A psychophysical theory of Shannon entropy.

Taiki Takahashi[1]

[1] Department of Behavioral Science, Hokkaido University

Corresponding Author: Taiki Takahashi

Email: taikitakahashi@gmail.com

Department of Behavioral Science, Hokkaido University

N.10, W.7, Kita-ku, Sapporo, 060-0810, Japan

TEL: +81-11-706-3057 FAX: +81-11-706-3066

Acknowledgements: The research reported in this paper was supported by a grant from the Grant- in-Aid for Scientific Research ("global center of excellence" grant) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Abstract

Connections between information theory and decision under uncertainty have been attracting attention in econophysics, neuroeconomics and quantum decision theory. This paper proposes a psychophysical theory of Shannon entropy based on a mathematical equivalence of delay and uncertainty in decision-making, and psychophysics of the perception of waiting time in probabilistic choices. Furthermore, it is shown that the well-known Shannon entropy is a special case of the general psychophysical entropy. Future directions in the application of the present theory to studies in econophysics and neuroeconomics are discussed.

Keywords: Entropy, Uncertainty, Neuroeconomics; Information theory; Risk; Psychophysics

1. **Introduction:**

Decision under risk and probabilistic uncertainty (probabilistic choice) has been a major topic in microeconomics (e.g., the expected utility theory, von Neumann and Morgenstern, 1947), behavioral neuroeconomics (e.g. the prospect theory, Kahneman and Tversky, 1977), and econophysics (Anteneodo et al., 2002). Studies in behavioral and neuro- economics have revealed that humans and non-human animals discount the value of probabilistic rewards as the receipt becomes more uncertain ("probability discounting", Rachlin and Raineri, 1991). In order to develop decision theory on probabilistic choice, recent efforts in econophysics have started to combine probabilistic choice processes with intertemporal choice process (Takahashi, 2007). Delay discounting in intertemporal choice refers to the devaluation of a delayed reward compared to the value of a sooner reward. In this line of investigations into the unification of delay and probability discounting, recent studies in behavioral psychology and neuroeconomics have demonstrated a mathematical equivalence of delay and probabilistic uncertainty (risk) in reward-seeking behavior (Takahashi, 2007). Furthermore, Takahashi (2005, 2006) proposed that perception of waiting time in intertemporal choice is logarithmic in physical time, which has recently been confirmed experimentally (Takahashi et al., 2008; Zauberman et al., 2009).

I introduce, in this paper, a novel theory for Shannon information entropy by utilizing psychophysics of waiting time based on the equivalence of delay and uncertainty (Rachlin and Raineri, 1991; Takahashi, 2007). Notably, I derive generalized Shannon entropy based on the psychophysical theory and demonstrate that conventional Shannon entropy is a special case for it.

This paper is organized in the following manner. In Section 2, I briefly introduce the mathematical equivalence of delay and probabilistic uncertainty, and psychophysics of logarithmic waiting time perception. In Section 3, I explain the psychophysical theory of information entropy. In Section 4, some conclusions from this study and future study directions by utilizing the present psychophysical theory of information entropy in neuroeconomics and neuropsychiatry of risky behavior are discussed.

2. **A mathematical equivalence of delay and uncertainty in decision making**

As noted above, it has been demonstrated that delay until receipt of gains in intertemporal choice and uncertainty of winning of gains in probabilistic choice may be equivalent. In unifying decision over time and under uncertainty, Rachlin et al (1991) hypothesized that a decrease in a probability of winning an uncertain reward corresponds to an increase in a delay until winning the reward. Specifically, an average waiting time until winning an uncertain reward is proportional to $(1/p)$-1 ("odds against"), where p is a probability of winning the uncertain reward. Therefore, according to Rachlin and colleagues' hypothesis, decision-making models in intertemporal choice (delay discounting) can straightforwardly be extended into probabilistic choice(s), after replacing a parameter of delay in intertemporal choice models with the odds against parameter. Hence, it appears to be a promising direction to establish information theory on the psychophysical equivalence of delay and probability (in terms of the odds-against=$\frac{1-p}{p}$) in decision under probabilistic uncertainty (risk).

With respect to psychophysics of time-perception, Takahashi (2005, 2006) proposed that "psychological time during intertemporal choice" (=τ) follows the Weber-Fechner law:

$$\tau(D)=a\ln(1+bD), \qquad (1)$$

where $\tau \in [0,\infty]$ is perceived/anticipated psychological time of physical delay D $\in [0,\infty]$, and a and b $\in [0,\infty]$ are free parameters (note that b is a nondimensionalization coefficient of physical time D). The theoretical proposal has been confirmed experimentally (Takahashi et al., 2008; Zauberman et al., 2009). Together, it may be a promising direction how information theory in decision theory and neuroeconomics could be formalized in terms of psychophysics of waiting time in probabilistic choices when D is proportional to the odds against $\frac{1-p}{p}$.

3. **Psychophysics of information entropy**

Let us suppose that the (average) inter-trial time in probabilistic choices is t. Then, the (average) waiting time in probabilistic choices for a single outcome is

$$D=\left(\frac{1-p}{p}\right)t. \qquad (2)$$

Therefore, psychological waiting time is (from equation 1),

$$\tau(D)=a\ln\left(1+bt\left(\frac{1-p}{p}\right)\right). \qquad (3)$$

Let us now consider the case with multiple N outcomes (x_i, p_i) where x_i, p_i are the magnitude of the i^{th} outcome and the probability of obtaining the outcome x_i (i is an integer $\in [1, N]$). Let us denote

$$\tau_i(D) = a_i \ln\left(1 + b_i t_i\left(\frac{1-p_i}{p_i}\right)\right), \qquad (4)$$

where the suffix i indicates each outcome i.

