Ascoli-type theorems in the cone metric space setting

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Abstract

We give some necessary and sufficient conditions for (global) continuity of the limit of a pointwise convergent net of cone metric space-valued functions, defined on a Hausdorff topological space, in terms of weak filter exhaustiveness. In this framework, we prove some Ascoli-type theorems, considering also possibly asymmetric and extended real-valued distance functions.

MSC: Primary: 26E50, 28A12, 28A33, 28B10, 28B15, 40A35, 46G10, 54A20, 54A40.

Secondary: 06F15, 06F20, 06F30, 22A10, 28A05, 40G15, 46G12, 54H11, 54H12.47H10.

Keywords:

lattice group; cone metric space; (free) filter; global continuity; filter convergence; (weak) filter exhaustiveness; Ascoli theorem; Lipschitz metric

1 Introduction

In the literature there have been several studies about cone metric spaces, namely abstract structures endowed with a distance function taking values in an ordered vector or a normed space, which includes in particular metric semigroups, whose an example is the set of fuzzy numbers, which is not a group. These structures are closely related with order vector spaces endowed with abstract convergences satisfying suitable axioms, but in which in general convergence of subsequences of convergent sequences is not required, like for instance filter convergence. There are several investigations about abstract convergences, distances with values in normed, solid or Hausdorff topological vector spaces and fixed point theory, in particular fixed point existence and uniqueness theorems and error estimates for the considered cone distance and contraction theorems, which have several applications to differential, functional and stochastic equations and reconstruction of signals. In this setting, there are also some other theorems related with fixed point theory, like for example continuity or semicontinuity of suitable functions. In this paper we continue the investigation on the extension of classical theorems to the context of cone metric spaces in connection with filter convergence.

In particular we focus our attention on properties of (global) continuity of the limit of a net of functions, taking values in cone metric spaces, in terms of weak filter exhaustiveness, and relate filter exhaustiveness with filter uniform convergence (on compact subsets). Moreover, we give some Ascoli-type theorems for lattice group-valued functions defined on metric or topological spaces, and consider also asymmetric distances and extended real-valued distances, like Lipschitz metrics, dealing with functions which are not necessarily contractions and extending earlier results proved for real-valued or metric space-valued functions. Asymmetric distance has different applications in several branches of Mathematics and in Physics, for example in gradient flow models, and is related also with the study of several semicontinuity properties of functions. Observe that extended Lipschitz metrics are complete and extended Lipschitz metric convergence is, in general, strictly stronger than uniform convergence on bounded sets. Moreover, extended Lipschitz metrics are equivalent with the supremum metrics when the topological space X in which the involved functions are defined is bounded and uniformly discrete (in its own metric), and they are also equivalent with the Sherbert and Weaver metrics in case X is just bounded. Furthermore note that, since in lattice groups the order convergence is in general not generated by any topology, in our context it is not advisable to deal with concepts like closedness and compactness in terms of topologies. So we formulate the corresponding notions directly in the setting of convergence and in terms of function nets, including the classical concepts as particular cases and giving some relations between filter pointwise convergence and filter uniform convergence on compact sets. In the literature, there have been several recent studies about abstract Ascoli-type theorems, which extend earlier results, different Ascoli-type theorems are proved, in connection with various kinds of convergence and exhaustiveness of function nets. Our approach is direct, simple and easy to handle in the context of our considered structures, that is when it is dealt with *nets* of functions taking values in *cone metric spaces*, defined in general *Hausdorff topological spaces* and with *filter exhaustiveness* instead of metric spaces and equicontinuity respectively, and it allows us to give direct necessary and sufficient conditions. We consider symmetric or asymmetric distances with values in lattice groups and use the tool of (weak) filter exhaustiveness in connection with (global) continuity of the limit function and uniform convergence on compact sets. One of the main used methods is to use some kinds of convergence of suitable subnets of the given net to deduce some compactness properties. This is given in a very abstract context, comparing two kinds of compactness for function nets, and after a particular case is presented, using compactness of suitable sets, properties of convergence and boundedness in metric spaces and the Tychonoff theorem. Furthermore we consider Lipschitz-type metrics using completeness properties an a “total boundedness” argument in terms of subsequence, without using a topological approach.

2 Preliminaries

We begin with some fundamental properties of convergence and continuity in the lattice group context.

