$

the selection rule $\tilde{R}$ for the symmetry groups $U(1)$, $SU(2)$ and $SU(3)$ gives

$$\begin{aligned}
U(1): &\quad l = \frac{1}{2}, 0, -\frac{1}{2} \;\Rightarrow\; s = 0, \frac{1}{2} \\
SU(2): &\quad l = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1 \;\Rightarrow\; s = 0, \frac{1}{2}, 1 \\
SU(3): &\quad l = \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2} \;\Rightarrow\; s = 0, \frac{1}{2}, 1, \frac{3}{2}
\end{aligned}$$

and in general, for $G(\mathbb{N})$, we have

$$l = \frac{\mathbb{Z}}{2} \;\Rightarrow\; s = \frac{\mathbb{N}}{2}, \tag{571}$$

i.e. we get all the $SU(2)$ representations, which proves that $\tilde{R}$ is a good selection rule. From this simple selection rule, it can be seen that the spin of the elementary particles about the symmetry groups of the subnuclear interactions are low, and this could explain, in broad outline, why quantum stable particles have not high values of s. We need to say the spins found by applying the selection rule $\tilde{R}$ to the groups $U(1)$, $SU(2)$, $SU(3)$, do not cover the spin ranges of the particles existing in Nature, because there are hadrons and resonances of higher spin. This could be due either to the roughness of the chosen selection rule or to the fact that, in the particle accelerators, energies greater than those which stabilized the present universe are reached.

This point of view finds confirmation in the circumstance that the particles with spin higher than those derived from the selection rule $\tilde{R}$ applied to the groups $U(1)$, $SU(2)$, $SU(3)$ are very unstable. This also suggests a larger symmetry group (big energy) existed, which spontaneously broke, generating the interactions that we know, and of which the unstable particles are not but a *trace fossil*. Then, the fact our selection rule always includes $s = 0$ could be the indication of the existence of a scalar field (Higgs type), which, through the Big-Break, gave mass to the gauge bosons. Among other things, a way for extending the selection rule $\tilde{R}$ to the case of larger groups, given as an example to the tensor product of unitary groups, could be that to let correspond to the tensor product the product of the natural numbers representing the order of groups (the product of two or more natural numbers always gives a natural number and so this does not dilute the selection rule $\tilde{R}$), namely

$$G(n_1) \otimes G(n_2) \otimes \cdots \otimes G(n_k) \Rightarrow n = n_1 n_2 \cdots n_k. \tag{572}$$

Due to (572), the group

$$G_1 = SU(2) \otimes SU(2) \simeq SO(4) \tag{573}$$

gives $n = 4$, from which, by applying $\tilde{R}$, one has

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{574}$$

while the group

$$G_2 = SU(3) \otimes SU(2) \otimes U(1) \tag{575}$$

gives $n = 6$, from which, by applying $\tilde{R}$, it can be obtained

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \tag{576}$$

and so on.

We see this simple extension of $\tilde{R}$ allows to consider larger symmetry groups, whose spontaneous breaking could have generated the set of elementary particles (with low spins) observed in our universe.

If effectively only a limited number of particles with low spins exists, what is the necessity of the α-Theory? In contrast to Klein-Gordon and Dirac theories, the α-Theory is a unified theory telling us, if confirmed, that the elementary particles, independently from their spin value, satisfy *in form* the same equation, and this represents a triumph for the physical knowledge, that could, as already said, lead to the unification of all the interactions present in Nature. This is the power and importance of the α-Theory.

7 α-Theory and Twentieth Century Physics

In the previous chapters, we developed the α-Theory, which, independently from its asymmetric or symmetric connotation, represents a new model for the physical description of quantum and cosmological phenomena. Since the α-Theory proposes itself of being a GUT, it is natural to make a comparison with the most well-known unified models of the twentieth century. This chapter wants just to do it, starting by the analysis of the second quantization and suggesting to replace this process with a different type of mathematical methodology, which does not look for quantizing a classical field, but rather it makes *to emerge* the classical fields from the quantum ones. This could concur to devise an effective Quantum Gravity thanks to the use of the α-Theory. After this, we will inquire on the relationship between Inflation Theory and α-Theory, proving that within this last there is the inflation phenomenon, so giving to it full justification. It will be demonstrated that inflation is a characteristic of tachyonic (IEP) as well as bradyonic (REP) universe and this could explain the acceleration of our universe, without using the "Dark Energy." In the last two sections of this chapter, we will compare, instead, Supersymmetry and String theory with the α-Theory, showing it not only concurs to exceed the super-symmetrical models conceptually, but allows also of defining two string actions, which could take to new developments of the research in this area.

7.1 α-Theory, Grand Unification, classical fields and Quantum Gravity. The expanding Universe and Dark Energy: the double inflation mechanism

The α-Theory wants, unifying the elementary particle physics in absence of interactions, to be a Grand Unified Theory. Strictly speaking, the α-Theory is already a GUT, because it concurs to put under a single formal equation all the particles of matter, which are subject to the four fundamental interactions. But, for achieving the whole unification, it is necessary to describe in detail the general process of spontaneous symmetry breaking (Big-Break) expected by the α-Theory, since only then we will know as the interactions of our universe (gauge bosons) were *born*. This takes us to conclude the Big-Break represents a universal principle of the physics. Nevertheless, in contrast to other principles, it is not built *ad hoc*, because it is not postulated and apodictic, but finds complete justification in the instability of the tachyonic universe and in the phase transition transforming this last in the bradyonic universe.[122] It is very likely that this global breaking off relates to the symmetry group $SU(5)$, which in literature is the most reasonable to generate the strong, electromagnetic and weak interactions [17]. From this point of view, such a group could be extended by using the general coordinate transformations, so making the space-time comes up by a process of Higgs condensation [18, 19].

[122] When a coherent Big-Break model will be developed, this will no longer be a "Principle," but rather a concrete physical cosmological process.

But all of this poses an important problem: what connection exists between the classical fields and the quantum ones? As is well known, in QFT for quantizing a classical field the so-called "second quantization" is applied, which has been used also for quantizing the theory exposed in this work. The second quantization is applied to the classical electromagnetic field for making it a quantum field, but also to the Dirac and Klein-Gordon fields, which are intrinsic quantum fields (precisely they are constructed for explaining some classes of elementary particles). Is it correct to indiscriminately use the second quantization? Is it right to transform a classical field in a quantum field by using a mathematical process and not a physical one? In order to focus this issue, we consider the homogeneous Proca equations, used for describing particles having spin 1, but that were developed for photons with nonzero mass

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu(x) = 0, \quad \partial_\mu A^\mu = 0. \tag{577}$$

As can be seen, these equations contain the electromagnetic field tensor $F^{\mu\nu}$ (which has as matrix elements the fields $\vec{E}$ and $\vec{B}$) and the four-potential $A^\nu(x)$. Therefore, for describing massive photons, *i.e.* quantum particles, classical fields are used and successively they are transformed in operator quantities, thanks to the second quantization. But this seems an abuse, because within the expressions of quantum particles should not be classical fields like $\vec{E}$ and $\vec{B}$. This takes to make an important consideration: can classical fields be a macroscopic effect of quantum fields? The discussion is identical to the one existing between Thermodynamics and Statistical Mechanics and which is based on the fact the thermodynamics state functions are macroscopic effects going out by considering gases of interacting particles. From this point of view, the quantization of a classical system is equivalent to assert the absolute temperature T, concerning a gas of atoms or molecules, is given by the sum of all the temperatures t_i of each atom or molecule, namely

$$T = \sum_i t_i. \tag{578}$$

We know this is incorrect, because the absolute temperature T is, instead, proportional to the average kinetic energy of all atoms or molecules of the gas, that is [123]

$$T \simeq \frac{\langle E_c \rangle}{\mathsf{k}}. \tag{579}$$

We can, therefore, think also within the QFT a similar thing happens, and that the classical fields (as the electromagnetic one) must *emerge* as macroscopic properties of the quantum fields or as coupling between quantum fields. For this reason, a direct construction of the classical fields components by the quantum fields components could be tried. But, doing this, it would be equivalent to identify quantum components with classical components, and this seems to be an abuse. Hence, it is desirable to find the law allowing, through the quantum fields, to reproduce the macroscopic

[123] Naturally, k is the Boltzmann constant.

properties today we identify with the classical fields.

The same logic previously applied to the electromagnetic field and massive photons (but also with $m = 0$) can be used for the gauge bosons of the gravitational interaction too. In order to explain such a concept, we must remember from the literature, in the weak-field approximation ($|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ = Minkowski metric) and in the absence of sources ($T_{\mu\nu} = 0$), the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{580}$$

become

$$\Box h_{\mu\nu} = 0; \quad h_{\mu\nu} = h_{\nu\mu}, \tag{581}$$

where we imposed the harmonic gauge conditions [124]

$$\partial^{\nu} h_{\mu\nu} = \frac{1}{2} \partial_{\mu} h. \tag{582}$$

The (581) describe the *undulating propagation* of the ripples in the curvature of space-time. The possibility to deal with gravitational waves has driven the physicists to relate electromagnetism and gravitation, so that it has been natural trying to quantize the Einstein gravity for finding the space-time quanta and the mediators of the gravitational interaction, which are said "gravitons." Then, an analysis on the degrees of freedom of $h_{\mu\nu}$ (number of independent components) established the helicity value of these hypothetical particles has to be ± 2 and, therefore, the spin of gravitons is fixed in 2. These considerations on the quantum nature of the space-time gave rise to a very successful area of research called "Quantum Gravity." So far, the attempts to quantize gravity with the "canonical" approach or with the "covariant" one revealed unfruitful. In particular, the covariant perturbative approach (Loop Quantum Gravity), wanting to deal gravity as a Yang-Mills field, it is a non-renormalizable theory, already to two-loop. The renormalization of the gravity to one-loop made to understand that a theory of gravity to the Planck era (high energies) must exist: the Einstein gravity, then, should be the low-energy aspect (*i.e.* to one-loop) of such a general theory. Until now, only the Supersymmetry and Superstring theory proposed to resolve the problem of the gravity to high energies, with no satisfactory result. Actually, there is also the problem to establish if the graviton (boson with spin 2 and $m = 0$) is really the mediator of the gravitational interaction. Basically, the properties of this hypothetical particle have been derived in the weak-field approximation, by using the gravitational waves. Nevertheless, it seems a hazard to use the ripples in space-time derived by stellar collapses, supernova explosions or colliding black holes, in order to find properties of particles whose dimension is comparable with the Planck length. Therefore, it should be more reasonable to find a suitable quantum gravity (high energy gravity), which is capable to beget at low energies the Einstein gravity, without using necessarily fields with spin 2, but even

[124] $h \equiv h^{\alpha}_{\alpha}$.

fields with different spins or sets of fields with fixed spins. Practically, the α-Theory wants to tell us that the quantization process of classical fields must be neglected for the benefit of a "classicization" process of quantum fields, which is able to give rise to the classical fields like macroscopic properties of quantum-dynamic processes happened in the after Big-Break. The α-Theory, therefore, requires that the correspondence

$$\text{classical field} \stackrel{\text{second quantization}}{\longrightarrow} \text{quantum field} \tag{583}$$

has to be replaced by the most probable and physically acceptable

$$\text{quantum field(s)} \stackrel{\text{classicization}}{\longrightarrow} \text{classical field.} \tag{584}$$

How can a classical field emerge by a quantum field or by a set of quantum fields? This is an open answer, and the solution could be found in the development of new mathematical methods, but, above all, in the formal explanation of the great process of spontaneous symmetry breaking (Big-Break), which presents itself as the *father* of all physical interactions.

Another important aspect we want now to treat is the relationship existing between the α-Theory and Inflation theory. In fact, this last model could just find full justification in the α-Theory. In order to understand it, we shortly recall the history and base concepts of the Inflation theory, that was proposed in 1981 by A. H. Guth [20], in order to resolve the anomalies due to the initial conditions of the Standard Hot Big-Bang model, which can be summarized in the horizon and flatness problems. Guth found the way to resolve both these problems, by supposing the early universe conducted itself to a supercooled state to temperatures 28 or more orders of magnitude below the critical temperature $T_c \simeq 10^{14}$ Gev, which is identified with the phase transition temperature from a universe with total symmetry $SU(5)$ to a universe with symmetry $SU(3) \otimes SU(2) \otimes U(1)$, according with the GUT model proposed by Georgi-Glashow. This hypothesis implies two remarkable consequences:

1. The pressure p of the early universe turns out negative and opposite to the matter density ρ, namely

$$p = -\rho. \tag{585}$$

This means the stress-energy tensor of the matter obeys to a covariant conservation equation, which, in the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{586}$$

is reduced to

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3). \tag{587}$$

Such an equation implies the energy of the co-moving volume decreases with the time when $p > 0$, while it grows with the time when $p < 0$. It is identically satisfied for $p = -\rho$ and this coincides with the beginning of the inflationary era, in which the negative pressure has carried to a fast expansion of the early universe.

2. The expansion with negative pressure implies a scale factor that evolves in an exponential way, according to the law

$$a(t) \approx e^{\chi t}, \quad \chi \equiv \left(\tfrac{8\pi G}{3}\rho_f\right) \simeq 10^{10}\text{Gev}. \tag{588}$$

Such a scale factor can only exist if one supposes of being in a flat universe ($k = 0$). The Friedmann-Robertson-Walker metric is reduced in this case to the de Sitter one

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2. \tag{589}$$

Guth assumed the Inflation was caused by a phase transition (to the first order), that conducted the early universe from a false vacuum to a true vacuum at temperatures below the critical temperature T_c, to which the spontaneous symmetry breaking of the gauge group $SU(5)$ occurred (anyway Guth did not exclude different symmetry groups). In order to explain this inflationary model, it can be supposed existing a like-Higgs scalar field $\phi(x)$ – called "inflaton" – with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{590}$$

The phase transition, originating the inflation, happens when the inflaton field passes from the *false vacuum* to the *true vacuum*, based on the diagram of fig. 11.

The original theory of the inflation ("Old Inflation"), although resolving the horizon and flatness problems, gave light to various inconsistencies, of which one of the most serious is the "bubble coalescence," *i.e.* the creation of *superdense bubbles* in the early universe, of which we are not able to explain the mutual energy exchange until thermal equilibrium (thermalization). In order to go around this problematic, another version of the Inflation Theory – known with the name of "New Inflation" (also called "Chaotic Inflation") – was born, for which in the Lagrangian density of the inflaton field are considered different types of scalar potentials, like the Coleman-Weinberg one

[21], having the characteristic to drive a slow phase transition (of the second order), called "slow rollover transition," regulated by random processes depending by the fluctuations of the scalar field. Naturally, such an approach resolves the problem of "bubble coalescence." The new (or chaotic) Inflation theory, even if resolves many of the problems of the "Old Inflation," introduces other ones. In fact, because of the exponential expansion of the scalar field ϕ, there are complex fluctuations of the vacuum, whose average amplitude is thus estimated [125]

$$|\delta\phi(x)| \approx \frac{H}{2\pi}. \tag{591}$$

These oscillations are so strong that could have created an eternal process of *self-reproduction* of the universe. What for this model practically occurs is that the quantum jumps, deriving from the fluctuations of the scalar field during inflation, can be divided into new infinite domains, in each of them there was a so devastating quantum jump to produce again inflation and, hence, a new universe, separated from all the others and with own physical laws. This model of self-reproduced multi-universes is not accepted by the most of the physicists, since it is not supported by any experimental evidence. Another issue, belonging to the old and new inflationary model, is of gnosiological type. How was the inflaton field born? When and why the period of inflation is finished? For giving answer to these questions, the Superstring theory and Supersymmetry are used, even if the current framework is very unclear and uneven [22, 23, 24].

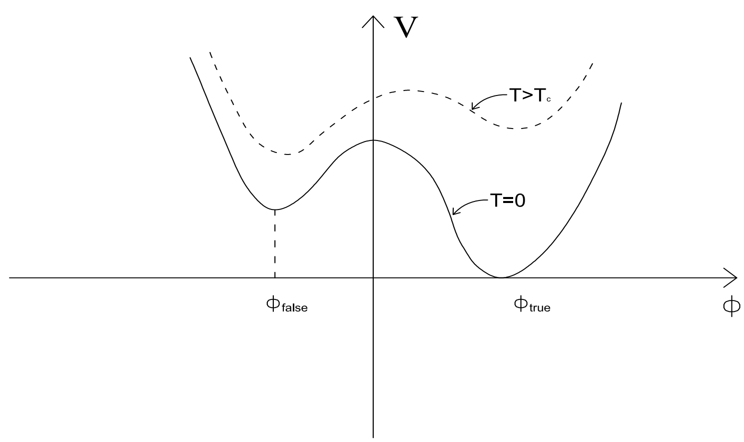

Figure 11: The typical potential for the inflationary universe: T_c is the critical temperature for a first-order phase transition. (Adapted from S. K. Blau and A. H. Guth in *300 Years of Gravitation*, 1987.)

[125] $H \equiv \dot{a}/a$.

