The cubic equation's relation to the fine structure constant, the mixing angles, and Weinberg angle

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(Dated: July 25, 2014)

A special case of the cubic equation is shown to possess three unusually economical solutions. A minimal case associated with these solutions is then shown to yield a congruous set of numbers that fit the fine structure constant, the sines squared of the quark and lepton mixing angles, as well as the Weinberg angle. Had Renaissance mathematicians probed the cubic equation's solutions more deeply these numbers might have formed a well-known part of algebra from the 16^{th} century.

I. INTRODUCTION

In [1] the author showed how a *slightly asymmetric equation* (that is, an equation whose left- and right-hand sides are very similar) produced the experimental value of the fine structure constant (approximately 1/137.036) [2, 3], which elsewhere was tied to the sines squared of the quark and lepton mixing angles [4, 5]. Here, a special case of the cubic equation is shown to possess three unusually economical solutions, where a minimal case associated with the above solutions produces a set of numbers that fit the fine structure constant, the sines squared of the quark and lepton mixing angles, and the Weinberg angle. This article extends the results of [6].

II. SPECIAL CASE OF THE CUBIC EQUATION

Let

$$Z = \left(\frac{m+x}{n}\right)^3 + (m+x)^2 \quad , \tag{2.1}$$

where x is a variable, Z a positive constant, and m and n are positive integer constants such that

$$m = \frac{n^3}{3} \quad . \tag{2.2}$$

III. FIRST SOLUTION

Then, by defining

$$W = \left(\frac{m}{n}\right)^3 + (m)^2 \tag{3.1}$$

$$u = 2\frac{Z}{W} - 1 \tag{3.2}$$

$$v = \sqrt[3]{u \pm \sqrt{u^2-1}} \tag{3.3}$$

$$w = v + \frac{1}{v} - 1 \tag{3.4}$$

while choosing Z and n so that $Z \geq W$ (and therefore $u^2 - 1 \geq 0$), we have

$$x = m\,(w-1) \quad , \tag{3.5}$$

which can be shown to be the first of the three solutions to be given for Eq. (2.1).

IV. MINIMAL CASE

The smallest positive integers fitting Eq. (2.2)

$$m = 9 \quad \text{and} \quad n = 3$$

are notable, simply because they are minimal.

V. FINE STRUCTURE CONSTANT

For the above minimal case the solution $x = 1$ to Eq. (2.1) requires that

$$\begin{aligned} Z &= \left(\frac{9+1}{3}\right)^3 + (9+1)^2 \\ &= 137.\overline{037} \quad . \end{aligned}$$

Here, the constant Z is close enough — within one thousandth of one per cent — to the reciprocal of the fine structure constant 137.036 [2, 3] to suggest that looking for a connection might turn up interesting mathematics.

In fact it does [7]. Now, if we let

$$m = 9 \qquad n = 3 \qquad Z = 137.036 \quad ,$$

then

$$\begin{aligned} W &= (m/n)^3 + (m)^2 \\ &= (9/3)^3 + 9^2 \\ &= 3^3 + 3^4 \\ &= 108 \end{aligned}$$

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$$\begin{aligned} u &= 2 \times Z/W - 1 \\ &= 2 \times 137.036/108 - 1 \\ &\approx 2 \times 1.268\,851\,851 - 1 \\ &\approx 1.537\,703\,703 \\ v &= \sqrt[3]{u \pm \sqrt{u^2 - 1}} \\ &\approx 1.393\,479\,170\,916^{\pm 1} \\ w &= v + 1/v - 1 \\ &\approx 1.111\,107\,407\,399 \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} x &= m\,(w - 1) \\ &\approx 9 \times (1.111\,107\,407\,399 - 1) \\ &\approx 0.999\,966\,666\,591 \\ &\approx 1 - \frac{1}{29\,999.932\,142\,743\,338} \\ &\approx 1 - \frac{1}{3 \times 10^4} \quad . \end{aligned} \tag{5.1}$$