A nonempty set $\Lambda=(\Lambda,\geq)$ is said to be *directed* iff $\geq$ is a reflexive and transitive binary relation on Λ, such that for any two elements λ_1, $\lambda_2\in\Lambda$ there is $\lambda_0\in\Lambda$ with $\lambda_0\geq\lambda_1$ and $\lambda_0\geq\lambda_2$.

A *cone metric space* is a nonempty set R endowed with a function $\rho: R\times R\to Y$, where Y is a Dedekind complete lattice group, satisfying the following axioms:

- $\rho(r_1,r_2)\geq 0$ and $\rho(r_1,r_2)=0$ if and only if $r_1=r_2$;
- $\rho(r_1,r_2)=\rho(r_2,r_1)$ (*symmetric property*);
- $\rho(r_1,r_3)\leq\rho(r_1,r_2)+\rho(r_2,r_3)$ (*triangular property*), for all $r_j\in R$, $j=1,2,3$.

Such a function ρ will be called a *distance function*. If ρ satisfies only the first and the third of the above axioms, but not necessarily the symmetric property, then we say that ρ is an *asymmetric distance function* and that (R,ρ) is an *asymmetric cone metric space*. Note that any Dedekind complete (ℓ)-group Y is a cone metric space: indeed, it is enough to take $\rho(y_1,y_2)=|y_1-y_2|$, y_1, $y_2\in Y$ (the *absolute value*).

When R is a semigroup and $Y=\mathbb{R}$, we say that R is a *metric semigroup*. An example of metric semigroup which is not a group is the set of the fuzzy numbers.

Let R be a cone metric space and Y be its associated Dedekind complete (ℓ)-group. A sequence $(\sigma_p)_p$ in Y is called an (O)*-sequence* iff it is decreasing and $\bigwedge_p\sigma_p=0$. A net $(x_\lambda)_{\lambda\in\Lambda}$ in R (that is an indexed system of elements of R such that the index set Λ is directed) is *forward order convergent* or *forward* (O)*-convergent* (resp. *backward order convergent* or *backward* (O)*-convergent*) to $x\in R$ iff there exists an (O)-sequence $(\sigma_p)_p$ in Y such that for every $p\in\mathbb{N}$ there is $\lambda\in\Lambda$ with $\rho(x,x_\zeta)\leq\sigma_p$ (resp. $\rho(x_\zeta,x)\leq\sigma_p$) for all $\zeta\in\Lambda$, $\zeta\geq\lambda$, and in this case we will write $(O)\lim_{\lambda\in\Lambda}x_\lambda=x$. We say that $(x_\lambda)_{\lambda\in\Lambda}$ *order converges* or (O)*-converges* to $x\in R$ iff it is both forward and backward (O)-convergent to x.

Let $\mathcal{F}$ be a (Λ)-free filter of Λ and choose $\overline{y}\in R$. A net $(s_\lambda)_{\lambda\in\Lambda}$ in R is said to be $\mathcal{F}$*-forward bounded* (resp. $\mathcal{F}$*-backward bounded*) with respect to $\overline{y}$ iff there is $k_0\geq 0$, $k_0\in Y$ such that $\{\lambda\in\Lambda:\ \rho(\overline{y},s_\lambda)\leq k_0\}\in\mathcal{F}$ (resp. $\{\lambda\in\Lambda:\ \rho(s_\lambda,\overline{y})\leq k_0\}\in\mathcal{F}$). We say that $(s_\lambda)_\lambda$ is $\mathcal{F}$*-bounded* with respect to $\overline{y}$ iff it is both $\mathcal{F}$-forward and $\mathcal{F}$-backward bounded with respect to $\overline{y}$, and that $(s_\lambda)_\lambda$ is *bounded* (resp. *forward bounded, backward bounded*) iff it is

$\mathcal{F}_{\text{cofin}}$ -bounded (resp. $\mathcal{F}_{\text{cofin}}$ -forward bounded, $\mathcal{F}_{\text{cofin}}$ -backward bounded).

Let X be a Hausdorff topological space. A function $f : X \to R$ is said to be *forward* (resp. *backward*) *continuous* at a point $x \in X$ iff there exists an (O) -sequence $(\sigma_p)_p$ in Y (depending on x) such that for every $p \in \mathbb{N}$ there is a neighborhood U_x of x with $\rho(f(x), f(z)) \le \sigma_p$ (resp. $\rho(f(z), f(x)) \le \sigma_p$) whenever $z \in U_x$.