After having shortly exposed the Inflation theory, we want now to understand if the α-Theory predicts in some way the presence of an exponential scale factor for the early universe, that, as seen, resolves the horizon and flatness problems. Only thus, in fact, we can demonstrate the absolute generality of the α-Theory, not only like elementary particle theory, but also like cosmological theory at high energies. However, we will do a very rough discussion, because, until the Big-Break mechanism will not be explained, we cannot make precise considerations on cosmological dynamics, and, therefore, our examination is useful only to trace a guideline for future developments. As previously seen, for having an inflationary phase, the inflaton field must take to an equation of state in which the pressure p of the *cosmological soup* has negative sign. Thence, it seems obvious to consider the kinetic terms of the Lagrangian densities, derived from $\mathcal{L}_{A\alpha}$ and $\mathcal{L}_{S\alpha}$, by adding an appropriate potential V, namely

$$\mathcal{L}^1_{A\alpha} = i\bar{\psi}_s(x)\chi^\mu\partial_\mu\psi_s(x) - V_{A\alpha}(\psi_s, \bar{\psi}_s, m) \tag{592}$$

$$\mathcal{L}^2_{S\alpha} = (\partial_\mu\psi^\dagger_s)\xi^\mu\xi^\nu(\partial_\nu\psi_s) - V_{S\alpha}(\psi_s, \psi^\dagger_s, m), \tag{593}$$

and to understand if one of these two theories takes to a negative pressure. For this purpose, we calculate the energy density ρ and the pressure p by the stress-energy tensor $T^{\mu\nu}$ of $\mathcal{L}^1_{A\alpha}$ and $\mathcal{L}^2_{S\alpha}$, by supposing of being in an expanding universe, with metric

$$g_{\mu\nu} \equiv \left(1, -a^2(t), -a^2(t), -a^2(t)\right). \tag{594}$$

In the hypothesis $V_{A\alpha}$ and $V_{S\alpha}$ do not depend by derivatives of the fields ψ_s and ψ_s (naturally of $\bar{\psi}_s$ or $\psi^\dagger_s$ too), we have

$$T^{\mu\nu}_1 = i\bar{\psi}_s\chi^\mu\partial^\nu\psi_s - i\bar{\psi}_s\chi^\alpha\partial_\alpha\psi_s g^{\mu\nu} + V_{A\alpha}g^{\mu\nu} \tag{595}$$

$$T^{\mu\nu}_2 = (\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^\mu(\partial^\nu\psi_s) + (\partial^\nu\psi^\dagger_s)\xi^\mu\xi^\alpha(\partial_\alpha\psi_s) - (\partial_\alpha\psi^\dagger_s)\xi^\alpha\xi^\beta(\partial_\beta\psi_s)g^{\mu\nu} + V_{S\alpha}g^{\mu\nu}. \tag{596}$$

If $\psi_s(t_i) = \text{constant}$ and $\psi_s(t_i) = \text{constant}$, it is straightforward to verify that for our theories the following equations of state are obtained

$$p_1 = -\rho_1 \tag{597}$$
$$p_2 = -\rho_2, \tag{598}$$

which take to an inflationary phase.

This is very important, because it proves that the α-Theory, asymmetric or symmetric, admits inflation. It must also be noticed, moreover, that if we consider the tachyonic theories from which the AαT and SαT are originated and we also add to their kinetic terms a potential V

$$\mathcal{L}^{V}_{M1} = \bar{\Psi}_s \chi^\mu \partial_\mu \Psi_s - V_1^{tach}(\Psi_s, \bar{\Psi}_s, \mu) \tag{599}$$

$$\mathcal{L}^{V}_{M2} = (\partial_\mu \psi_s^\dagger)\xi^\mu \xi^\nu (\partial_\nu \psi_s) - V_2^{tach}(\psi_s, \psi_s^\dagger, \mu), \tag{600}$$

we have for them, like for the Lagrangian densities $\mathcal{L}^1_{A\alpha}$ and $\mathcal{L}^2_{S\alpha}$, negative pressures too. Therefore, also the tachyonic universe admits an inflationary phase. This result, although obtaining without specifying the form of the potentials at stake, opens a very interesting cosmological scenario. In fact, being also the tachyonic universe compatible with an inflationary phase, we can conclude the inflation not only is gushed from Big-Bang characterizing the unstable tachyonic universe in virtue of a negative pressure, but it continued, maybe in a more soft way, also in the bradyonic universe thanks to Big-Break, which transformed the potentials V_1^{tach} or V_2^{tach} in $V_{A\alpha}$ or $V_{S\alpha}$ respectively, according with the pattern

$$V_1^{tach} \stackrel{\text{Big-Break}}{\longrightarrow} V_{A\alpha} \tag{601}$$

$$V_2^{tach} \stackrel{\text{Big-Break}}{\longrightarrow} V_{S\alpha}. \tag{602}$$

Therefore, the α-Theory, asymmetric or symmetric, is compatible with a process of "double inflation" that, as it is easy to understand, could resolve many open problems of modern cosmology. In fact, this hypothetical mechanism could not only synthesize the old and new inflationary model, but it could explain us the nature of the cosmological constant Λ and why the observed universe is accelerated, and this without using exotic ideas like the "Dark Energy" one. In order to understand what was just asserted, we want to draw a cosmological scenario coherent with the α-Theory and with the double inflation which it expects: after the hot Big-Bang a universe characterized by a square negative energy is generated, which did not respect the principle of causality. This universe, extremely unstable and chaotic, was characterized by a negative pressure, that produced an exponential pressure. At some temperature T (very high) and time t (very small), such a universe is collapsed, producing a phase transition, which took it to a stability condition and generated the separation of the fundamental interactions (Big-Break). In this context, the initial potential transformed itself, passing from false to true vacuum. Nevertheless, the new universe was always characterized by a negative pressure, which has given life to an accelerated phase.

The proposed scenario is an half-way between the old and new inflation, because it could make coexist both visions. This takes to think the "Tachyonic Inflation" ("IEP Inflation") is able to resolve the horizon problem (and perhaps also the flatness one), while the "Bradyonic Inflation" ("REP Inflation") is able to resolve the problem of the accelerated universe (and/or perhaps also the flatness one). Naturally, all the problems of the "Old Inflation," like for example the "bubble coalescence," are not in a chaotic universe as the tachyonic one.

The scenario emerging by the double inflation is charming, even if, for being completely demonstrated, it will have to understand if all the particles have contributed to this process or only those with spin $s = 0$ and, above all, appropriate potentials V must be defined and it has to be explained the way in which these potentials are transformed thanks to the Big-Break. This is not only important for the elementary particle physics, because the Big-Break will reveal as the fundamental interactions were created, but mostly for the understanding of our universe. In particular, like already said, the *secret* of the cosmological constant could be explained through relations of the type [69]

$$\Lambda = 8\pi G V. \tag{603}$$

If the bradyonic inflation will be able to explain the acceleration of our universe, this should be a great result, because such an acceleration will not be derived by a phantom Dark Energy, but will be simply a *residue* of the great inflation within the tachyonic world (tachyonic inflation), that our universe has inherited from Big-Break.

7.2 α-Theory and Supersymmetry

The theory exposed in this work allows to describe the particle fields with arbitrary spin and makes it under the hypothesis that – after the hot Big-Bang – there was another catastrophic cosmological event (Big-Break), from which our universe is generated. Like said in the previous section, this could explain the birth of all the forces and unify them just thanks to the α-Theory, thus wanting to be a Grand Unified Theory. Naturally, for having a complete and exhaustive framework, the micro-macro problem should be resolved, explaining in detail as, from the quantum fields, the classical fields (the gravitational and electromagnetic ones) came up, and to describe in a general way the Big-Break, representing in this model the most important event of our universe. It is good to underline we arrived to the development of the α-Theory, asymmetric or symmetric, and to the above considerations, only making use of the specific technologies of the QFT, without using difficult transformations or multi-dimensional spaces. This becomes important when we want to compare the α-Theory with the super-symmetric Theory, called simply "Supersymmetry" (often abbreviated SUSY), which was developed around 1970s with the main purpose to describe in a unitary way fermions and bosons (even then it seemed strange a theory describing such particles in the same way did not exist).

The Supersymmetry, unlike α-Theory, proposes a unified model that *mixes* fermions and bosons through spinorial transformations based on dotted and undotted spinors, which satisfy the Grassmann algebra. The simplest super-symmetrical model is the Wess-Zumino (simplified) one, whose Lagrangian density is [126]

[126] In this case, we define $\overleftrightarrow{\partial} \equiv \overrightarrow{\partial} - \overleftarrow{\partial}$.

$$\mathcal{L}_{W-Z} = (\partial_\mu \phi)^* (\partial^\mu \phi) + \frac{i}{4} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial_\mu} \psi, \tag{604}$$

where ψ is a Majorana spinorial field, while ϕ is a scalar field such that [127]

$$\phi(x) \equiv \frac{L(x) - iM(x)}{\sqrt{2}}. \tag{605}$$

Like we promptly see, the (604) characterizes a massless field theory. For obtaining a massive super-symmetrical theory, we must consider

$$\tilde{\mathcal{L}}_{W-Z} = (\partial_\mu \phi)^* (\partial^\mu \phi) + \frac{i}{4} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m^2 \phi^* \phi + \frac{m}{2} \bar{\psi} \psi. \tag{606}$$

The scalar fields and the spinorial field ψ are connected by the following transformations (said of supersymmetry)

$$\begin{cases} \delta L = \bar{\varepsilon} \psi \\ \delta M = i \bar{\varepsilon} \gamma^5 \psi \\ \delta \psi = -i \gamma^\mu \varepsilon (\partial_\mu L) + \gamma^\mu \gamma^5 \varepsilon (\partial_\mu M) \\ \delta \bar{\psi} = i \bar{\varepsilon} \gamma^\mu (\partial_\mu L) - \bar{\varepsilon} \gamma^5 \gamma^\mu (\partial_\mu M), \end{cases} \tag{607}$$

where ε, the parameter allowing to pass from the scalar fields to the spinorial field (*i.e.* that *mixes* bosons and fermions), is a Majorana spinor. The Supersymmetry peculiarity, just deriving from the super-symmetrical transformations, is that each boson with fixed rest mass has a supersymmetric partner, characterized by a fermion with the same rest mass, and vice-versa. Moreover, the spin of any supersymmetric particle will be $|s - 1/2|$, where s represents the spin of the particle of which the supersymmetric partner has to be considered. This means the following illustrative outline is valid

$$\begin{gathered} 0 \longleftrightarrow \frac{1}{2} \\ 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{3}{2} \\ \cdots\cdots\cdots \end{gathered}$$

[127] $L(x), M(x)$ are scalar fields.

The short summary of the α-Theory and Supersymmetry tells us this two models cannot live together, in the sense that if one is right the other is wrong. It is simply to admit that, from a theoretical point of view, the α-Theory is better than Supersymmetry, for simplicity and elegance. In the long run, a deeper analysis leads us to conclusion that the α-Theory is just the right supersymmetrical Theory tried in the 1970s, because it realizes the dream to describe the elementary particles of integer and half-integer spin with only an equation. Is it, therefore, necessary to neglect the Supersymmetry? As all the theories based on the scientific method, SUSY has within it interesting ideas too, which can also be used in other theories. For example, we can continue to study those transformations sending bosons in fermions and vice-versa. However, these transformations have no longer to be of the type [128]

$$B \longrightarrow B' = B + \delta B, \quad \delta B = \bar{\varepsilon} F \tag{608}$$

$$F \longrightarrow F' = F + \delta F, \quad \delta F = \varepsilon \partial B, \tag{609}$$

which are the canonical SUSY transformations, because, having the AαT and SαT a different mass dimensions for the fields ψ_s and ψ_s, these two theories have a mass dimension of ε not equal to $M^{-1/2}$. In order to prove it, we note in a d-dimensional space one obtains

$$[\psi_s] = M^{\frac{d-1}{2}} \tag{610}$$

$$[\psi_s] = M^{\frac{d-2}{2}}, \tag{611}$$

i.e. the field ψ_s has the same mass dimension of the Dirac field, while the field ψ_s has the same mass dimension of the Klein-Gordon field and this suggests that, in general terms, the mass dimension of a field also depends on the order n of the (partial) differential equation which it is subject to. Naturally, for $d = 4$, we have

$$[\psi_s] = M^{3/2} \tag{612}$$

$$[\psi_s] = M. \tag{613}$$

Thanks to these considerations, the transformations mixing boson fields (integer spins) with fermion fields (half-integer spins) are written, for the AαT and SαT, in the following way [129]

$$B \longrightarrow B' = B + \delta B, \quad \delta B = \varepsilon F \tag{614}$$

[128] Note that $[\varepsilon] = M^{-1/2}$.
[129] In this case, $[\varepsilon] = M^0$.

$$F \longrightarrow F' = F + \delta F, \quad \delta F = \varepsilon B, \tag{615}$$

where ε must be a dimensionless quantity, because, for the AαT and SαT, the fields B and F have equal dimension.

The problem at this point we have is to establish the nature (matrix or scalar) of parameter ε. If, for example, B is a field with spin 1 and F a field with spin 1/2, we have that B is described by a matrix 3×1, while F is described by a matrix 2×1. From this fact, it promptly follows that ε must be a matrix of dimension 3×2. Therefore, established the parameter ε is a matrix, we have to consider also its conjugate transpose $\bar{\varepsilon}$, which, in such a case, have dimension 2×3. For this reason, the transformations (614) and (615) must correctly be written

$$B \longrightarrow B' = B + \delta B, \quad \delta B = \varepsilon F \tag{616}$$

$$F \longrightarrow F' = F + \delta F, \quad \delta F = \bar{\varepsilon} B. \tag{617}$$

We understand this reasoning can be repeated not only for mixing the fields with integer spin (bosons) and the fields with half-integer spin (fermions), but also the fields with arbitrary spin (also equals, and in such a case ε and $\bar{\varepsilon}$ are square matrices). This means the AαT and SαT, in general, admit the following transformations (for the first we have obviously $\phi_s = \psi_s$, while for the second $\phi_s = \psi_s$. The spins s_1 and s_2 are fixed, and, naturally, it also can be $s_1 = s_2$)

$$\phi_{s_1} \longrightarrow \phi'_{s_1} = \phi_{s_1} + \delta\phi_{s_1}, \quad \delta\phi_{s_1} = \varepsilon\phi_{s_2} \tag{618}$$

$$\phi_{s_2} \longrightarrow \phi'_{s_2} = \phi_{s_2} + \delta\phi_{s_2}, \quad \delta\phi_{s_2} = \bar{\varepsilon}\phi_{s_1}, \tag{619}$$

where ε have to be a matrix of dimension $(2s_1 + 1) \times (2s_2 + 1)$ and $\bar{\varepsilon}$ a matrix of dimension $(2s_2 + \ 1) \times (2s_1 + 1)$.

Why have we to study the transformations between fields with arbitrary spin? Beyond merely mathematical reason, this could save and maybe extend the fundamental property of the Supersymmetry, consisting in the fact that when local SUSY models are constructed, these models *live* in a curved space-time, *i.e.* they include the gravitation (as macroscopic interaction). This aspect is known as "Supergravity." It can be argued such a property derives from the presence of the gradient in the transformation (609) concerning the fermionic field (this creates a connection between Supersymmetry and translations, namely between local SUSY transformations and (super)gravity), while in the transformations above seen, having the fields the same mass dimension (and hence ε as dimensionless quantity), the gradient operator does not appear. But this is not a problem, because it is clear that, within the α-Theory, the transformations (618) and (619) are not useful for constructing a supergravity model like the Einstein-Rarita-Schwinger one, given by the Lagrangian density [130]

$$\mathcal{L}_{E-R-S} = \left(-\frac{1}{2\chi}\sqrt{-g}R + \frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}\bar{\psi}_\mu\gamma_5\gamma_\nu\nabla_\alpha\psi_\beta \right), \tag{620}$$

but rather of allowing to the same gravity to appear by a connection between fields with different spins. In fact, in the previous section, we explained the α-Theory could concur to give life to a quantum gravity based on a real field of particles, which does not suffer of the process of canonical quantization commonly applied to the classical fields. Then, we also said this involves some problems about the number of degrees of freedom to really take into account, that, if it is bound with the components of the particle fields, it is equivalent to an uncertainty on the spin value of the field which created the gravitational interaction. What should happen if such an interaction was really due to a connection process between particle fields (*i.e.* if the gravity depends at quantum scale on various fields)? This eventuality can be taken into consideration just studying those models which are unchanged under the transformations (618) and (619) and which allow to make visible macroscopic properties, according with the Einstein field. As an example, by considering the degrees of freedom of the linearized Einstein field, we know the tensor $h_{\mu\nu}$ has 10 independent components, which are reduced to 6 imposing the 4 conditions of the harmonic gauge. But, by imposing other gauge conditions, these conditions are reduced to 2, corresponding to the degrees of polarization that are thought to be the graviton ones. Beyond the fallacy of such a reasoning, that – as said – does not distinguish between the peculiar properties of a classical and quantum field (wrong relationship macro-micro and its reversibility), we can, however, try to construct some models of coupling between particle fields, having at least the same number of degrees of freedom required by the linearized gravitational field. As an example, in order to have 6 independent fields (which, then, would be the independent components of $h_{\mu\nu}$) could be considered the coupling of a field with $s = 2$ (5 independent components) and a field with $s = 0$ (1 independent component). Instead, if we want 2 freedom degrees, these could be obtained from the coupling of two scalar fields. Also in such a case, the scalar fields seem to cover a primary role in the birth of the interactions. If these coupling models between fields with different spins will able to construct the gravitational interaction with the consequent curvature of the space-time, only future studies will be able to establish.

What we want to emphasize is that the α-Theory, asymmetric or symmetric, concurs to study complex models of unification, bringing also those conceptual simplifications which are always wished within the modeling of the Nature. Therefore, the α-Theory wants to be an effective alternative to the Supersymmetry, by showing no small capacities.

[130] $\chi \equiv \frac{8\pi G}{c^4}$.