Substituting the values for m, etc. into Eq. (2.1) gives

$$\begin{aligned} 137.036 = &\left[\left(\frac{9+1}{3}\right) - \frac{1}{3 \times 29\,999.932\ldots}\right]^3 \\ &+ \left[(9+1) - \frac{1}{29\,999.932\ldots}\right]^2 \quad . \end{aligned} \tag{5.2}$$

VI. AN IMPORTANT MINIMUM

Now suppose that Z takes the form

$$Z = \frac{M^3 - M^{-3}}{n^3} + M^2 - M^{-3} \quad , \tag{6.1}$$

where

$$M = m + 1 \quad .$$

Then, a surprisingly accurate and simple approximate solution to Eq. (2.1) follows

$$x \approx 1 - \frac{1}{3 \times M^4} \tag{6.2}$$

(see Theorem 2 in [7]). For $M = 10$ this equation recovers Eq. (5.1), so it should come as no surprise that, for the minimal case introduced in Sec. IV, Eq. (6.1) gives

$$\begin{aligned} Z &= \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} \\ &= 137.036 \quad . \end{aligned}$$

In the previous section for the minimal case we assigned 137.036 to Z “by hand”; now in this section we learn that the value 137.036 would have been assigned to Z *automatically* had we combined the minimal case with Eq. (6.1). It follows that 137.036 is itself an important minimum for the cubic equation, and hence of purely mathematical interest (i.e., apart from its role as an key constant of physics).

VII. QUARK AND LEPTON MIXING ANGLES

Moreover, the following four quantities seen in Eq. (5.2)

$$\begin{array}{cc} \dfrac{10}{3} & \dfrac{1}{3 \times 29\,999.932\ldots} \\ 10 & \dfrac{1}{29\,999.932\ldots} \end{array}$$

which can be reproduced from the sines squared of the quark and lepton angles, are also of purely mathematical interest, independent of *their* role in physics. Specifically, values such as 10/3, 10, etc. can be produced from the quark and lepton mixing angles $L12$, $L13$, $L23$, $Q12$, $Q13$, $Q23$, as follows

$$\left.\begin{aligned} 10/3 &\approx 1/\sin^2 L12 \\ 1/3 \times 29\,999.932 &\approx \sin^2 Q13 \\ 10 &\approx \sin^2 L23 \times 1/\sin^2 Q12 \\ 1/29\,999.932 &\approx \sin^2 Q23 \times \sin^2 L13 \end{aligned}\right\} \tag{7.1}$$

where a mathematical model conforming to the above relations, and predicting mixing angles (in degrees) of 33.210 911, 8.034 394, and 45 for leptons — and 12.920 966, 0.190 986, and 2.367 442 for quarks — is detailed in [5]; there, matrix algebra is used to impose three constraints on mixing, just one of which is independent of the four constraints imposed by Eq. (7.1). It is this additional constraint, along with the further constraint that $L23 = 45°$, that allows the *four* constraints of Eq. (7.1) to produce the *six* mixing angles predicted above, which are all within the limits of experimental error.

VIII. WEINBERG ANGLE

Consider the similarity of Eq. (2.1) to Eq. (3.1): One cannot help but notice that the constants W and Z appear on similar footing mathematically. Hence, one might expect W and Z to appear on similar footing phenomenologically. In this section we show that W/Z can be used to fit economically the ratio of the W- and Z-boson masses.

At the outset, the variable *names* W and Z were chosen in anticipation of their use in a formula reproducing the ratio of the W- and Z-boson masses — that is, M_W/M_Z. That there appears to be some relationship between W/Z and M_W/M_Z is shown by

$$\begin{aligned} \frac{W}{Z} &\approx \left(\frac{M_W}{M_Z}\right)^2 \\ &\approx \cos^2\theta_W \quad , \end{aligned} \tag{8.1}$$

where $W = 108$ and $Z = 137.036$, as before, and θ_W is the simplest of the Weinberg angle’s definitions [8, 9].