In this case, the average of anticipatory waiting time for N outcomes is

$$H_p = \sum_{i=1}^{N} p_i \tau_i(D). \qquad (5)$$

By inserting equation 4 into 5, we obtain

$$H_p = p_i \sum_{i=1}^{N} a_i \ln[1 + b_i t_i (\frac{1-p_i}{p_i})]. \qquad (6)$$

In the special case for equation 6 in which $a_i = b_i t_i = 1$, we obtain

$$H_s = -p_i \sum_{i=1}^{N} \ln p_i ,$$

which is (conventional) Shannon information entropy (Shannon, 1948). Therefore, it is demonstrated that the conventional Shannon entropy is a special case for the generalized psychophysical entropy (equation 6).

4. **Conclusions and implications for neuroeconomics and econophysics**

Decision under probabilistic uncertainty (risk) has been attracting strong attention in neuroeconomics (Paulus et al., 2006; Weber and Huettel, 2008; Hsu et al., 2009; Christopoulos et al., 2009; Gianotti et al., 2009; Levy et al., 2010). However, neuroeconomic studies on risk have not paid enough attention to information theory, while mainstream neuroscience has been strongly connected to information theory (e.g., Nemenman, 2008). A recent econophysical study introduced a model based on Tsallis' entropy (Takahashi, 2009). Therefore, future studies in econophysics and neuroeconomics should more extensively utilize information theory by adopting the present psychophysical theory of information entropy, because psychophysical theory has also been widely exploited in theoretical neuroscience (Dayan and Abbott, 2001). These lines of studies may help understand neuroeconomic and neurochemical bases of risky decision making by neuropsychiatric patients (Paulus, 2007) such as bipolar disorder patients (Chandler et al., 2009).

References

Anteneodo C, Tsallis C, Martinez AS. Risk aversion in economic transactions. 2002 ***Europhysics Lett.*** 59 (5): 635-641.

Chandler RA, Wakeley J, Goodwin GM, Rogers RD. (2009) Altered risk-aversion and risk-seeking behavior in bipolar disorder. ***Biological Psychiatry.*** 66(9):840-846.

Christopoulos GI, Tobler PN, Bossaerts P, Dolan RJ, Schultz W. (2009) Neural correlates of value, risk, and risk aversion contributing to decision making under risk. **Journal of Neuroscience.** 29(40):12574-12583.

Dayan P and Abbott LF (2001) Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems, MIT press

Gianotti LR, Knoch D, Faber PL, Lehmann D, Pascual-Marqui RD, Diezi C, Schoch C, Eisenegger C, Fehr E. (2009) Tonic activity level in the right prefrontal cortex predicts individuals' risk taking. ***Psychological Science*** 33-38.

Hsu M, Krajbich I, Zhao C, Camerer CF.(2009) Neural response to reward anticipation under risk is nonlinear in probabilities. ***Journal of Neuroscience***. 29(7):2231-2237.

Kahneman D, Tversky A: Prospect Theory: An Analysis of Decision under Risk. ***Econometrica*** 1979, 47:263-292.

Levy I, Snell J, Nelson AJ, Rustichini A, Glimcher PW (2010) Neural representation of subjective value under risk and ambiguity. ***Journal of Neurophysiology***.103(2):1036-1047.

Nemenman I, Lewen GD, Bialek W, de Ruyter van Steveninck RR. (2008) Neural coding of natural stimuli: information at sub-millisecond resolution. ***PLoS Computational Biology***. 4(3):e1000025.

Paulus MP, Frank LR. (2006) Anterior cingulate activity modulates nonlinear decision weight function of uncertain prospects. ***Neuroimage.*** 30(2):668-677

Paulus MP. (2007) Decision-making dysfunctions in psychiatry--altered homeostatic processing? ***Science***. 318(5850):602-606.

Rachlin H, Raineri A, Cross D. Subjective probability and delay. ***J Exp Anal Behav.*** 1991 Mar;55(2):233-244.

Shannon, C.E. (1948), "A Mathematical Theory of Communication", ***Bell System Technical Journal***, 27, pp. 379–423 & 623–656, July & October, 1948.

Takahashi T. (2005) Loss of self-control in intertemporal choice may be attributable to logarithmic time-perception. ***Med Hypotheses.*** 65(4):691-693.

Takahashi T. (2006) Time-estimation error following Weber-Fechner law may explain subadditive time-discounting. ***Med Hypotheses.*** 67(6):1372-1374.

Takahashi, T. (2007). A probabilistic choice model based on Tsallis' statistics, ***Physica A: Statistical Mechanics and its Applications,*** 386: 335-338.

Takahashi T., Oono H., Radford M.H.B. (2008) Psychophysics of time perception and intertemporal choice models. ***Physica A: Statistical Mechanics and its Applications,*** 387 (8-9), pp. 2066-2074.

Takahashi, T. (2009). Tsallis' non-extensive free energy as a subjective value of an uncertain reward, ***Physica A: Statistical Mechanics and its Applications,*** 388: 715-719.

von Neumann J, Morgenstern O: Theory of Games and Economic Behavior. Princeton, NJ Princeton Univ. Press; 1947.

Weber BJ, Huettel SA. (2008) The neural substrates of probabilistic and intertemporal decision making. ***Brain Research*** 1234:104-115.

Zauberman, Gal, B. Kyu Kim, Selin Malkoc, and James R. Bettman (2009). Discounting Time and Time Discounting: Subjective Time Perception and Intertemporal Preferences. ***Journal of Marketing Research***. 46 (4), 543-556.