7.3 String Theory and α-Theory

As we have shown in the previous section, the α-Theory represents a valid alternative to the Supersymmetry. It goes without saying that in all the cases in which SUSY is used, or one wants to use it, the α-Theory can be utilized. One of the most intensive use of the Supersymmetry is within the String theory, which is the model that, by supposing to assimilate elementary particles to vibrational modes of filiform objects (strings), wants to realize the desired unification of the four fundamental interactions of the Nature.

The supersymmetry transformations enter the string model when it is attempted to construct an action describing *vibrational modes* of bosonic and fermionic type. In short, such an action is given by [131]

$$S_{\text{super}} = -\frac{1}{4\pi\alpha'}\int_{W\subset M} d^2\sigma \left\{\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right\}, \tag{621}$$

where $2\alpha'$ is the square of the characteristic length of string, given by $l \equiv \sqrt{2\alpha'}$.

The action (621), for the reasons above explained, is the starting grid of all modern string theories, or better of "superstring." The weak point of S_{super} is that the use of SUSY transformations for connecting X^μ and ψ^μ renders too complex the theoretical discussion on the Superstring theory, taking it to results often farthest from the physical reality. The previous section showed the α-Theory concurs to work without the disagreeable transformations of supersymmetry and, therefore, a string theory founded on the α-Theory could not only describe all elementary particles at the same time (without distinguishing between bosonic and fermionic strings), but it should render the string model *slimmer* and more physically acceptable. Moreover, by using the α-Theory, the String theory should not more suffer from the presence of the tachyons (this happens precisely in the bosonic string theory), but, from this, it could draw new life too, because their materialization should be a sign the strings were the first element of our universe, being present also in the unstable *tachyonic soup* of the after Big-Bang. Naturally, in this section, we will not treat in a detailed analysis the new String theory constructed through the α-Theory ("α-String theory"), since this would involve an encyclopedic work, which is outside of our purposes. However, what we will make is to lay the foundations of this new String theory, going to write its actions. We correctly used the plural, because not knowing what between the AαT and SαT better physically describes the elementary particles, we must necessarily introduce two different string actions, of which only one will be saved, when we will understand what between asymmetric and symmetric α-Theory is the best to describe the quantum particles.

To write a string action by using the AαT or SαT is quite simple. Obviously, for future developments, it is opportune to avoid imposing some gauge or initial condition. As we just asserted, the string actions based on the AαT and SαT, respectively, are easy to write, since they are not other but a sort of *holographic projection* of the kinetic terms concerning the actions of these two theories on the world-sheet W. Then, one has to consider the fact the world-sheet W is immersed

[131] The latin indices are those on the world-sheet W, while the greek ones are the Lorentz indices, therefore: $a \in \{1, 2\}$; $\mu \in \{0, 1, 2, 3\}$.

in the Minkowski space-time M. Hence, it is not difficult to understand the wanted actions are [132]

$$S^{A\alpha}_{\text{string}} = -\frac{i}{4\pi\alpha'}\int_{W\subset M} d^2\sigma\sqrt{h}g_{\mu\nu}\bar{\Phi}^{\mu}_{s}\theta^{a}\partial_{a}\Phi^{\nu}_{s} \tag{622}$$

$$S^{S\alpha}_{\text{string}} = -\frac{1}{4\pi\alpha'}\int_{W\subset M} d^2\sigma\sqrt{h}g_{\mu\nu}(\partial_a\Psi^{\mu}_{s})^{\dagger}\pi^{a}\pi^{b}(\partial_b\Psi^{\nu}_{s}), \tag{623}$$

where $g_{\mu\nu}$ is the metric tensor of Minkowski space-time and $h = -det_{ab}(h_{ab})$, with h_{ab} metric tensor of the world-sheet (the line element of W is so $ds^2_W = h_{ab}d\sigma^a d\sigma^b$). The fields Φ^{μ}_{s} and Ψ^{μ}_{s} totally describe, in the case of $S^{A\alpha}_{\text{string}}$ and $S^{S\alpha}_{\text{string}}$, respectively, the world-sheet dynamics and, therefore, the string dynamics. In this case, with s we do not indicate the spin, but an intrinsic degree of freedom on the string, which could be called "s-index." A most deep consideration should make us to think this intrinsic property, within the generic string, has really generated the spin of the elementary particles. This could be an amazing result, because so the α-String theory would be able to explain the spin origin, resolving in this way an open problem of the modern physics.

It remains to understand what is the explicit form of the matrices θ^a and π^a. Let us drive ourselves, for such an aim, to the fermionic string and in particular to the matrices ρ^a. According to the Green-Schwarz-Witten notation [79], they are given by

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{624}$$

namely they are two-dimensional. Moreover, since for the fermionic string

$$\bar{\psi}^{\mu} \equiv \psi^{\mu\dagger}\rho^0, \tag{625}$$

it is also defined a two-dimensional matrix ρ^0 in the following way

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{626}$$

We quickly notice

$$\rho^1 = i\sigma_1, \quad \rho^2 = -i\sigma_2, \quad \rho^0 = \sigma_2 \tag{627}$$

[132] In principle, these actions can be defined with + sign too.

i.e. the ρ^a are just equal to the first two Pauli matrices multiplied by i and $-i$. Therefore, by remembering that $\chi^\mu \equiv (\delta, \delta\varepsilon^i)$ and $\xi^\mu \equiv (\delta, i\varepsilon^i)$, we can define

$$\begin{cases} \theta^1 \equiv i\chi^1 = i\delta\varepsilon^1 \\ \theta^2 \equiv -i\chi^2 = -i\delta\varepsilon^2, \ \text{ hence } \theta^a \equiv (\theta^1, \theta^2) = (i\chi^1, -i\chi^2) \end{cases} \tag{628}$$

$$\begin{cases} \pi^1 \equiv i\xi^1 = -\varepsilon^1 \\ \pi^2 \equiv i\xi^2 = \varepsilon^2, \ \text{ hence } \pi^a \equiv (\pi^1, \pi^2) = (-\varepsilon^1, \varepsilon^2). \end{cases} \tag{629}$$

With regard to $\bar{\Phi}^\mu_s$, defined

$$\bar{\Phi}^\mu_s \equiv \Phi^{\mu\dagger}_s \theta^0, \tag{630}$$

we can devise θ^0 in two different ways. By taking

$$\theta^0 \equiv \delta \ \text{ or } \ \theta^0 \equiv \chi^2 = \delta\varepsilon^2. \tag{631}$$

It is clear the choice of the matrices θ^a and π^a (comprised θ^0) is only indicative, and it can change by virtue of future necessities (for example these matrices could be defined also in this way: $\theta^1 \equiv \chi^1$, $\theta^2 \equiv -\chi^2$, $\pi^1 \equiv \xi^1$, $\pi^2 \equiv -\xi^2$, *i.e.* without the imaginary unit i).

What we want to underline is the variability of the matrices θ^a and π^a in contrast with the ρ^a. In fact, although all these matrices represent world-sheet vectors, the matrices θ^a and π^a have an own dimension, dependent on the representations of $LieSU(2)$, and they are not fixed in two dimensions like the ρ^a. The mistake, indeed, is to believe that the matrix elements – being the components of vectors defined on the world-sheet – must have two dimensions, just like the space where the vectors live, whose components they are. But this is arbitrary, because one thing is the dimension of the space in which the ρ^a represent a vector, and another thing is the dimension of the matrices which are components of this vector, since their dimension could depend on additional degrees of freedom, as the s-index. Therefore, the α-Theory introduces a new element in the String theory, consisting in the fact to have a different string action for any $s \in \mathbb{N}/2 - \{0\}$, although each of these actions always has a common form defined on the world-sheet.[133]

We note that the two-dimensional representation for the θ^a and π^a (comprised θ^0) is obtained when $s = 1/2$. In such a case, we have

[133] This is valid for any $s \in \mathbb{N}/2$, unless $s = 0$. Naturally, it is possible to define the matrices θ^a and π^a so that writing the string actions (622) and (623) when the s-index is equal to zero ($s = 0$) too. However, it is necessary to understand if this is according to the physics of the early universe. Therefore, the question is: does exist a physical reason for which we cannot write the $S^{A\alpha}_{\text{string}}$ and $S^{S\alpha}_{\text{string}}$ for $s = 0$?

$$\theta^1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \theta^2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{632}$$

$$\theta^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \theta^0 = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{633}$$

$$\pi^1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; \pi^2 = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{634}$$

Since the α-Theory is a GUT, what is the necessity to construct through it a string theory (α-String theory), which is proposed of being a GUT too? The fact is that in physics an absolute model of knowledge does not exist. The α-Theory wants to unify the elementary particle physics and, when the Big-Break will be explained in detail, maybe this theory could realize the dream to unify the fundamental interactions completely. However, it is not said that sometime, this complete unification achieved, a more still accurate models on the physical universe investigation could be constructed, enabled to describe those phenomena seeming now, at our eyes, to be meta-physical. The α-String theory, described in this section, could be one of these models. It is for such a reason the development of the string model, based on the formalism of the α-Theory, is hopefully expected.

8 The α-Theory Philosophically

In this chapter, we briefly want to study the α-Theory from the philosophical point of view. This type of discussion is necessary whenever one proposes a new model for the physical world description. In fact, a theory cannot be made without worrying to establish its relationship with the fundamental principles of the former theories and above all its predictive power, that becomes manifest through the "falsifiability" notion introduced by the philosopher Karl Popper. It is precisely on these points this last chapter is based. In the first section, the foundations within the α-Theory will be discussed and we will see that the two fundamental principles of the past century physics – the uncertainty principle and constancy of the speed of light – come out in a natural way by the same structure of the theory. Moreover, referring to the formalism developed in the appendix A, we will give a matter for the resolution of those paradoxes and inconsistencies that plague the quantum mechanics again. In the second and last section, we will discuss about the falsifiability of the α-Theory, based on the Popper vision. On that point, it will be attempted to generalize this key concept, by introducing the "good theory" one.

8.1 Paradoxes and principles into a fundamental theory

The aim within any physical theory should be that to explain (and expect) the largest number of phenomena, on the base of a little fundamental suppositions. Therefore, the "perfect theory" is the one able to explain (and expect) all the physics, starting by an extremely small number of primary properties, *i.e.* it should be animated by simplicity [86].

The α-Theory, described in this work, fits neatly in this sort of thing, because it starts with few assumptions and makes to gush out global characteristics of our universe, by showing a possible unification of the four interactions, if it will be suitably developed.

What are the main ingredients of the α-Theory? We obtained the equations of the theory by reasoning on the Pauli equation and using the equation (44), in order to find the unknown value of the matrix $\tilde{\delta}$. Really, as we already explained in the footnote 7, the α-Theory can be developed – in the hypothesis to know the constituent elements of the spin four-vector s_μ – simply by the specific properties of a quantum particle, which are the four-momentum p^μ and spin four-vector s_μ. The first is related to the energy and momentum of a quantum particle and the second to its *spontaneous rotation*. Therefore, the α-Theory basic ingredients are s_μ and p^μ, whose scalar product expresses the indissoluble characteristic of such physical quantities into a quantum particle. The "bakeware" in which these ingredients are cooked is the Schrödinger equation, in the general form

$$H\psi(\vec{x},t) = i\hbar\frac{\partial\psi}{\partial t}. \tag{635}$$

Practically, the Schrödinger equation is the *instrument* which allows us to study dynamics of quantum particles characterized by s_μ and p^μ. Therefore, it seems the α-Theory, being identified by few primary elements, has the prerequisite for being a fundamental theory.

The characterization of a quantum particle through s_μ and p^μ does not indicate that the α-Theory can be merely defined "fundamental theory," but it could allow us to exceed some inconsistencies of the quantum world and to explain the most important principles of the modern physics, like the uncertainty principle and constancy of the speed of light. First of all, we notice that, having coupled the spin four-vector s_μ to the four-momentum p^μ, has allowed us to construct a quantum field theory with arbitrary spin and intrinsic degrees of freedom (the spin ones), which have not changed the dimension of the physical space, but simply act on the number of components belonging to any field of particles with fixed spin. Therefore, the Lorentz group $L_+^\uparrow$ has not been expanded or deformed, but simply "coupled" to the group $SU(2)$ (or better to its Lie algebra representations). The spin four-vector s_μ, varying with the representations of $LieSU(2)$, imposes the particle fields to be automatically column matrices inserted in matrix equations, *i.e.* they have to be operator representations.[134] This means within the α-Theory it seems useless to prescribe the second quantization condition, which establishes the physical quantities must be operators.[135] In fact, due to the product $s_\mu p^\mu$, the α-Theory is an intrinsically quantum model and this miracle appears to be possible thanks to the spin four-vector. Therefore, such a core property of elementary particles is the *noumenon* able to make them so different from the classic ones. Hence, the spin of quantum particles seems to impose a mechanical treatment avoiding the definition of postulates or principles. And that is not all, because in the long run the simple energetic analysis on the studied systems is able to conduct in a natural way into a non-commutative algebra and so to an uncertainty principle. In fact, we have seen the imposition of commutation and anti-commutation rules on creation and annihilation operators was born by the request to obtain a positive-definite Hamiltonian for particles having arbitrary spin ($s = 0$ included). Therefore, it seems to be this physical requirement to lead us to the definition of non-commutative algebras and not the second quantization process. Thence, we can also proceed without imposing by the beginning that normal modes of the fields are transformed in creation and annihilation operators. At the end of our process, by evading to make any hypothesis on the nature of the particle fields, it can be observed that, for having definite-positive Hamiltonians, the most general condition the normal modes of such fields must satisfy is to be elements of a C^*-algebra. With regard to the AαT, such a C^*-algebra have to satisfy the following anti-commutation rules [136]

$$\begin{cases} \{b_\alpha(k), b^*_\beta(k')\} = \{d_\alpha(k), d^*_\beta(k')\} = (2\pi)^3\omega_k\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ \{b_\alpha(k), b_\beta(k')\} = \{b^*_\alpha(k), b^*_\beta(k')\} = 0 \\ \{d_\alpha(k), d_\beta(k')\} = \{d^*_\alpha(k), d^*_\beta(k')\} = 0, \end{cases} \tag{636}$$

while, for the SαT, the C^*-algebra of the normal modes must obey the following commutation rules [137]

[134] Only for $s = 0$ such a reasoning is not valid, since we have a scalar representation. But, as it will be explained, this is not a problem, due to the energetic analysis on the Hamiltonians of the studied systems.

[135] Remember that, in general, we speak about operators, but, practically, we use representations of operator algebras.

[136] The * indicates the involution. In such a case, we have: $||b^*_\alpha(k)b_\alpha(k)|| = ||b_\alpha(k)||^2$, $||d^*_\alpha(k)d_\alpha(k)|| = ||d_\alpha(k)||^2$.

[137] In this case, we have: $||a^*_\alpha(k)a_\alpha(k)|| = ||a_\alpha(k)||^2$, $||b^*_\alpha(k)b_\alpha(k)|| = ||b_\alpha(k)||^2$.

$$\begin{cases} [a_\alpha(k), a^*_\beta(k')] = [b_\alpha(k), b^*_\beta(k')] = 2\omega_k(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta} \\ [a_\alpha(k), a_\beta(k')] = [a^*_\alpha(k), a^*_\beta(k')] = 0 \\ [b_\alpha(k), b_\beta(k')] = [b^*_\alpha(k), b^*_\beta(k')] = 0. \end{cases} \tag{637}$$

This includes, as already announced, the possibility of not using for our theory (asymmetric or symmetric) the second quantization process, since it has an intrinsic quantum frame, namely a non-commutative structure.[138] This leads us to think the non-commutativity represents a *basic condition* for the elementary particles and it is not a *quid* derived by measurement problems or strange postulates. It is easy to understand this justifies the Heisenberg uncertainty principle, since it is a directed consequence of the non-commutativity [85]. Hence, the α-Theory is able to exceed the dogmatic condition of the quantum physics, because it proposes to transform a principle in a corollary.

What we said here and in the appendix A sheds new light on the interpretation and paradoxes of the Quantum Mechanics (QM). The first point on which reflecting concerns the circumstance that the quantum particles, being relativistic particles, must be dealt with the formalism of the QFT. Therefore, the non-relativistic quantum mechanics appears nothing but a *toy model* and not a complete theory of elementary particles. If, instead, we think the α-Theory and formalism developed into appendix A, based on the occupation numbers, are the most suitable way for dealing with the physics of elementary particles, we could obtain a rationalization of the QM, which could lead us to the resolution of its apparent paradoxes. Furthermore, by using the results of the α-Theory and formalism of the occupation numbers, which is related to the concept of *information* contained in a physical state more than on the localization of a single particle or a group of particles, it is possible to propose a methodology in order to exceed the paradoxes of the quantum mechanics and to make the particle physics a logical system, as the classic one. Naturally, what we said in this section is only aimed to give bases for future physical and philosophical speculations.