Using the precisely-measured mass $M_Z = 91.1876 \pm 0.0021$ GeV, we can calculate the value of M_W with the aid of Eq. (8.1), giving $M_W \approx 80.951$ GeV. Experimentally, $M_W = 80.385 \pm 0.015$ GeV [10]. This calculated M_W differs from experiment by 1 part in 142 and is out of range of experiment; but uncertainty over the best definition of θ_W suggests that a modified, but still valid, Eq. (8.1) might give a better fit [8, 9].

In summary, the above mathematics fits the fine structure constant, the Weinberg angle, and (with the aid of two additional constraints) the six mixing angles $L12$, $L13$, $L23$, $Q12$, $Q13$, $Q23$. This says something for the congruity and efficiency of the above "cubic" framework in modeling fundamental constants. Moreover, as will be shown in the next two sections, the cubic equation responsible for all this possesses two unusually compact alternative solutions.

IX. SECOND SOLUTION

The second solution to Eq. (2.1) requires defining

$$\cos\theta_C = \sqrt{\frac{W}{Z}} \quad , \tag{9.1}$$

where $0 \le \theta_C < \pi/2$, so that

$$\sin\theta_C = \sqrt{1 - \frac{W}{Z}} \quad . \tag{9.2}$$

It can then be shown that Eq. (3.3) can be restated

$$v = \sqrt[3]{\frac{1+\sin\theta_C}{1-\sin\theta_C}} \quad , \tag{9.3}$$

so that, for real-valued v, Eq. (3.4) gives

$$w = \sqrt[3]{\frac{1+\sin\theta_C}{1-\sin\theta_C}} + \sqrt[3]{\frac{1-\sin\theta_C}{1+\sin\theta_C}} - 1 \quad . \tag{9.4}$$

Substituting into Eq. (3.5) gives the second of three solutions to Eq. (2.1)

$$x = m\left(\sqrt[3]{\frac{1+\sin\theta_C}{1-\sin\theta_C}} + \sqrt[3]{\frac{1-\sin\theta_C}{1+\sin\theta_C}}\right) - 2m \quad . \tag{9.5}$$

Now, given the round numbers

$$\left.\begin{array}{l} x = 1 - 1/30\,000 \\ n = 3 \end{array}\right\} \text{Key solution}$$

Eq. (2.1) gives

$$\left.\begin{array}{l} \alpha \approx 1/Z \\ \theta_W \approx \theta_C \end{array}\right\} \text{Key approximations}$$

or, equivalently,

$$\left.\begin{array}{l} \alpha \approx 1/137.036\,000\,002 \\ \theta_W \approx 27.407\,157\,329^\circ \end{array}\right\} \text{Key approximations}$$

so that $\sin^2\theta_C \approx 0.211\,886$. These values are again surprisingly close to those obtained from experiment. Moreover, the above θ_W and θ_C are similar in multiple ways: *They are not only close numerically, but they are also similar in how they are defined.* Each depends on ∼137.036 for its value in the same way — with ∼137.036 implicitly appearing as a value (in the form of electron charge) in the Weinberg angle's definition [8, 9]. In addition, when one works out the dimensional details of θ_W, Eq. (8.1) emerges naturally.