Nevertheless, the α-Theory does not concur to introduce an acceptable formalism to the resolution of the QM paradoxes and to justify the Heisenberg uncertainty principle only, but it says something new also regarding the special Relativity (SR) and in particular about the constancy of the speed of light. In fact, from the α-Theory turns out that the particles of our universe (REP universe) cannot exceed the speed of light c, since such a universe was born through a great spontaneous symmetry breaking – that we called Big-Break – from an unstable universe characterized by particles having negative square energy (we called them "ieps") and not $v > c$. Therefore, the α-Theory justifies also the principle of invariant light speed that, with the principle of relativity, is the basis for the SR. But it goes further, because, unlike the special Relativity, it establishes an arrow for the tachyon condensation, which overcomes the "interchangeability" between tachyons and bradyons within the SR, which caused a lot of misunderstandings, by producing unrealistic science fiction visions. Naturally, we have not to image other types of space for the α-Theory developments (in particular for the gravity description), since the Minkowski space is completely able to describe the high energy physics. In fact, as explained over this work, it is possible that – after Big-Break and the rise of the interactions – the space-time of macroscopic phenomena has shown, at least locally,

[138] Such a reasoning – as said – is valid for $s = 0$ too, since the C^*-algebra has to be defined also in this point for having positive-definite Hamiltonians.

a non-vanishing curvature $R \neq 0$.[139]

Only the accurate study of the α-Theory and the construction of a reliable Big-Break model will be able to clear these mysteries. What we want once again to emphasize is the power and importance of the α-Theory. In fact, the goal of any fundamental theory is to justify those principles and/or mathematical models at basis of the former theories. This means a fundamental theory must explain *why* some hypotheses are right, without taking any further. The circumstance that the α-Theory could justify the most important Quantum Mechanics and special Relativity principles has to make us happy, since it is the proof this theory should create a bridge between the two speculative giants of the past century physics, which many people believed to be totally incompatible. Everything suggests that α-Theory is the theory which we all waited and for this we must work in order to improve and comprise it in the best possible way. Maybe, making it, we will more fully know the universe, our home.

8.2 Is the α-Theory a Popperian model?

According to the falsificationist conception of the philosopher Karl Popper, a scientific theory for being significant has not to be necessarily correct (expectations in perfect agreement with the experiments), but it must be falsifiable, that is the scientific community must be able to verify, with its instruments (theoretical and experimental ones), if the proposed theory is true or false. In any case, this sort of theory in Popper's opinion is remarkable, because, if it is true, it allows a step forward for the human knowledge, while, if it is false, supplies however important informations about the inapplicability to the natural phenomena of certain theoretical speculations.

All the theories developed in the first part of the past century, to which obviously adding the Newton's theory of Gravitation and Maxwell's theory of Electromagnetism, were falsifiable theories, because their characteristic equations allowed of subjecting them to direct tests, which could confirm or invalidate them. The second part of the 1900s saw, instead, the birth of not falsifiable theories, such as the Supersymmetry and String theory.[140] Why, then, whole generations of theoretical physicists are engaged in the development of these two theories? The answer is deep and it goes to modify the Popperian idea of falsification. In fact, a physical theory, in order to induce the scientific community to take it into consideration, must be consistent, *i.e.* it should have an inner coherence and it should aim at important results. The Supersymmetry and String theory satisfy such requests, because they both have solid mathematical bases and are proposed with the scope of unifying the physics laws, which represents the higher objective for any modern theory.

This consideration induces, therefore, to extend the Popper's concept of "critical rationalism" in the following way:

Any physical Model, constructed through coherent mathematical methodologies, having an inner consistency and wanting to extend or unify the pre-existing knowledge, if it is falsifiable too, represents a "good theory."

[139]Remember that Minkowski space is a pseudo-Riemannian manifold of signature $(3,1)$ having null curvature.

[140]Really, there is the Quantum Gravity too, since, for the moment, also this theory cannot be experimentally tested.

Now it must be specified what is meant by "good theory." A "good theory" is a model describing the physical reality. The accuracy degree of this description is the index for understanding what, between the theories candidates for modeling a certain phenomenon, is the one to take like reference. But it must be admitted that other theories too, satisfying the previous postulate, may be *good theories*. For example, the Newton's theory of Gravitation is a "good theory," because it satisfies the postulate previously enunciated. However, the Einstein's general Relativity is better, because the accuracy it achieves, in order to describe the gravitational phenomena, is higher-up than Newton's theory. This does not mean the Newton's theory of Gravitation must be to discard, like all the students of the Analytical Mechanics course can testify. Therefore, a "good theory" is often a useful theory, also when a more accurate one is developed (*good theories*, from this point of view, can be considered like many matryoshka dolls, maybe infinite). How can we consider the α-Theory? The α-Theory is a QFT, in its asymmetric or symmetric connotation, *i.e.* it supplies equations which are useful to qualitative and quantitative research through methodologies commonly used, as the perturbation theory and the path integral formulation. In fact, the study of effective scattering processes will be able certainly to put in relation the predictions of the α-Theory with experimental data. Therefore, the α-Theory represents a Popperian model, because its theoretical structure concurs to legitimate or invalidate it. On this subject, one must advise who wants to undertake the construction of the perturbation α-Theory that the coupling of the free field of such a theory, asymmetric or symmetric, to a gauge field (interaction) is not banal, because, as already explained, due to the supposed Big-Break, this coupling could reveal itself very different from the one we have for Dirac and Klein-Gordon fields. In that sense, it should be proceeded before to the generalization of the α-Theory (explanation of the *big* process of SSB and of the micro-macro problem) and, then, to develop its perturbation theory. In any case, the α-Theory is falsifiable, and so, according to the Popper definition, it is a model to take into consideration. But there is more, because the α-Theory respects also the previous postulate of "good theory," since it is not falsifiable only, but it has also an inner consistency, based on the advanced mathematics of the QFT, and, above all, it unifies the physical description of elementary particles not subject to interaction (free particles), wanting to be the fundamental element for the "Grand Unification." Therefore, the α-Theory, asymmetric or symmetric, is a "good theory." It remains to understand only its accuracy degree, namely if the forecasts it puts on the plate are better than those of the Dirac and Klein-Gordon theories.

Strictly speaking, the α-Theory already supplies remarkable results, and who studied this work knows there exists the freedom of choice on the theoretical wide range, which goes from the generalized Pauli principle to the double inflation passing for the multi-statistics. But, probably, the most amazing result of the α-Theory was seen when – through this model – dynamics of left- and right-handed particles were studied. In this case, the α-Theory, without any theoretical forcing or mathematical illusion game, has taken to two equations, for the AαT and SαT, respectively, changing under parity and with nonzero mass. This means our theory predicted, in an elegant way and without *ad hoc* hypothesis, the characteristic properties associating to the neutrinos and anti-neutrinos, thing that neither the Weyl theory nor the Majorana one is able to make. For this, the theoretical triumph could be declared.

Another fundamental issue will be to understand what, between the AαT and SαT, is the right α-Theory. In the course of the following work, we tried to analyze not only the mathematical and physical differences of these two theories, but, above all, their impact on the vision of the physical world, going from quantum mechanics to cosmology. In fact, it will be fundamental to deepen the multi-statistics and double inflation concepts, in order to find the form of potentials we have defined in the section 7.1, which could give answer to fundamental issues, like the acceleration and/or the missing mass of our universe. Moreover, the different energetic distributions concerning the AαT and SαT can solve other cosmological problems too, like the supremacy of matter on anti-matter, which characterizes our universe (what happens if such an asymmetry were responsible on large-scale for the gravitational force or vice-versa?). Nevertheless, as already explained, this phenomenon could not imply AαT is better than SαT, but to depend on other causes of collective nature, which are outside of the energetic states of elementary particles. In fact, it must be admitted the symmetry between particles and anti-particles, in the SαT, seems to be more in touch with the logical hypothesis that in the infinitely small there is a perfect equilibrium between matter and anti-matter, probably failed on large-scale due to physical phenomena which for the moment are unknown. Furthermore, it must be said that the AαT lets emerge in a simply way the Pauli Principle just from the energetic distribution of its particles and this does not happen within the SαT. Nevertheless, this last theory seems better than the AαT regarding the interaction with a gauge field. In fact, the SαT, unlike the AαT, when it is coupled with an external gauge field, gives a conserved four-current, in which there are the contribution of the particle field and the gauge one, like rightly it should be for a theory in interaction with an external field. For that and many other reasons, being aware that ulterior and deeper studies must be completed, I preferred to abstain from imposing my opinion on the prominence concerning one of these theories, describing both and waiting for future theoretical and experimental speculations.

This long journey is ended. It has shown us the physics can be improved with some revisions concerning the foundations, which unfortunately are often neglected and undervalued, in the name of an empty and alienating technicality. But a skyscraper cannot be built if the bases are wobbly: inexorably it will fall. I hope the α-Theory can open the road to new and important physical developments, in brief that this theory has not been in vain. Only the force of this conviction supported me in these years, between deprivations and snubs, so that the end of this section really can be a beginning: the dawn of a new physics, illuminated by the α-Theory.

A Occupation Numbers and Statistics. A New Way Towards the α-Theory

In this appendix, we want to study the quantum states and their occupation numbers, intended as the amount of particles characterizing them. This approach, based essentially on the concept of *information* that any quantum state takes, will concur to place under a new light the idea of symmetry and anti-symmetry of quantum states, and it will allow to establish the non-connection between Pauli exclusion principle and anti-symmetry of the wave function under the application of the exchange operator P_{ij} (and more generally of the permutation operator P), which sends the generic i-particle in the j-particle and vice-versa. In particular, we will show that, if also a Slater determinant for totally anti-symmetric states can be defined, this has no connection with the exclusion principle. From that reason, a redefinition of distinguishable and indistinguishable quantum particles will follow, in tune with the Quantum Mechanics (QM) principles, which are outside from the classical definition of (elementary) particle and that are, instead, based on the information referring to the amount of particles a state possesses, which leads to the probability of measuring or not a certain event. In the end, we will analyze the Pauli's article of 1940 about the relationship between spin and statistics, which is known with the name of "spin-statistics theorem." The inadequacy of the Pauli's demonstration will be proved, from physical and mathematical point of view, putting thus in crisis the idea that a relationship between the Bose-Einstein (or the Fermi-Dirac) statistics and canonical commutation (or anti-commutation) relations, exists. However, our examination will not be destructive, meaning it will not have the purpose to invalidate the spin-statistics theorem, but, instead, to show the fallacy of the Pauli's demonstration and, therefore, prompting the researchers to review those concepts which were believed to be correct up to now. This should open the road to a redefinition of the relationship between spin and statistics within quantum systems. For this reason, it will be made to see that, although a relationship between spin and statistics is uncertain, a relationship between occupation numbers of quantum states and the two known types of statistics already exists. Such a connection should lead towards new horizons for the study of the elementary particle physics.

In order to begin, we consider a generic quantum state $|A\rangle$ of an abstract Hilbert space $\mathcal{H}$. We have

$$|A\rangle = \sum_{n=1}^{\infty} a_n |e_n\rangle, \tag{638}$$

where $a_n \equiv \langle e_n|A\rangle$, and $\{|e_n\rangle\}_{n=1}^{\infty}$ is an orthonormal basis of vectors of $\mathcal{H}$ satisfying the relations

$$\begin{cases} \langle e_n|e_m\rangle = \delta_{nm} \\ \sum_{n=1}^{\infty} |e_n\rangle\langle e_n| = \mathbb{I} \in \mathcal{H}. \end{cases} \tag{639}$$

The existence of the state $|A\rangle \in \mathcal{H}$ ensures (convergence of the series)

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty. \tag{640}$$

On the contrary, if the condition (640) is valid, based on the Riesz-Fischer theorem, the convergence of the series $\sum_{n=1}^{\infty} a_n|e_n\rangle$ to a generic element $|A\rangle \in \mathcal{H}$ is ensured. The wave function associated to the quantum state $|A\rangle$ of $\mathcal{H}$, that we indicate with $\psi_A(x)$, is defined in the following way

$$\psi_A(x) \equiv \langle x|A\rangle, \quad x \in \mathbb{R}. \tag{641}$$

Practically, the wave function $\psi_A(x)$ allows to the state $|A\rangle$, and to its amount of particles, to move along the real line (or in any interval $(a, b) \subset \mathbb{R}$). In virtue of the normalization condition

$$\int_{\mathbb{R}} |\psi_A(x)|^2 dx = 1, \tag{642}$$

deriving from the probability interpretation of $\psi_A(x)$, we can assert our wave function belongs to the space $L^2(\mathbb{R})$ of the square-integrable functions, characterized from the inner product

$$(f, g) \equiv \int_{\mathbb{R}} f^*(x)g(x)dx \quad \forall f, g \subset L^2(\mathbb{R}). \tag{643}$$

Thanks to the relation (638) and by using the definition (641), we can write

$$\psi_A(x) \equiv \langle x|A\rangle = \sum_{n=1}^{\infty} a_n \langle x|e_n\rangle, \tag{644}$$

that is, defined [141]

$$\psi_n(x) \equiv \langle x|e_n\rangle \quad \forall n \in [1, \ldots, \infty), \tag{645}$$

we have

$$\psi_A(x) = \sum_{n=1}^{\infty} a_n \psi_n(x), \tag{646}$$

[141] They are the wave functions relative to the states of basis.

and so $|A\rangle$ and $\psi_A(x)$ are characterized by the same decomposition coefficients a_n, which induce therefore the isomorphism

$$|A\rangle \longrightarrow \psi_A(x), \tag{647}$$

which moreover is guaranteed, in general terms, by the isomorphism of $\mathcal{H}$ with $L^2(\mathbb{R})$. Of course, also the correspondences exist

$$\begin{aligned} |e_1\rangle &\longrightarrow \psi_1(x) \\ |e_2\rangle &\longrightarrow \psi_2(x) \\ &\ldots\ldots\ldots\ldots \\ |e_n\rangle &\longrightarrow \psi_n(x) \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{648}$$

which say there is a one-to-one correspondence between any state of basis and *elementary* wave functions (it is obvious that $\{\psi_n(x)\}_{n=1}^{\infty}$ represents a basis of $L^2(\mathbb{R})$).

The question now we want to deal with is: how many particles can contain, in principle, the quantum state $|A\rangle$ (and so also its associated wave function $\psi_A(x)$)? The answer is: infinite, by and large. These particles can go all in an elementary state $|e_i\rangle$, or, block by block, in all the states of the basis $\{|e_n\rangle\}_{n=1}^{\infty}$. The number of particles that can be found in a generic state of basis is called "occupation number" (or "filling number") α_n. Hence, we have

$$\begin{aligned} |e_1\rangle &\longrightarrow \alpha_1 = 0, 1, 2, \ldots, \infty \\ |e_2\rangle &\longrightarrow \alpha_2 = 0, 1, 2, \ldots, \infty \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ |e_n\rangle &\longrightarrow \alpha_n = 0, 1, 2, \ldots, \infty \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{649}$$

and so the generic state of basis $|e_i\rangle$ can be indicated in a much better way, by writing

$$|e_i\rangle_{\alpha_i} \quad \forall i \in [1, \ldots, \infty),\ \alpha_i \in \{0, 1, 2, \ldots, \infty\}. \tag{650}$$

Therefore, the total number of particles into the state $|A\rangle$ is [142]

[142]For a physical consistency, such a series should be convergent, *i.e.*

$$k = \sum_{n=1}^{\infty} \alpha_n < \infty.$$

$$k = \sum_{n=1}^{\infty} \alpha_n. \tag{651}$$

The occupation numbers formalism concurs to better characterize the quantum states or the wave functions of particle systems. In fact, we have

$$|A\rangle_k = \sum_{n=1}^{\infty} a_n |e_n\rangle_{\alpha_n} \tag{652}$$

$$\psi_A^k(x) = \sum_{n=1}^{\infty} a_n \psi_n^{\alpha_n}(x). \tag{653}$$

We notice that the wave function $\psi_A^k(x)$ causes the motion of the state $|A\rangle_k$ in $L^2(\mathbb{R})$ (and this thanks to the probability interpretation of the wave function, otherwise $\psi_A^k(x)$ would have moved, like already said previously, on the real line). In general terms, if we want to move $|A\rangle_k$ in $L^2(\mathbb{R}^m)$, we must consider

$$\langle x_1, \ldots, x_m | A\rangle_k \equiv \psi_A^k(x_1, \ldots, x_m) = \sum_{n=1}^{\infty} a_n \psi_n^{\alpha_n}(x_1, \ldots, x_m), \quad (x_1, \ldots, x_m) \subset \mathbb{R}^m. \tag{654}$$

Is there a relation between the m-tuple $(x_1, \ldots, x_m)$ and the generic occupation number of α_l? No, since the m-tuple says *where* is the generic state ψ_l in $L^2(\mathbb{R}^m)$ (independently from how many particles it contains), while α_l says *how many* particles it contains. Therefore, expressions of the type: "*the particle a is in x_1 and the particle b in x_2,*" or: "*the wave function of two particles is* $\psi(x_1, x_2)$" are conceptually wrong.

We want now to express, in the formalism as soon as developed, the "indistinguishability principle" of quantum particles. First of all, instead of simply writing

$$|A\rangle_k \text{ and } \psi_A^k(x_1, \ldots, x_m),$$

we make clear in these quantities the dependence from their occupation numbers, namely

$$\begin{cases} |A\rangle_k \equiv |A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k \\ \psi_A^k(x_1, \ldots, x_m) \equiv \psi_A^k(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_n, \ldots). \end{cases} \tag{655}$$

Practically, $|A\rangle_k$ and $\psi_A^k(x_1, \ldots, x_m)$, with the above definitions, now directly depend on the *amount* of particles existing into each of their elementary states (of basis). What happens when two particles which are in the states i and j are exchanged? In order to answer this question, we need to concentrate on the distinguishability and indistinguishability concept of quantum particles. As it is known, the quantum particles, according with the uncertainty principle, are indistinguishable, *i.e.* for them is valid the "indistinguishability principle." The formalism we are using contains this principle in itself, being able to exceed it too. In fact, to define an "indistinguishability principle" for the quantum particles is really without sense, because this treats the particles as single objects on which is asked if it is possible or not to consider their trajectory. But, in QM the same concept of "particle" comes to fall, since the utilized physical quantity is the wave function, which does not represent a particle or a group of particles, but the *physical information* which is derived from this particle or group of particles. Therefore, to use a formalism based essentially on the *amount of particles*, namely on the information resulting from such an amount, and not on the particles only, seems to be in full agreement with the spirit of the Quantum Mechanics. Hence, speaking about the indistinguishability of identical particles is incorrect, since it is a concept still connected with the classical particle idea, which in the philosophy of quantum physics is meaningless.

This as soon as asserted makes us understand that wondering what succeeds when a i-particle is exchanged with a j-particle is wrong. It is, instead, right to ask what happens when the amounts of the elementary states i and j are exchanged, *i.e.* when α_i is exchanged with α_j. In order to answer this question, we define an exchange operator P_{ij} such that

$$P_{ij}|A;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots\rangle_k = |A;\alpha_1,\ldots,\alpha_j,\ldots,\alpha_i,\ldots\rangle_k \tag{656}$$

$$P_{ij}\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots) = \psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_j,\ldots,\alpha_i,\ldots). \tag{657}$$

It is easy to see that, if we apply (on the left) once again the operator P_{ij} to the previous equations, we get

$$P_{ij}^2|A;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots\rangle_k = P_{ij}|A;\alpha_1,\ldots,\alpha_j,\ldots,\alpha_i,\ldots\rangle_k = |A;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots\rangle_k \tag{658}$$

$$P_{ij}^2\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots) = P_{ij}\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_j,\ldots,\alpha_i,\ldots) = \psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_j,\ldots), \tag{659}$$

and so $P_{ij}^2 = \mathbb{I}$, where with $\mathbb{I}$ we indicate the identity operator.

The above property is very important, because it allows us to identify the eigenvalues of the operator P_{ij}. In fact, if we consider

$$P_{ij}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \lambda|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k \tag{660}$$

$$P_{ij}\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = \lambda\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots), \tag{661}$$

by multiplying (on the left) both the members by P_{ij}, we have

$$P_{ij}^2|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = P_{ij}\lambda|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \lambda P_{ij}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \lambda^2|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k \quad (662)$$

$$P_{ij}^2\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = P_{ij}\lambda\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = \lambda P_{ij}\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = \lambda^2\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots), \quad (663)$$

from which, by remembering $P_{ij}^2 = \mathbb{I}$, it quickly follows $\lambda^2 = 1$, that is

$$\lambda = \pm 1. \quad (664)$$

Therefore

$$P_{ij}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \pm|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k \quad (665)$$

$$P_{ij}\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = \pm\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots). \quad (666)$$

It is well to underline that the eigenstate $|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k$ of P_{ij} (thus like the wave function $\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots)$) is, of course, eigenstate of the Hamiltonian H of the system, thanks to the indistinguishability principle of quantum particles. Hence

$$H|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = E|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k \quad (667)$$

$$H\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots) = E\psi_A^k(x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots). \quad (668)$$

Since P_{ij} and H are characterized by a common eigenstate, we can assert P_{ij} is a constant of motion, namely

$$[P_{ij}, H] = 0. \quad (669)$$

Usually, the eigenstates of P_{ij} with $\lambda = 1$ are said "symmetric," while those with $\lambda = -1$ are said "anti-symmetric." Since the evolution of a physical state does not change the sign of λ, symmetry and anti-symmetry of a quantum state (or wave function), about the exchange of α_i with α_j, are conserved, respectively. Therefore, if a physical state is eigenstate of P_{ij}, it is necessarily "symmetric or anti-symmetric," regarding the exchange of α_i with α_j. Since $P_{ij}|A\rangle_k$, thus like $P_{ij}\psi_A^k$, is eigenstate of H belonging to the same eigenvalue E of $|A\rangle_k$ (or ψ_A^k), there is a degeneration phenomenon (exchange degeneracy), for which the totality of the states (or wave functions), deriving from all the possible exchanges of the occupation numbers concerning the set $(\alpha_1, \ldots, \alpha_n, \ldots)$, belong to the same energy. This logic is not only valid for the exchange of two occupation numbers α_i and α_j, but for any permutation P, even or odd, of the occupation numbers $(\alpha_1, \ldots, \alpha_n, \ldots)$. However, we must be quick-witted, because all the permutations do not commute and so a system of common eigenvectors will not be. In reality, this problem is avoided by noticing any permutation P can be expressed as the product of more exchange operations and, therefore, the common eigenvectors of all the permutations P are just those relative to all the generic exchange operators P_{ij}, always with eigenvalues $\lambda = \pm 1$. Also in such a case, these states (or wave functions) are eigenstates of H with eigenvalue E and so a degeneration phenomenon appears, for which N_P states (or wave functions) belong to the same energy.[143] Therefore, we can write

$$P|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k = \pm|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k \tag{670}$$

$$P\psi_A^k(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_n, \ldots) = \pm\psi_A^k(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_n, \ldots). \tag{671}$$

As said, the eigenvalues are always two, *i.e.*

$$\lambda_1 = +1, \quad \lambda_2 = -1 \quad \text{(or conversely)} \tag{672}$$

and this independently by the fact P supports even or odd permutations. Also for the permutation operator P, as for the exchange operator P_{ij}, eigenstates belonging to the eigenvalue $+1$ are said symmetric and those belonging to the eigenvalue -1 are said anti-symmetric and this specific feature does not change with the evolution of the system.

Now we want to write the relations (670) and (671) in the following more compact form, which will be useful in the next pages [144]

$$P|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k = (\lambda_1, \lambda_2)_P|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k \tag{673}$$

$$P\psi_A^k(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_n, \ldots) = (\lambda_1, \lambda_2)_P\psi_A^k(x_1, \ldots, x_m; \alpha_1, \ldots, \alpha_n, \ldots). \tag{674}$$

[143] N_P is the total number of the permutations that can be made on the set $(\alpha_1, \ldots, \alpha_n, \ldots)$.

[144] Naturally, $(\lambda_1, \lambda_2)_P \equiv (\lambda_{1,P}, \lambda_{2,P})$.

Does a physical reason exist why the eigenvalues of P (or P_{ij}) just assume the values $+1$ and -1? We have demonstrated this depends on the fact $P^2 = \mathbb{I}$ (or $P_{ij}^2 = \mathbb{I}$) and so it should derive only by a mathematical property of the generic permutation (or exchange) operator. Nevertheless, we can make some hypotheses with regard to. As an example, it could be thought this depends on the type of permutation, even or odd, which works on the system, that is we can assume

$$\lambda_i \equiv (-1)^P \quad \forall i \in \{1, 2\}. \tag{675}$$

If P is even then we have $+1$, while if it is odd we have -1. From what previously seen, we note that the (675) is wrong, because P, independently from the fact it is even or odd, always must have like eigenvalues $+1$ and -1 (and not only one!). Alternatively, we can suppose the eigenvalues λ_1 and λ_2 depend on the sum of the occupation numbers involved in the permutation P. However, such a sum will be even or odd and, therefore, we have $+1$ or -1 only. But this is not a problem, because it can be considered [145]

$$\sum_\alpha P(\alpha) \text{ and } \sum_\alpha P(\alpha) + 1, \tag{676}$$

since the first summation is even when the second one is odd, and vice-versa. Therefore, we can reasonably define

$$\lambda_1 \equiv (-1)^{\sum_\alpha P(\alpha)}, \quad \lambda_2 \equiv (-1)^{\sum_\alpha P(\alpha)+1}. \tag{677}$$

It is simple to notice that the (677) are well-defined, since they do not fix *a priori* the values $+1$ and -1, but make depend them on time after time by $\sum_\alpha P(\alpha)$. Hence, in general, if we want by necessity to give a physical meaning to λ_1 and λ_2, it is possible to use the definitions (677) in the expressions (673) and (674).

Now we want to find the most general expression of $|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k$ as a function of all the possible permutations of the set $(\alpha_1, \ldots, \alpha_n, \ldots)$. For making this, we must before analyze the concept of symmetry and anti-symmetry of a quantum state (or wave function), that in literature is misunderstood. As seen in the previous pages, the property of symmetry and anti-symmetry of a quantum state consists of assigning to such a state, eigenstate of the operator P (or P_{ij}), the eigenvalue $+1$ or -1, respectively. And this happens *all the time* we apply a permutation P (or an exchange P_{ij}) on a quantum state. Therefore, we can assert the property of symmetry or anti-symmetry of a quantum state depends on the exchange or permutation operated *once only* on the system and not by the coupling (sum and/or difference) of all the possible exchanges or permutations applied on the system. This means that, referring to a single exchange or a single permutation, it makes sense to say a quantum state (or wave function) must be symmetric or anti-symmetric and it must conserve this characteristic in the time, but it makes no sense to say that symmetry and anti-symmetry, which in short depend on assigning the eigenvalue $+1$ and the eigenvalue -1 to the eigenstate of a single permutation P (or exchange P_{ij}), are an intrinsic property of a quantum state

[145] We indicate with $P(\alpha)$ the set of α_i on which the permutation is done.

(or wave function), when one wants to express this state like combination of all the exchanges or permutations which we can make on the occupation numbers of the system (or on the particles, if we use the old formalism). Thence, we understand the mistake – commonly committed – is to impose that a typical property of a single permutation (or exchange) is conserved also when we consider the sum of all the exchanges or permutations, that would reasonably give us a different result.

Therefore, what we want to treat is a new mathematical problem, which consists of expressing the quantum state $|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k$ (or the wave function ψ_A^k) like a combination of all the permutations we can do on the set $(\alpha_1,\ldots,\alpha_n,\ldots)$. At the end, we will confront the obtained result with the totally symmetric state and the totally anti-symmetric state that are possible to construct taking the sum of all the eigenstates with eigenvalue $+1$ and the sum of all the eigenstates with alternating eigenvalues $(-1)^P$, where, with P, we indicate the even and odd permutations. In order to resolve our problem, we notice by the (673) the following eigenvalue equations are valid

$$\begin{aligned}
P_1|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_1}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
P_2|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_2}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_i|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_i}|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\tag{678}$$

From which, by multiplying both sides of the first expression by $(\lambda_1,\lambda_2)_{P_1}$, both sides of the second expression by $(\lambda_1,\lambda_2)_{P_2}$, *etc.*, we obtain [146]

$$\begin{aligned}
|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_1}P_1|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_2}P_2|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k &= (\lambda_1,\lambda_2)_{P_i}P_i|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\tag{679}$$

In order to estimate the general expression of $|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k$ in function of all the permutations operated on such a state, we must, obviously, add all the terms. At left-hand side certainly will appear $N_P|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k$, where with N_P is indicated the total number of permutations made on the set $(\alpha_1,\ldots,\alpha_n,\ldots)$. On the contrary, to estimate the right-hand side is much more complicated. In fact, it is not possible to consider the simple sum of the several terms, because all the possible combinations between the λ_1 and λ_2 of any permutation must be considered.[147] For

[146] It is banal to notice that, in general, $(\lambda_1,\lambda_2)^2_{P_l}=1$ for any $l\in\mathbb{N}$.

[147] Naturally, we can only couple the λ concerning to different permutations, since the λ of a same permutation cannot be added. In other words, we cannot consider the amount $\lambda_{1,P_l}+\lambda_{2,P_l}$, for any fixed $l\in\mathbb{N}$.

understanding this problem, we consider some simple cases of coupling. To such a purpose, we start from the 2 couples

$$(+,-)_1,\ (+,-)_2 \tag{680}$$

and matching all distinct elements in each couple, we have

$$++,\ +-,\ -+,\ --\,. \tag{681}$$

Therefore, from 2 couples it is possible to draw 4 elements. In the case of the 3 couples

$$(+,-)_1,\ (+,-)_2,\ (+,-)_3 \tag{682}$$

we have the combinations

$$+++,\ +-+,\ +--,\ -++,\ -+-,\ ---,\ ++-,\ --+. \tag{683}$$

Hence, from 3 couples it is possible to draw 8 elements. In the case of the 4 couples

$$(+,-)_1,\ (+,-)_2,\ (+,-)_3,\ (+,-)_4 \tag{684}$$

we have the combinations

$$\begin{aligned} &++++,\ +-++,\ +--+,\ +---,\ -+++,\ --++,\ ---+,\ ----,\\ &-+-+,\ +-+-,\ +++-,\ --+-,\ -+--,\ -++-,\ ++-+,\ ++--\,. \end{aligned} \tag{685}$$

Hence, from 4 couples it is possible to draw 16 elements. In general terms, therefore, we have n^q elements, where n is the number of the elements of the couple ($n = 2$), while q is the number of the couples ($q = 2, 3, 4 \Rightarrow 2^2 = 4, 2^3 = 8, 2^4 = 16$). This means that, in order to calculate all the elements generated from the matching of the couples $(\lambda_1, \lambda_2)_{P_i}$, we have to consider nothing but the ordered selections with repetition, two by two different within them, of class N_P on the set of the $2N_P$ elements identified by the previously written equations. At the end, the elements of these ordered selections with repetition must be added between them and this gives the state $N_P|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k$. The total number concerning this type of states is 2^{N_P}.

In order to formalize that as soon as asserted, we develop a general formalism. For that reason, we consider the following classes of elements

$$
\begin{aligned}
C_1 &= \{X_1^1, X_1^2, \ldots, X_1^n\} \\
C_2 &= \{X_2^1, X_2^2, \ldots, X_2^n\} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
C_q &= \{X_q^1, X_q^2, \ldots, X_q^n\}.
\end{aligned} \tag{686}
$$

The set of the ordered selections with repetition, two by two different, of all the n elements of the classes $C_1, C_2, \ldots, C_q$ are given by

$$
D_{n,q}(C_1, \ldots, C_q) = \left\{ D_{n,q}^1(C_1, \ldots, C_q), \ldots, D_{n,q}^{n^q}(C_1, \ldots, C_q) \right\}, \tag{687}
$$

i.e. it is constituted, like we expected, by n^q different ordered selections.

The problem now we want to resolve is that to find a writing able to express the sum of all the elements which represent each one of the ordered selections of the set $D_{n,q}(C_1, \ldots, C_q)$. The expression

$$
\sum_{C_1, \ldots, C_q} D_{n,q}^l(C_1, \ldots, C_q), \quad \forall l \in \{1, \ldots, n^q\} \tag{688}
$$

seems to satisfy this wish.

We specialize such a formalism to the state $N_P|A; \alpha_1, \ldots, \alpha_n, \ldots\rangle_k$. Since [148]

1. $n = 2$, $q = N_P$

[148] It is possible to construct a more explicit and compact formalism than the above one. For doing it, we must note that the (686) can be written in this fashion

$$
C_j = \{X_j^i\}_{i=1}^n \quad \forall j \in \{1, \ldots, q\},
$$

from which the (687) becomes

$$
D_{n,q}\left[\{X_1^i\}_{i=1}^n, \ldots, \{X_q^i\}_{i=1}^n\right] = \left\{ D_{n,q}^1\left[\{X_1^i\}_{i=1}^n, \ldots, \{X_q^i\}_{i=1}^n\right], \ldots, D_{n,q}^{n^q}\left[\{X_1^i\}_{i=1}^n, \ldots, \{X_q^i\}_{i=1}^n\right] \right\}.
$$

Now it is banal to see that the (688) can be so written

$$
D_{n,q}^l\left[\{X_1^i\}_{i=1}^n + \ldots + \{X_q^i\}_{i=1}^n\right] = D_{n,q}^l\left[\sum_{j=1}^q \{X_j^i\}_{i=1}^n\right] = \sum_{j=1}^q D_{n,q}^l\left[\{X_j^i\}_{i=1}^n\right].
$$

2. $C_1 = (\lambda_1, \lambda_2)_{P_1}, \dots, C_i = (\lambda_1, \lambda_2)_{P_i}, \dots$

3. $(\lambda_1, \lambda_2)_{P_1} P_1 |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k + (\lambda_1, \lambda_2)_{P_2} P_2 |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k =$
$[\delta_{1j}(\lambda_1, \lambda_2)_{P_1} + \delta_{2j}(\lambda_1, \lambda_2)_{P_2}] \sum_{r=1}^{2} \delta_{rj} P_j |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k$,[149]

and, by using in place of $\sum_{C_1,\dots,C_q}$ the writing $\sum_{\lambda_P}$, for indicating the sum of all the λ of any ordered selection without repetition, we can immediately write [150]

$$N_P |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k = \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1, \lambda_2)_{P_1}, \dots, \delta_{ij}(\lambda_1, \lambda_2)_{P_i}, \dots\right] \delta_{rj} P_j |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k, \tag{689}$$

from which it follows [151]

$$|A; \alpha_1, \dots, \alpha_n, \dots\rangle_k = \frac{1}{N_P} \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1, \lambda_2)_{P_1}, \dots, \delta_{ij}(\lambda_1, \lambda_2)_{P_i}, \dots\right] \delta_{rj} P_j |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k \tag{690}$$

$\forall l \in \{1, \dots, 2^{N_P}\}$.

---

[149] Naturally, $\delta_{1j}\delta_{1j} = 1$, $\delta_{1j}\delta_{2j} = 0 = \delta_{2j}\delta_{1j}$, $\delta_{2j}\delta_{2j} = 1$.