X. THIRD SOLUTION

The third solution to Eq. (2.1) is the most obvious. One merely rewrites Eq. (2.1) in the form of the general cubic equation, and then solves it using the general cubic's classical solution. The classical solution to

$$ax^3 + bx^2 + cx + d = 0 \tag{10.1}$$

is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - r \quad , \tag{10.2}$$

where

$$\left.\begin{array}{l} p = \dfrac{c}{a} - \dfrac{b^2}{3a^3} \\ q = -\dfrac{2b^3}{27a^3} + \dfrac{bc}{3a^2} - \dfrac{d}{a} \\ r = \dfrac{b}{3a} \quad . \end{array}\right\} \tag{10.3}$$

Though complicated, tellingly, these complexities largely vanish when a, b, c, and d derive from Eq. (2.1). So, when Eq. (2.1) is expanded into the general cubic equation we get these coefficients

$$\left.\begin{array}{ll} a = 1 & b = 6m \\ c = 9m^2 & d = m(4m^2 - 3Z) \end{array}\right\} \tag{10.4}$$

in terms of m and Z. Substituting the coefficients of Eq. (10.4) into Eq. (10.3) allows *simplifying* Eq. (10.2) to get

$$x = \sqrt[3]{t + \sqrt{t^2 - m^6}} + \sqrt[3]{t - \sqrt{t^2 - m^6}} - 2m \quad , \tag{10.5}$$

where

$$\begin{aligned} t &= m^3 - \frac{d}{2} \\ &= m\left(1.5Z - m^2\right) \quad . \end{aligned} \tag{10.6}$$

This is the third of the three solutions given for Eq. (2.1), a solution that is notably economical.

XI. SOLUTION ECONOMY

The complexity of the above three solutions can be objectively assessed by comparing them against the exceptionally-simple classical solution to the *depressed cubic*, a natural off-the-shelf benchmark. This will help clarify just how "basic" Eq. (2.1) really is. The depressed cubic is merely the general cubic equation without its quadratic term (i.e., $b = 0$). Below, the solution to the depressed cubic will depend on just two constants: c and d — just as earlier the solution to Eq. (2.1) depended on just two constants: n and Z.

Assume the coefficient of the depressed cubic's leading term to equal one (i.e., $a = 1$); then for the depressed cubic we have

$$x^3 + cx + d = 0 \quad . \tag{11.1}$$

Substituting the depressed cubic's coefficients into Eq. (10.3) gives

$$\left.\begin{aligned} p &= c \\ q &= -d \\ r &= 0 \end{aligned}\right\} \tag{11.2}$$

so that Eq. (10.2) gives this compact solution to Eq. (11.1)

$$x = \sqrt[3]{\frac{-d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} \quad , \tag{11.3}$$

which dates back to Cardano's Renaissance masterwork *Ars Magna*.

This solution is roughly as complicated as the second and third solutions to Eq. (2.1), showing that Eq. (2.1) and its solutions are sufficiently fundamental to be of interest to mathematicians. But perhaps Eq. (2.1) should also engage the interest of physicists, given its apparent connections to physical quantities?

XII. SUMMARY AND CONCLUSION

To help shed light on this, consider that it was the proximity of α to $1/137.\overline{037}$ that earlier suggested that Eq. (2.1) might produce interesting mathematics for the minimal case — as it does in Secs. V and VI. In the same way, θ_W helped the author find the second solution to Eq. (2.1) — the one using θ_C — with θ_W providing the primary clue that such a solution existed. The precise way that α and $(M_W/M_Z)^2$ mapped over to the "cubic" constants 1/137.036 and 108/137.036 appeared to require that θ_W have a correlate among the cubic equation's solutions — as it does in the second solution. But why should this second solution to the cubic equation relate to empirical constants such as α and θ_W?

Yet another way to consider the above solutions is in their historical context: In the 16$^{\text{th}}$ century the Italian mathematicians Scipione del Ferro, Niccolò Tartaglia, and Gerolamo Cardano did pioneering work on the solution to the cubic equation. Had they probed more deeply, a congruous set of numbers fitting

- the fine structure constant
- the Weinberg angle
- $1/\sin^2 L12$
- $\sin^2 Q13$
- $\sin^2 L23 \times 1/\sin^2 Q12$
- $\sin^2 Q23 \times \sin^2 L13$

might have formed an integral part of their work from the outset, with these numbers waiting several centuries till their eventual re-discovery as part of 20$^{\text{th}}$ century physics.

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