[150] By using the formalism introduced in the footnote 148, we have

$$D^l_{2,N_P} \left[\sum_{j=1}^{N_P} (\lambda_1, \lambda_2)_{P_j}\right] = \sum_{j=1}^{N_P} D^l_{2,N_P} \left[(\lambda_1, \lambda_2)_{P_j}\right],$$

and so the (689) can be written

$$N_P |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k = \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1, \lambda_2)_{P_j}\right] \delta_{rm} P_m |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k.$$

[151] Or else

$$|A; \alpha_1, \dots, \alpha_n, \dots\rangle_k = \frac{1}{N_P} \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1, \lambda_2)_{P_j}\right] \delta_{rm} P_m |A; \alpha_1, \dots, \alpha_n, \dots\rangle_k,$$

$\forall l \in \{1, \dots, 2^{N_P}\}$.

Adding all these 2^{N_P} states, we obtain the super-state [152]

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{N_P 2^{N_P}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right] \delta_{rj} P_j |A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k, \tag{691}$$

which represents the most general eigenstate of H expressed through the eigenstates obtained from all the permutations made on the set of occupation numbers $(\alpha_1,\ldots,\alpha_n,\ldots)$. If we want to normalize this eigenstate, it must be written [153]

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right] \delta_{rj} P_j |A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k. \tag{692}$$

Similarly, in terms of wave function [154]

$$\psi^k_A(\underset{\sim}{x};\underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right] \delta_{rj} P_j \psi^k_A(\underset{\sim}{x};\underset{\sim}{\alpha}), \tag{693}$$

where, for convenience, one has defined

[152] Or else

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{N_P 2^{N_P}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right] \delta_{rm} P_m |A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k.$$

[153] Or else

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right] \delta_{rm} P_m |A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k.$$

[154] Or else

$$\psi^k_A(\underset{\sim}{x};\underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right] \delta_{rm} P_m \psi^k_A(\underset{\sim}{x};\underset{\sim}{\alpha}).$$

$$(\underset{\sim}{x};\underset{\sim}{\alpha}) \equiv (x_1,\ldots,x_m;\alpha_1,\ldots,\alpha_n,\ldots). \tag{694}$$

At this point, we consider the two states

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P}}\sum_P P|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k \tag{695}$$

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P}}\sum_P (-1)^P P|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k, \tag{696}$$

where

$$(-1)^P = \begin{cases} +1 \text{ for even permutations} \\ -1 \text{ for odd permutations.} \end{cases} \tag{697}$$

We want to know what relationship exists between the above two states and super-state (692). The answer is: none, because in no way the states (695) and (696) can be derived from the super-state (692). However, if we consider one of the l-states composing the (normalized) grand-state[155]

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P}}\sum_{r=1}^{N_P}\sum_{\lambda_P} D^l_{2,N_P}\left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right]\delta_{rj}P_j|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k, \tag{698}$$

we sure have within these 2^{N_P} states there are the states (695) and (696) too. This means that $(2^{N_P}-2)$–states different from (695) and (696) exist. This fact is essential for trying to clarify the role of the state (695), which is called "totally symmetric," and the role of the state (696), which is called "totally anti-symmetric," within those quantum systems in which the generic state, eigenstate of H, is expressed through the sum of the states obtained from this generic state by making all the possible permutations on the set of occupation numbers $(\alpha_1,\ldots,\alpha_n,\ldots)$. The result to which we arrive is that the states (695) and (696) have no relation with the super-state (692) and, at most, they can be considered two ordinary states within the 2^{N_P} states assembling the normalized grand-state (698). This means it is absolutely wrong to think that the quantum state (or the wave function) of a system of identical particles can be totally symmetric or anti-symmetric under the

[155]Or else

$$|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k = \frac{1}{\sqrt{N_P}}\sum_{r=1}^{N_P}\sum_{j=1}^{N_P} D^l_{2,N_P}\left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right]\delta_{rm}P_m|A;\alpha_1,\ldots,\alpha_n,\ldots\rangle_k.$$

exchanges or permutations that we can make on the occupation numbers of the set $(\alpha_1, \ldots, \alpha_n, \ldots)$.

It could be objected this result has been obtained because, in contrast to what is made in literature, we have not reasoned on the permutation of $(1, 2, 3, \ldots, N)$ identical particles, but on the occupation numbers of each elementary (quantum) state constituting the generic state $|A\rangle$. Although, like already widely explained, to consider the set of $(1, 2, 3, \ldots, N)$ identical particles and their permutations has not much sense in the quantum physics (among other things, all the particles could be found in one of the infinite elementary states of $|A\rangle$), we will show that also this (bad) approach takes to analogous results like those obtained previously. With this aim, we consider the system of eigenvalue equations

$$\begin{aligned}
&P_1|A; 1, \ldots, N\rangle = \pm|A; 1, \ldots, N\rangle; \quad [H, P_1] = 0 \\
&P_2|A; 1, \ldots, N\rangle = \pm|A; 1, \ldots, N\rangle; \quad [H, P_2] = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P_i|A; 1, \ldots, N\rangle = \pm|A; 1, \ldots, N\rangle; \quad [H, P_i] = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{699}$$

which can also be written in such a way

$$\begin{aligned}
&P_1|A; 1, \ldots, N\rangle = (\lambda_1, \lambda_2)_{P_1}|A; 1, \ldots, N\rangle; \quad [H, P_1] = 0 \\
&P_2|A; 1, \ldots, N\rangle = (\lambda_1, \lambda_2)_{P_2}|A; 1, \ldots, N\rangle; \quad [H, P_2] = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P_i|A; 1, \ldots, N\rangle = (\lambda_1, \lambda_2)_{P_i}|A; 1, \ldots, N\rangle; \quad [H, P_i] = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{700}$$

where we have defined

$$\lambda_1 \equiv +1, \quad \lambda_2 \equiv -1. \tag{701}$$

Now, by multiplying both sides of the first expression of the (700) by $(\lambda_1, \lambda_2)_{P_1}$, both sides of the second expression by $(\lambda_1, \lambda_2)_{P_2}$, *etc.*, we have [156]

$$\begin{aligned}
&|A; 1, \ldots, N\rangle = (\lambda_1, \lambda_2)_{P_1} P_1|A; 1, \ldots, N\rangle \\
&|A; 1, \ldots, N\rangle = (\lambda_1, \lambda_2)_{P_2} P_2|A; 1, \ldots, N\rangle
\end{aligned}$$

[156] Of course, in general, $(\lambda_1, \lambda_2)^2_{P_l} = 1$ for any $l \in \mathbb{N}$.

$$\begin{aligned} &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &|A;1,\ldots,N\rangle=(\lambda_1,\lambda_2)_{P_i}P_i|A;1,\ldots,N\rangle \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{702}$$

from which, following the argumentations made in the previous pages and by remembering in such a case $N_P = N!$ (number of permutation of N particles), we get [157]

$$|A;1,\ldots,N\rangle=\frac{1}{N!}\sum_{r=1}^{N!}\sum_{\lambda_P}D^l_{2,N!}\left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right]\delta_{rj}P_j|A;1,\ldots,N\rangle, \tag{703}$$

and therefore, by making the sum on all the $2^{N!}$ states, we obtain the (normalized) super-state [158]

$$|A;1,\ldots,N\rangle=\frac{1}{\sqrt{N!2^{N!}}}\sum_{l=1}^{2^{N!}}\sum_{r=1}^{N!}\sum_{\lambda_P}D^l_{2,N!}\left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right]\delta_{rj}P_j|A;1,\ldots,N\rangle, \tag{704}$$

and in terms of wave function (always by supposing it depends on N particles) [159]

$$\psi_A(1,\ldots,N)=\frac{1}{\sqrt{N!2^{N!}}}\sum_{l=1}^{2^{N!}}\sum_{r=1}^{N!}\sum_{\lambda_P}D^l_{2,N!}\left[\delta_{1j}(\lambda_1,\lambda_2)_{P_1},\ldots,\delta_{ij}(\lambda_1,\lambda_2)_{P_i},\ldots\right]\delta_{rj}P_j\psi_A(1,\ldots,N). \tag{705}$$

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[157] Or else

$$|A;1,\ldots,N\rangle=\frac{1}{N!}\sum_{r=1}^{N!}\sum_{j=1}^{N!}D^l_{2,N!}\left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right]\delta_{rm}P_m|A;1,\ldots,N\rangle.$$

[158] Or else

$$|A;1,\ldots,N\rangle=\frac{1}{\sqrt{N!2^{N!}}}\sum_{l=1}^{2^{N!}}\sum_{r=1}^{N!}\sum_{j=1}^{N!}D^l_{2,N!}\left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right]\delta_{rm}P_m|A;1,\ldots,N\rangle.$$

[159] Or else

$$\psi_A(1,\ldots,N)=\frac{1}{\sqrt{N!2^{N!}}}\sum_{l=1}^{2^{N!}}\sum_{r=1}^{N!}\sum_{j=1}^{N!}D^l_{2,N!}\left[\delta_{jm}(\lambda_1,\lambda_2)_{P_j}\right]\delta_{rm}P_m\psi_A(1,\ldots,N).$$

This proves that, by utilizing an (incorrect) formalism which does not use the occupation numbers, but only the particles which a state hypothetically contains, we always arrive at the result that the states

$$|A; 1, \ldots, N\rangle_S = \frac{1}{\sqrt{N!}} \sum_P P|A; 1, \ldots, N\rangle \tag{706}$$

$$|A; 1, \ldots, N\rangle_A = \frac{1}{\sqrt{N!}} \sum_P (-1)^P P|A; 1, \ldots, N\rangle \tag{707}$$

are not the only ones (if we reason in terms of l-th states) and are incoherent from the theoretical point of view too (if we reason in terms of super-state, which then is the more general state resolving our problem). This irreparably damages the idea that the wave function of an identical system of particles can have only two alternatives under the permutation of all its particles, *i.e.* the fact of being totally symmetric or totally anti-symmetric.

What now we want to make is the study of the Pauli exclusion principle, in terms of the formalism of the occupation numbers we introduced in these pages. In particular, we want to inquire on the validity of the consequences of such a principle, in order to understand its real logical and physical subsistence. The exclusion principle, formulated by Pauli in 1925, can be enunciated in the following way:

An elementary quantum state, characterized in a unique way by the quantum numbers (n, l, m, s), can maximum contain a single electron. Therefore, an elementary quantum state can be empty or having one electron at the most.

Figuratively, this can be expressed in the following way:

$$\square = \text{elementary quantum state}, \quad \cdot = \text{electron}$$

- $\square$ the state is empty.
- $\boxdot$ the state contains one electron, *i.e.* the state is *complete*.

Thence, an elementary quantum state can be imagined like a box which no more than one electron enters into.[160] This means that, by using the formalism of the occupation numbers, given a system

[160] Pauli, rather than quantum state, spoke about "atomic orbital," since its principle was born for just explaining the disposition of electrons into energy levels of the periodic table elements.

of electrons, or of particles subjected to the Pauli principle, the occupation numbers $(\alpha_1, \ldots, \alpha_n, \ldots)$ of this system can singularly assume value 0 or 1, namely

$$\forall \alpha_i \in (\alpha_1, \ldots, \alpha_n, \ldots) \;\Rightarrow\; \alpha_i = 0, 1. \tag{708}$$

Except this (not banal) difference, all the results obtained in the previous pages continue to be valid. It is important to observe the Pauli principle gives physical coherence to the quantum systems, because it shows the elementary quantum states must have a *finite capacity*.

What now we want to do is to discuss some fundamental results following more or less directly from the Pauli principle. They, in order, can thus be enumerated

1. Based on the Pauli exclusion principle, the wave function of a system of electrons (and more in general of fermions) is anti-symmetric under the exchange of two particles of the system.
2. The Pauli principle is valid for all the fermions, namely also the particles with spin 3/2, 5/2, 7/2, *etc.*, have elementary quantum state equal to the atomic one.
3. Particles with integer spin follow the Bose-Einstein statistics (bosons) and have totally symmetric wave function under any particle permutation, while particles with half-integer spin follow the Fermi-Dirac statistics (fermions) and have totally anti-symmetric wave function under any particle permutation.
4. The fermions (half-integer spins) are characterized by field operators following anti-commutation rules, while the bosons (integer spins) are characterized by field operators following commutation rules.

Let us discuss the point 1. The expression (693) [161]

$$\psi^k_A(\underset{\sim}{x}; \underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{\lambda_P} D^l_{2,N_P} \left[\delta_{1j}(\lambda_1, \lambda_2)_{P_1}, \ldots, \delta_{ij}(\lambda_1, \lambda_2)_{P_i}, \ldots\right] \delta_{rj} P_j \psi^k_A(\underset{\sim}{x}; \underset{\sim}{\alpha}),$$

which is naturally valid even if the occupation numbers can assume values 0 or 1 (Pauli principle), automatically excludes the possibility of anti-symmetric wave functions and, therefore, it impair the point 1. Then, if we also consider the exchange P_{ij} of any two electrons (or better of two occupation numbers), a reason why the eigenvalue -1 must be assigned to such eigenstate does not exist, and so the eigenvalue $+1$ cannot *a priori* be excluded (symmetric state). This takes to conclude the

[161] Or else

$$\psi^k_A(\underset{\sim}{x}; \underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P 2^{N_P}}} \sum_{l=1}^{2^{N_P}} \sum_{r=1}^{N_P} \sum_{j=1}^{N_P} D^l_{2,N_P} \left[\delta_{jm}(\lambda_1, \lambda_2)_{P_j}\right] \delta_{rm} P_m \psi^k_A(\underset{\sim}{x}; \underset{\sim}{\alpha}).$$

Pauli principle has nothing to do with the anti-symmetry of a wave function of a system of electrons under the exchange of two particles of this system.

For giving ulterior credibility to such an assertion-result, we analyze the procedures that in literature lead to the alleged relationship between anti-symmetry and exclusion principle. For such a purpose, we start from the so-called "Slater determinant," which represents a kind of theorem that proves (or would have to prove) the validity of the point 1. It can be defined by the most general wave function (or quantum state) totally anti-symmetric for particle exchange, under the (weak) hypothesis the particles of system do not interact between them. We try to construct this determinant before applying the formalism of the occupation numbers and then the commonly used formalism, in order to study the relationship with the Pauli principle. As already previously observed, this is made for the sake of completeness, because, like demonstrated, it makes no sense to define a totally anti-symmetric (or symmetric) state. All this presupposes the theoretical groundlessness of the Slater determinant and so its non-connection with the Pauli exclusion principle. Therefore, with this in mind, we must emphasize that all what will be said below about the Slater determinant is based on the *ad absurdum* assumption that totally anti-symmetric states exist. That said, let us suppose existing the quantum state characterized by the wave function

$$\psi_A^k(\underset{\sim}{x};\underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P}}\sum_P (-1)^P P \psi_A^k(\underset{\sim}{x};\underset{\sim}{\alpha}), \tag{709}$$

where the (697) is valid. If we presume the following factorization be correct

$$\psi_A^k(\underset{\sim}{x};\underset{\sim}{\alpha}) = \phi_1^{\alpha_1}(\underset{\sim}{x})\phi_2^{\alpha_2}(\underset{\sim}{x})\cdots\phi_n^{\alpha_n}(\underset{\sim}{x})\cdots, \tag{710}$$

the formula (709) can be written through the following determinant, whose development just represents the right-hand side

$$\psi_A^k(\underset{\sim}{x};\underset{\sim}{\alpha}) = \frac{1}{\sqrt{N_P}}\begin{vmatrix} \phi_1^{\alpha_1}(\underset{\sim}{x}) & \phi_1^{\alpha_2}(\underset{\sim}{x}) & \dots & \phi_1^{\alpha_n}(\underset{\sim}{x}) & \dots \\ \phi_2^{\alpha_1}(\underset{\sim}{x}) & \phi_2^{\alpha_2}(\underset{\sim}{x}) & \dots & \phi_2^{\alpha_n}(\underset{\sim}{x}) & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ \phi_n^{\alpha_1}(\underset{\sim}{x}) & \phi_n^{\alpha_2}(\underset{\sim}{x}) & \dots & \phi_n^{\alpha_n}(\underset{\sim}{x}) & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \end{vmatrix}. \tag{711}$$

Based on the properties of determinants, the above expression is null if two rows or two columns are equal, *i.e.* if

<u>Rows</u>:

$$\{\phi_i^{\alpha_k}\}_{k=1}^{\infty} = \{\phi_j^{\alpha_k}\}_{k=1}^{\infty}, \quad \forall \text{ fixed } i,j \in \mathbb{N}. \tag{712}$$

Columns:

$$\{\phi_i^{\alpha_k}\}_{i=1}^{\infty} = \{\phi_i^{\alpha_l}\}_{i=1}^{\infty}, \quad \forall \text{ fixed } k, l \in \mathbb{N}. \tag{713}$$

Let us control if these conditions infringe the Pauli principle, *i.e.* if they give its indirect demonstration. Concerning the first condition, it practically tells us that two different states (or wave functions) cannot have the same occupation number. This really depends on the fact we constructed these states associating in general to the i-th one the i-th number of occupation and so on, and therefore the first condition is not but a mathematical requirement that is necessary to confirm the adopted formalism. Either way, this condition never violates the Pauli principle. For what concerns the second condition (on the columns), it simply tells us that a fixed state (or wave function) cannot have different occupation numbers[162] and this, like seen, must be true for construction. Therefore, also the second condition is a pure formal requirement and it has nothing to do with the Pauli principle. Thence, by using the formalism of the occupation numbers and by supposing *ab absurdo* the existence of a totally anti-symmetric state (or wave function) by which is possible to define a Slater determinant, we demonstrated the conditions for which such a determinant is null have no relationship with the Pauli principle and so the assumption according to which this principle is in relation with the anti-symmetry of the wave function must be thought wrong.

It could be objected, as previously, this is true with the formalism of the occupation numbers, while it is not so with the formalism commonly used in literature. Although we have shown such a formalism is not correct and that however also for it totally symmetric or totally anti-symmetric states do not exist, we now want to prove that, if also a Slater determinant can be defined, it is false that the annihilation conditions thereby arising are in agreement with the Pauli principle. For this aim, we consider

$$\psi_A(1,\ldots,N) = \frac{1}{\sqrt{N!}}\sum_P (-1)^P P\psi_A(1,\ldots,N), \tag{714}$$

which, under the factorization hypothesis

$$\psi_A(1,\ldots,N) \equiv \phi_1(1)\phi_2(2)\cdots\phi_N(N), \tag{715}$$

can be written

[162] Otherwise, if it is preferred, it tell us that different occupation numbers cannot belong to the same state (or wave function).

$$\psi_A(1,\ldots,N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(1) & \phi_1(2) & \ldots & \phi_1(N) \\ \phi_2(1) & \phi_2(2) & \ldots & \phi_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(1) & \phi_N(2) & \ldots & \phi_N(N) \end{vmatrix}. \tag{716}$$

Such a determinant is null if

Rows:

$$\{\phi_i(k)\}_{k=1}^N = \{\phi_j(k)\}_{k=1}^N, \quad \forall \text{ fixed } i,j \in \{1,\ldots,N\}. \tag{717}$$

Columns:

$$\{\phi_i(k)\}_{i=1}^N = \{\phi_i(l)\}_{i=1}^N, \quad \forall \text{ fixed } k,l \in \{1,\ldots,N\}. \tag{718}$$

Let us verify, if also in this case, the above conditions infringe the Pauli principle and what are their consequences. The condition on the rows tells us that two different states (or wave functions) cannot describe the same particle and this depends on the factorization condition (715), for which in the state ϕ_1 must be the particle 1, in the state ϕ_2 the particle 2 and so on. Therefore, this condition is not but a mathematical requirement needing to preserve the hypothesis (715), and so it has nothing to do with the Pauli principle. Among other things, the fact that into different states cannot be the same particle absolutely does not represent an alternative definition of the exclusion principle, since this last does not want to establish a correspondence between states and particles, but it simple fixes the number of particles which can occupy an elementary quantum state, according to the indistinguishability principle of the particles. With regards to the condition on the columns, it tells us that different particles cannot be described by the same state (or wave function), and this is nearly another formal condition based on the hypothesis of the factorization (715). Therefore, for the just explained reasons, also the condition on the columns has nothing to do with the Pauli principle. Hence, also using the *classic* formalism and the idea that totally anti-symmetric states exist, we arrive at the conclusion that the Slater determinant and, above all, the conditions for which it is null have no relationship with the Pauli exclusion principle. Moreover, it is better to underline the condition of factorization (715) evidently violates the indistinguishability principle of the quantum particles, because to suppose the particle 1 is in the state ϕ_1, and the particle 2 in the state ϕ_2, *etc.*, means to be in a position to distinguish (and to isolate) the particles one without the other, against the principles of the Quantum Mechanics. Therefore, we demonstrated that, also by using the formalism common in literature, it is not correct to associate the Pauli principle to the anti-symmetry of the wave function (or of a generic quantum state $|A\rangle$), because at most it describes a particle system just subjected to such a principle. More precisely, the exclusion principle only fixes the maximum occupation number of the generic elementary state of a particle system (therefore all the other elementary quantum states must have the same maximum occupation number) and it puts no veto on the symmetry or anti-symmetry of the system wave function.[163] All this invalidates the point 1.

Let us now discuss the point 2. It asserts the Pauli principle must be valid for all the particles with half-integer spin (fermions), namely such particles have elementary quantum state equal to the atomic one. Where is this property proved? The Pauli principle is a heuristic principle, in the sense it was enunciated thanks to the study about the atomic orbitals of the chemical elements in the periodic table. But this only concerns the electrons, which are the particles with spin 1/2. For other particles having half-integer spin, like so for those with integer spin, any experimental observation exists, neither a logical theory,[164] which tells us as the elementary quantum states of particles with half-integer spin (but also integer) different from the electron are made. Therefore, to assert for these particles (*i.e.* for fermions) the Pauli principle is valid, means that the elementary quantum states associated with them can have occupation numbers equal to 0 and 1, and this is completely arbitrary. Hence, without experimental tests or a logical theory proving the contrary, also the point 2 is wrong.

Now it remains to discuss the points 3 and 4, that maybe are the most important, since they represent one of the cornerstones of modern physics, and in particular of the QFT, which consists of thinking the spin of particles inextricably is linked to one of two quantum statistics, that are those of Fermi-Dirac and Bose-Einstein. One of the corollaries of this result sanctions that the particles with integer spin can be only described by field operators following an algebra having commutators as Lie brackets and the particles with half-integer spin can be only described by field operators following an algebra having anti-commutators as Lie brackets (point 4). All these fundamental facts, summarized in the points 3 and 4, are derived from what in technical jargon is called "spin-statistics theorem," which Pauli demonstrated in a famous article of 1940. Thanks to this Pauli's work and to the result as soon as mentioned, the physicists have divided the elementary particles, independently from the interaction which they are subjected to, in two great classes: those with integer spin, linked to the Bose-Einstein statistics (bosons), and those with half-integer spin, linked to Fermi-Dirac statistics (fermions). From what said, it is therefore obvious, in order to correctly analyze the points 3 and 4, we must resume from the attic the original Pauli's article and to pass it at the microscope, line by line. Only thus, in fact, it could be cleared what this article has effectively demonstrated and what not. However, our examination has to be made with great respect, but without complexes, otherwise one risks of being obfuscated by the excellence of the signature, transforming the results of this article in pure dogmatic rules, rather than physical and mathematical unequivocal facts.

Then, it is with this free spirit, without psychological conditioning, we want now to go through the original Pauli's article with the title "The Connection Between Spin and Statistics," published in 1940 by Physical Review [25]. For the complexity of the Pauli's work and for the several arguments covered, our attempt will not be simple, and so we must elaborate an opportune strategy which

[163] Among other things, we demonstrated it is not correct, from the theoretical point of view, to consider totally symmetric or totally anti-symmetric states under permutation of occupation numbers or particles.

[164] The α-Theory, dealt in this work, could resolve such a problem.

concurs to analyze the Pauli's article in the most complete way. For this reason, it has been decided to subdivide our analysis in different points, in each of which, before we write a statement or result cited or demonstrated by Pauli in its article (in cursive), and then we estimate its coherence and validity. Let us begin:[165]

a) *If we now want to determine the spin value of particles which belong to a given field it seems at first that these are given by $l = j + k$.* `Phy.Rev.58 pag.717 column 2 lines 9-11`

In Pauli's opinion, *it seems* reasonable to think that $l = j + k$, concerning the representations of the subgroup $SO(3)$ of $L_+^\uparrow$, is equal to the spin of the particles on whose fields such representations act. In reality, we must remember the spin of the elementary particles just represents an intrinsic property of particles and not a physical characteristic deriving from space-time transformations. Then, we have not to forget that $SU(2)$, whose Lie algebra is isomorphic to that of $SO(3) \subset L_+^\uparrow$, is not the reference group of the spin only, but also the group which the quantum angular momentum is subjected to, which instead can be naturally interpreted in a space-time way. The Pauli mistake is to use the properties of the Lorentz group at will, without considering the physics of the quantum systems. This is as if someone assert flavour and colour of the quarks are the same thing, just because both are adapted to the irreducible representations of the group $SU(3)$. It is worth emphasizing Pauli himself is not very satisfied about his hypothesis, since in the next line he admits: "*Such a definition would, however, not correspond to the physical facts, for there then exists no relation of the spin value with the number of independent plane waves,* [...]." However, what is disconcerting is the fact that Pauli constructs all his article on a hypothesis he himself judges to be not physical.

b) *The number of quantities $U(j, k)$ which enter the theory is, however, in a general coordinate system more complicated, since these quantities together with the vector k_i have to satisfy several conditions.* `Phy.Rev.58 pag.717 column 2 lines 38-42`

What are the conditions that the quantities $U(j, k)$ must satisfy in a general system of coordinates? Pauli does not say it.

c) *Particularly is this the case for the current vector s_i. To the transformation $k_i \to -k_i$ belongs for arbitrary wave packets the transformation $x_i \to -x_i$ and it is remarkable that from the invariance of Eq.(1) against the proper Lorentz group alone there follows an invariance property for the change of sign of all the coordinates. In particular, the indefinite character of the current density and the total charge for even spin follows, since to every solution of the field equations belongs another solution for which the components of s_k change their sign. The definition of a definite particle density for*

[165]The quantities defined by Pauli will remain with their original numeration.

even spin which transforms like 4-component of a vector is therefore impossible.
`Phy.Rev.58 pag.719 column 1 lines 11-25`

Pauli associates physical quantities, such as the current vector s_i, the current density and the total charge, to the tensors S and/or T, without a detailed proof. He seems to assume the transformation (2) are right for s_i too, from that follows the indefinite character of the current density and the total charge for *even spin*. This fact, not supported by an accurate demonstration, seems to be debatable. Lastly, Pauli speaks about field equations (for particles having *even spin*). What are these equations? By any chance, are they the equations (1)? Therefore, shall we consider them complete and correct?

d) *In case of half-integral spin, therefore, a positive definite energy density, as well as a positive definite total energy, is impossible. The latter follows from the fact, that, under the above substitution,*[166] *the energy density in every space-time point changes its sign as a result of which the total energy changes also its sign.* `Phy.Rev.58 pag.720 column 1 lines 5-11`

What are in Pauli's opinion the general expressions of the energy density and the total energy for particles having *half-integral spin*?

e) *This method is especially convenient in the absence of interaction, where all fields $U^{(r)}$ satisfy the wave equation of the second order*

$$\Box U^{(r)} - k^2 U^{(r)} = 0,$$

where

$$\Box \equiv \sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2} = \Delta - \frac{\partial^2}{\partial x_0^2}$$

and k is the rest mass of particles in units $\hbar/c$.
`Phy.Rev.58 pag.720 column 2 lines 16-21`

Pauli chooses to use the Klein-Gordon equation for the fields $U^{(r)}$. But, since he wants to do a reasoning for fields describing particles with integer and half-integer spin, he had to use the Dirac equation too, and this can be due to the fact that Pauli – as he asserts in a footnote of his article – does not think the Dirac equation to be definitive, above all for what concerns its differential order, that Pauli would have liked second rather than first. However, this does not justify his choice of using the Klein-Gordon equation for describing particle fields with integer and half-integer spin and

[166] $T \to -T$, $S \to S$.

it makes misleading and woolly the whole demonstration.

f) *This is also true for brackets with the + sign, since otherwise it would follow that gauge invariant quantities, which are constructed bilinearly from the* $U^{(r)}$*, as for example the charge density, are noncommutable in two points with a space-like distance.* `Phy.Rev.58 pag.721 column 2 lines 14-19`

The issue is much more complex and deeper than the one described by Pauli. The Microcausality condition, which he is referring to, is valid for the bosonic fields but not for the fermionic ones, which instead satisfy anti-commutation rules.[167] This means, within the modern QFT, the fermionic fields are thought non-observable, although naturally the physical quantities, constructed through them, continue to be observable, like for example the energy density, the charge density or the cross sections. The problem is that Pauli seems to use the equation

$$\Box U^{(r)} - k^2 U^{(r)} = 0 \tag{719}$$

for both fields, with integer and half-integer spin, by hoping to generalize his discussion to theories which, on the contrary, he does not know. Indeed, it is false to claim the above equation is valid for fields with arbitrary spin, like so it is false that its Green's function, which totally depends on the associated differential equation, can be adapted to particles describing fields with integer and half-integer spin. The Pauli's mistake is, therefore, to expose physical and mathematical argumentations with respect to unknown theories. Hence, his discussion seems only hypothetical, and so misleading.

g) *We consider especially the bracket expression of a field component* $U^{(r)}$ *with its own complex conjugate*

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')].$$

We distinguish now the two cases of half-integral and integral spin. In the former case this expression transforms according to (8) under Lorentz transformations as a tensor of odd rank. In the second case, however, it transforms as a tensor of even rank. Hence we have for half-integral spin

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \textit{odd number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'') \tag{19a}$$

and similarly for integral spin

[167] Let us still use the terminology associating the commutators to the bosons and the anti-commutators to fermions, in order to not anticipate the result of this appendix.

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \textit{even number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0''). \quad (19b)$$

This must be understood in such a way that on the right-hand side there may occur a complicated sum of expressions of the type indicated. `Phy.Rev.58 pag.722 column 1 lines 1-14`

This is a very delicate point to deal with, because it is one of the pillars of the Pauli's demonstration. It founds itself on the properties of transformation under the Lorentz group $L_+^\uparrow$ of the commutator

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')]. \tag{720}$$

But one has to wonder: can a commutator (or anti-commutator) be transformed under the Lorentz group? The question is not banal, since it seems reasonable the commutator (or anti-commutator) of two physical quantities must be invariant under Lorentz transformations, otherwise if two quantities commute (or anti-commute) in an inertial frame of reference, they could not commute (or anti-commute) in another inertial frame of reference, at odds with the equivalence principle. After all, this would have to be also a general principle of the Quantum Mechanics, since Wigner demonstrated the states of two systems O and O' are always connected by unitary transformations, which, within an arbitrary phase factor, leave the observational results unchanged. What should happen if the commutator of two physical quantities is not invariant under Lorentz? As an example, the Microcausality condition

$$[O(x), O'(y)] = 0, \text{ where } (x-y)^2 < 0 \tag{721}$$

would ruin, because an inertial frame of reference in which the above commutator is not null could be found. This should involve that two observable quantities in an inertial frame of reference would not be observable quantities in another inertial frame of reference and this is absurd. After all, the same SR teaches us if an interval is space-like, time-like or light-like, it does not change its nature, in the sense that it is never possible to find a Lorentz transformation transforming an interval of a type into an interval of another type. This should confirm the commutator of two physical quantities is not transformed under $L_+^\uparrow$. From the mathematical point of view, this could be tried based on what is made in Classic Mechanics as far as the Poisson brackets is concerned. It is known, in fact, the Poisson bracket $\{\ ,\ \}$ of two physical quantities is invariant under canonical transformations, differential and invertible, such that the Jacobian matrix A of the change of basis (about the two chosen frames of reference) is symplectic, *i.e.* such that

$$A^T \tilde{I} A = \tilde{I}, \text{ with } A \in \mathcal{M}(2l, \mathbb{R}); \quad \tilde{I} = \begin{pmatrix} \mathbb{O}_l & \mathbb{1}_l \\ -\mathbb{1}_l & \mathbb{O}_l \end{pmatrix}. \tag{722}$$

Therefore, by using the correspondence

$$\{\ ,\ \} \longrightarrow \frac{1}{i\hbar}[\ ,\ ]_{\pm} \tag{723}$$

and constructing a symplectic group on the field of the complex numbers, it could be taken advantage of the fact that such a group, which usually it is indicated with $Sp(2l,\mathbb{C})$, is isomorphic to $SL(2l,\mathbb{C})$ for $l = 1$.[168] Hence, by remembering $SL(2,\mathbb{C})$ is isomorphic to $L_+^\uparrow$, the assertion is obtained, and so the commutators (or anti-commutators) do not change under Lorentz group transformations. Aside from this observation, that on its own could invalidate the (19a) and (19b), another unclear thing of these relations is why (and Pauli does not demonstrate it) the even and odd number of derivatives of the function $D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'')$ depend on the rank of the (tensorial) transformation operated on them. Therefore, the weakness of such a point is that the demonstration of (19a) and (19b) is not given, although these relations are necessary, in order to demonstrate the relationship between spin and statistics. All these facts take only to ulteriorly weakening the Pauli's demonstration.

h) *We have therefore the result for integral spin*

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = 0. \tag{21}$$

`Phy.Rev.58 pag.722 column 1 lines 25-26`

If the above relation (21) is true, we would have

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = -[U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = [U^{*(r)}(\mathbf{x}', x_0'), U^{(r)}(\mathbf{x}'', x_0'')], \tag{724}$$

from which follows [169]

$$U^{(r)} = U^{*(r)} \quad \forall x \in M, \tag{725}$$

and so the particle fields with integer spin must be always real fields. The theory of the complex Klein-Gordon field, fundamental for the characterization of the electric charge, demonstrates this

[168] In such a case, the symplectic condition (722) is verified *iff* the determinant of A is equal to 1.
[169] M is the Minkowski space.

is absurd.

i) *So far we have not distinguished between the two cases of Bose statistics and the exclusion principle. In the former case, one has the ordinary bracket with the − sign, in the latter case, according to Jordan and Wigner, the bracket*

$$[A, B]_+ = AB + BA$$

with the + sign. `Phy.Rev.58 pag.722 column 1 lines 27-30 column 2 lines 1-2`

Pauli uses the conception for which the commutator is associable to the particle fields subjected to Bose-Einstein statistics, and the anti-commutator to the exclusion principle, *i.e.* to the particle fields subjected to Fermi-Dirac statistics. Nevertheless, no formal demonstration, proving it, exists and indeed, by remembering that the exclusion principle – like seen in the previous pages – has nothing to do with the anti-symmetry of the wave function (or with the field function associated to particles) and, hence, with the anti-commutators, it seems such a conception be completely groundless.

j) *On the other hand, it is formally possible to quantize the theory for half-integral spins according to Einstein-Bose-statistics, but according to the general result of the preceding section the energy of the system would not be positive. Since for physical reasons it is necessary to postulate this, we must apply the exclusion principle in connection with Dirac's hole theory.*
`Phy.Rev.58 pag.722 column 2 lines 12-19`

This statement is based on some misconceptions. The first one concerns the relationship between commutator and Bose-Einstein statistics, exclusion principle (anti-commutator) and Fermi-Dirac statistics. In the previous pages, we demonstrated such an association does not exist. Secondly, Pauli introduces as general result the fact that, if a particle system with half-integer spin is quantized through (canonical) commutation relations, one has an energy density not defined positive. This is true for the Dirac theory. But such a theory is developed for particles with spin 1/2 and not for all the particles having half-integer spin. Therefore, in the absence of a general theory for particles with half-integer spin, this affirmation seems faulty. From that fact, it appears the Pauli's attempt of selling hypotheses like concrete facts.

k) *In conclusion we wish to state, that according to our opinion the connection between spin and statistics is one of the most important applications of the special relativity theory.*
`Phy.Rev.58 pag.722 column 2 lines 30-33`

Pauli makes this observation since he used the Lorentz group $L_+^\uparrow$, even if it has been contracted to $SU(2)$, by using the "dangerous correlation"

$$\boxed{j = \text{spin of the fields}}$$

Apart from that, the Pauli's work does not utilize directly the special Relativity and, therefore, this last statement seems having the only goal to give credit to his *demonstration*.

The text analysis of the Pauli's work, which we have subdivided in the previous most remarkable eleven points, shows us, in an absolutely clear way, that the Pauli's demonstration about the alleged relationship between quantum statistics (Fermi-Dirac and Bose-Einstein) and the spin of elementary particles leaks as a sieve. This should be enough to convince any researcher with a minimum of intellectual honesty about the logical fallacy of the so-called "spin-statistics theorem." However, always a lot of supporters of the past ideas exist, even if groundless, and so we now inflict the *final blow* to the Pauli's demonstration, thus to dispel any doubt. This will not be made by criticizing the *modus operandi* of Pauli, like in previous eleven points, but by supposing correct his result and making to see this takes to a contradiction (reduction *ad absurdum*). We start, thence, from the relations

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \text{odd number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'') \qquad (19a)$$

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \text{even number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0''), \qquad (19b)$$

where the $(19a)$ is valid for fields with half-integer spin, while the $(19b)$ is valid for fields with integer spin. By supposing (to absurdity) the $(19a)$ and $(19b)$ are true, we consider as Pauli the quantity

$$X = [U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')]. \qquad (20)$$

According to the Pauli's opinion, since the function D, given by

$$D(\mathbf{x}, x_0) = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin k_0 x_0}{k_0}, \qquad (11)$$

is even in the *space coordinates* and odd in the *time coordinate*, it follows

$X =$*even number of space-like times odd numbers of time-like derivatives of* $D(\boldsymbol{x}' - \boldsymbol{x}'', x_0' - x_0'')$.

This Pauli's conclusion, he however does not prove, seems to depend on the (19*a*) and (19*b*) (by supposing these relations are true, of course), *i.e.* by the fact $[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')]$ is equal to an even or odd number (and this, for Pauli, depends on the spin) of derivatives of the function $D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')$, independently from the fact D is even in the *space coordinates* and odd in the *time coordinate*. Therefore, also X, which is defined through the sum of two commutators, would have to be expressed through even derivatives of the functions $D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')$ and $D(\mathbf{x}''-\mathbf{x}', x_0''-x_0')$ or through odd derivatives of the same functions and this for whatever its symmetry relation (unless it is not explicitly demonstrated, but Pauli does not make it). The shocking thing is that, the conclusion

$X =$*even number of space-like times odd numbers of time-like derivatives of* $D(\boldsymbol{x}' - \boldsymbol{x}'', x_0' - x_0'')$,

seems, on the contrary, to derive by the fact D is an even function in the *space coordinates* and odd in the *time coordinate*, as if the property of a function of being even or odd could be transformed in an even or odd number of derivatives under appropriate operations (which Pauli does not demonstrate). If this were correct the same (19*a*) and (19*b*) would be wrong and should be approximately replaced with

A) <u>half-integer spin</u>:

$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] =$ odd number of time derivatives of the function $D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'')$.

B) <u>integer spin</u>:

$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')]$=even number of space derivatives of the function $D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'')$.

Based on A) and B), the quantity X is always well-defined, making obviously attention to specify if we are considering fields with integer or half-integer spin. In the first case, there are only an even number of space derivatives of the function D, while, in the second case, there are only an odd number of time derivatives of the function D. Therefore, it is not true $X = 0$ for integer spin, and so, by using the same Pauli's arguments, we proved his conclusion is wrong. But there is a still more serious consequence in the expressions (19*a*) and (19*b*) postulated by Pauli. In fact, by supposing (*ad absurdum*) the (19*a*) and (19*b*) are true, one must admit existing a law $\mathcal{F}_s$, dependent by the spin s, such that to the commutator a certain number of derivatives of the function $D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')$ correspond.[170] Therefore, in general terms, it can be written

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right], \tag{726}$$

and, naturally, we also have

$$[U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = \mathcal{F}_s\left[D(\mathbf{x}''-\mathbf{x}', x_0''-x_0')\right]. \tag{727}$$

By the definition of X, we obtain

$$X \equiv [U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = \mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right] + \mathcal{F}_s\left[D(\mathbf{x}''-\mathbf{x}', x_0''-x_0')\right]. \tag{728}$$

Now, by remembering D is even in the *space coordinates* and odd in the *time coordinate*, we get

$$D(\mathbf{x}''-\mathbf{x}', x_0''-x_0') = D\left[-(\mathbf{x}'-\mathbf{x}''), -(x_0'-x_0'')\right] = -D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0''), \tag{729}$$

from which, it promptly follows

$$\mathcal{F}_s\left[D(\mathbf{x}''-\mathbf{x}', x_0''-x_0')\right] = -\mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right], \tag{730}$$

where we have taken account that $\mathcal{F}_s$, being a number of derivatives, does not absorb the sign of the function on which it acts. From that reason, it is straightforward to see

$$X \equiv [U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = \mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right] + \mathcal{F}_s\left[D(\mathbf{x}''-\mathbf{x}', x_0''-x_0')\right] = \mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right] - \mathcal{F}_s\left[D(\mathbf{x}'-\mathbf{x}'', x_0'-x_0'')\right] = 0. \tag{731}$$

[170]If s is integer, there is an even number of derivatives, while, if s is half-integer, there is an odd number of derivatives.

Hence, if the Pauli's considerations are true, the amount X, which he defines, would be identically null for any spin value s and not for the integer spins only.[171] Therefore, the final Pauli's thesis have to be considered wrong. This involves that also the consequence deriving from such a mistake – practically consisting in associating particles having integer spin to the Bose-Einstein statistics (with commutation rules) and particles having half-integer spin to the Fermi-Dirac statistics (with anti-commutation rules) – comes to fall, thus like all the Pauli's deductive castle.

In conclusion, we can assert the Pauli's demonstration about the relationship between spin and statistics is groundless. This requires an upheaval in the actual theoretical background, which thinks the spin-statistics theorem and its consequences be untouchable and unassailable bedrocks.

Before asking what can involve the sunset of the spin-statistics theorem and looking for possible alternatives, we want to inquire on the reasons which pushed the scientific community to accept, without reserve, the Pauli's demonstration of 1940. The first thing must be asked is: was Pauli aware about the inadequacy of its demonstration? As we have seen, Pauli constructed a muddled and unclear article, that through the distorted use of complex arguments wants to prove the relationship between the quantum statistics and spin of particles. And, without a doubt, he succeeds in its attempt, since everyone who reads for the first time his article is distracted from the lots of technical he uses, by losing the cognition on the legitimacy or less of his task. Without wanting to encourage libelous hypothesis regarding Pauli, it can be thought his approach is all grounded in the positivist desire of the physics of those years, which collided with the uncertainty and distrust characterizing the Second World War beginning. However, it has to be said Pauli simply gave a theoretical justification to what by time was thought to be a undoubted fact, and *i.e.* that the quantum statistics were connected to the spin of elementary particles and to the commutators and anti-commutators of the fields describing such particles. The same Jordan, Wigner and Fierz [26] wrote articles oriented towards this direction, also based on the wrong relationship between exclusion principle and anti-commutators. Therefore, Pauli, with his article of 1940, made nothing but putting the icing on this theoretical cake. However, it has not to charge to these actors too many blames, because their aim was simply of finding the laws for the elusive elementary particles, which represented something very similar to *magical* than physical objects. The heavy responsibility belongs to those who came later, that, instead of accepting these fundamental principles dogmatically, would have had to inquire about their formal correctness. After all, to have also called "spin-statistics theorem" the Pauli's result is completely misleading, since many times the author just specifies lots of his assumptions were only hypotheses and nothing more. Therefore, at most, the scientific community should have called the Pauli's result "spin-statistics postulate" and, maybe, the things would be changed.

Have we, after all, to waive the old conception connecting quantum statistics with the spin of elementary particles? As long as it will never be a correct and exhaustive demonstration about this connection, of course it cannot be thought right. What should happen in the particle physics if the relationship between spin and statistics would change? The more obvious thing is that the association between the two great particle classes (*i.e.* the fermions and bosons) and quantum statistics of Fermi-Dirac and Bose-Einstein would be lost. This would require a necessary review of some underlying concepts, but neither the fundamental results achieved from the QFT nor its

[171] However, in such a case, we have seen in the point h) this leads to the absurd that the particle fields, having integer spin, must be real.

mathematical formalism would change, at the end.

However, beyond further studies which on this argument have to be made, it is well to underline physical science is in continuous evolution and the end of old concepts can open the way to new and more exciting visions, which not are able to include the past ideas only, but, above all, to make forecasts seeming impossible before. It could be objected the Bose-Einstein condensate, regarding boson systems at temperatures very close to absolute zero, are a kind of proof of the "spin-statistics theorem" and association between fermions and bosons to the two quantum statistics. But the Bose-Einstein statistics could concern the gauge bosons only, as the photons just are. Moreover, recent studies have put in evidence also fermionic systems can be subject to condensation phenomena [39, 40]. This is not other but a hint on the possible blunders made in the statistical discussion of elementary particles. For example, it is not said the Fermi-Dirac and Bose-Einstein statistics are the only ones. If the spin-statistics theorem is not valid, it is possible to find a relationship between occupation numbers and quantum statistics. Such a relation is well-known and, in fact, the Fermi-Dirac and Bose-Einstein statistics can be both obtained by the thermodynamic potential

$$\Omega_\ell = -\mathsf{k}T \ln \sum_{\alpha_\ell} \left(e^{\frac{\mu - E_\ell}{\mathsf{k}T}} \right)^{\alpha_\ell}, \tag{732}$$

putting once $\alpha_\ell = 0, 1$ and another time $\alpha_\ell = 0, 1, 2, \ldots, \infty$, for any $\ell \in \mathbb{N}$. This does not only show the importance of the occupation numbers, but also as they are effectively connected to quantum statistics. Therefore, it could be defined, instead of the "spin-statistics theorem," the "α_n-statistics postulate," where the α_n are the occupation numbers of the quantum states concerning the systems we want to study. Hence, if a particle theory, able to characterize – for any spin – the occupation number of the fundamental quantum state, were found, a relationship between spin and statistics could be constructed again, still based on the concept of "occupation number." This would lead to multi-statistics, which could include also the Fermi-Dirac and Bose-Einstein ones, delivering unexpected experimental predictions.[172]

Lastly, we shortly recall the results obtained in this appendix:

- Superiority of the occupation numbers formalism compared with the current one, also thanks to the fact it intrinsically satisfies the "indistinguishability principle" of quantum particles.

- Non-existence of totally symmetric and totally anti-symmetric wave functions (or quantum states), by using the formalism of the occupation numbers or the current one.

- Absence of a solid and demonstrable relationship between the exclusion principle and anti-symmetry of the wave functions (or the quantum states) under particle permutations. Uselessness of the Slater determinant.

- Wrong extension of the exclusion principle to other particles with half-integer spin different from the electron.

- Inadequacy of the "spin-statistics theorem" constructed by Pauli in 1940.

[172] Look at the α-Theory dealt in this work. In particular, in the chapter 5.

- Investigation on the relationship between occupation numbers and statistics, which a theory for arbitrary spin should be able to make clear. Multi-statistics and new connection with the spin, just through the occupation numbers.

I hope the facts dealt in this appendix not only will support the theory studied in this work, but they should be an interesting matter for reflection.

Acknowledgements

When an exalted work, covering several years of the own life, comes to the end, one turns back and it is difficult to believe that it is the fruit of his own effort. If ten years ago someone had said me that I would have written the α-Theory, I would have never thought true it. Instead, there is *our* just born theory. It is beautiful. For me it is a real dream, a miracle. Really this result did not come by me only, but above all by those tens of thousands of researchers in the world bravely work and worked for obtaining a whole theory in physics. Many of their publications are precursors of the α-Theory and so I cited them in the bibliography. It is as if the α-Theory was in the air, needed to be listen, needed to be loved and written down on paper.

We live difficult times, especially in my country. I should have liked to work out an exhaustive Big-Break model, thus to generalize the α-Theory, but unfortunately it has not been possible. The social Italian background not allowed it. In these long years, I fought against corruption and mafia, that in my nation have seized all the bodies which should defend and ensure the equality of the citizens. In great loneliness, I struggled for the natural environment and human rights. I paid my personal fight against *state-corporate crime* suffering a lot of intimidations, so thinking to be killed at any moment. In this condition I wrote our theory, standing my ground and trying to not despond. I would like the people, and in particular the youths, understand from my little sacrifice that we must always battle for our dreams, with the knowledge that the same life can be left. We cannot combat without getting our hands dirty, without any hardship. I called this work "α-Theory," because I think it is the base-model of the physics, which starts from Big-Bang and explains as we arrived to our universe. The beginning theory, in other words. I should like my dedication for writing it be able to drive the young generations to give a new beginning for our sick society too. I should like the α-Theory be able to inspire an "α-generation," that understand the importance of the alliance, fraternity and mutual respect. This does not want to be rhetoric, but it is the only way we have for escaping all together from the abyss going forward.

And now let me give thanks those *good people* that in these years have supported me, making to feel me a human being and rendering my solitude less heavy. Nevertheless, the first person I want to thank is not a my close friend, even if without him you could not read the α-Theory here. His name is Philip Gibbs and he is the viXra founder. Due to his science passion he created a true free website, where everybody can record his own work. Then, it is up to professional scientists to be able of distinguishing what is good and what is not. This is the minimum demanding their qualification. I am still a student and so without an academic powerful affiliation. For that reason, it would have been impossible to show my work to the world without the real danger of plagiarism by some already well-known university professor, to whom necessarily I had to address myself for an endorsement. Science must be rigorous, but free. All the people have to be able to cooperate for the human knowledge evolution. Without this condition the science is at risk of becoming a power game, arid and harmful.

The other person I think he deserves a big thank-you is Prof. G. M. Briganti, that I consider as a father. In these my difficult years he always supported me, prompting to believe in my ideas. He gave as a present some of the books on which I studied for realizing the α-Theory and that I was not able to buy otherwise. I will never forget our talks about life and faith, which inspired my thought and my job. Another person I feel to thank is my old university friend Dr. Davide Guadagnuolo. I am grateful to him for the relaxing walks made in these years and for introducing me to the markup language LaTeX. Furthermore, I want to thank Dr. Pasquale Mormile of the CNR in Pozzuoli for his kindness and the time reserved to me. Then, I am greatly indebted to the very gentle Eng. Luigi D'Oriano for the realization in AutoCAD of my hand-pictures. A further thank must be given to Nicolino Rossi, tenured professor of Esperanto in Naples. I am grateful to him for according me hospitality, but above all for the review of my English version of the α-Theory. In any case, I want to emphasize of being the only liable for any linguistic mistake escaped to the examination and for every other inaccuracy present in my work. That is why I will be grateful to all the readers able to bring to my attention the possible misprints or other errors survived in the text.

The last person I would like to finish of thanking is really the first in my heart. She is my fiancée Margherita Rossi (Maggie), who in these years combated, cried and suffered with me. She relieved me during the most difficult moments and without ever being in fear she always helped me to lift myself up. Pampering me like a child, she took care of me as no other never could, so giving me the sensation of not feeling a castaway. Without her love the α-Theory would not be. This theory is the daughter we desire and that we not still have.

I am honored to dedicate this work to the memory of Peppino Impastato[173] and Enzo Tortora,[174] innocent victims of the Italian mafia.

[173] *Eroe eccelso, vittima della mafia e dello stato italiano che troppo spesso coincidono, esempio di cittadino che ha difeso il valore della libertà, dell'equità e della partecipazione sociale solo in nome della verità e dell'encomiabile senso civico. Tu adesso sei luce e resterai per sempre luce, i tuoi assassini sono ombre e resteranno per sempre ombre.*

[174] *Vittima di quel potere giudiziario subdolo e tribale di cui non abbiamo bisogno, fiaccato da un vergognoso e violento linciaggio mediatico ed infine ucciso dal male incurabile della falsità e dell'ipocrisia, mentre i suoi carnefici, glorificati ed omaggiati, non hanno mai pagato per nulla. Tuo è il paradiso, loro è l'inferno.*

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