A function $f : X \to R$ is *globally forward* (resp. *backward*) *continuous on* X iff there is an (O) -sequence $(\sigma_p)_p$ in Y such that for any $p \in \mathbb{N}$ and $x \in X$ there is a neighborhood U_x of x with $\rho(f(x), f(z)) \le \sigma_p$ (resp. $\rho(f(z), f(x)) \le \sigma_p$) for each $z \in U_x$. We say that $f \in R^X$ is (*globally*) *continuous on* X iff it is both (globally) forward and (globally) backward continuous on X.

Let Λ be any nonempty set, and $\mathcal{P}(\Lambda)$ be the class of all subsets of Λ. A family of sets $\mathcal{I} \subset \mathcal{P}(\Lambda)$ is called an *ideal* of Λ iff $A \cup B \in \mathcal{I}$ whenever A, $B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$. A class of sets $\mathcal{F} \subset \mathcal{P}(\Lambda)$ is a *filter* of Λ iff $A \cap B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ and for every $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ (resp. a filter $\mathcal{F}$) of Λ is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $\Lambda \notin \mathcal{I}$ (resp. $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$).

Let $(\Lambda, \ge)$ be a directed set. A non-trivial ideal $\mathcal{I}$ of Λ is said to be (Λ) *-admissible* iff $\Lambda \setminus M_\lambda \in \mathcal{I}$ for each $\lambda \in \Lambda$, where $M_\lambda := \{\zeta \in \Lambda : \zeta \ge \lambda\}$.

A non-trivial filter $\mathcal{F}$ of Λ is (Λ) *-free* iff $M_\lambda \in \mathcal{F}$ for every $\lambda \in \Lambda$.

Given an ideal $\mathcal{I}$ of Λ, we call *dual filter* of $\mathcal{I}$ the family $\mathcal{F} = \{\Lambda \setminus I : I \in \mathcal{I}\}$. In this case we say that $\mathcal{I}$ is the *dual ideal* of $\mathcal{F}$ and we get $\mathcal{I} = \{\Lambda \setminus F : F \in \mathcal{F}\}$.

When $\Lambda = \mathbb{N}$ endowed with the usual order, the $(\mathbb{N})$-admissible ideals and the $(\mathbb{N})$-free filters are called simply *admissible ideals* and *free filters* respectively. The filter $\mathcal{F}_{\text{cofin}}$ is the filter of all subsets of $\mathbb{N}$ whose complement is finite, and its dual ideal $\mathcal{I}_{\text{fin}}$ is the family of all finite subsets of $\mathbb{N}$. The filter $\mathcal{F}_{\text{st}}$ is the filter of all subsets of $\mathbb{N}$ having asymptotic density 1, while its dual ideal $\mathcal{I}_{\text{st}}$ is the family of all subsets of $\mathbb{N}$, having null asymptotic density. Note that $\mathcal{F}_{\text{st}}$ is a P *-filter*, namely a filter $\mathcal{F}$ of $\mathbb{N}$ such that for every sequence $(A_n)_n$ in $\mathcal{F}$ there is another sequence $(B_n)_n$ in $\mathcal{F}$, such that the symmetric difference $A_n \Delta B_n$ is finite for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{F}$.

A nonempty family $\mathcal{B}' \subset \mathcal{P}(\Lambda)$ is said to be a *filter base of* Λ iff for every A, $B \in \mathcal{B}'$ there is an element $C \in \mathcal{B}'$ with $C \subset A \cap B$. Note that, if $\mathcal{B}'$ is a filter base of Λ, then the family $\mathcal{F} = \{A \subset \Lambda :$ there is $B \in \mathcal{B}'$ with $B \subset A\}$ is a filter of Λ. We call it the *filter generated by* $\mathcal{B}'$.

If $\mathcal{B}' = \{M_\lambda : \lambda \in \Lambda\}$, then $\mathcal{B}'$ is a filter base of Λ, and the filter $\mathcal{F}_\Lambda$ generated by $\mathcal{B}'$ is a (Λ) -free filter of Λ.

We now give the fundamental notions of filter convergence and related topics in the cone metric space setting.

A net $(x_\lambda)_{\lambda \in \Lambda}$ in a cone metric space R $(O\mathcal{F})$ *-converges to* $x \in R$ (shortly, $(O\mathcal{F})\lim_\lambda x_\lambda = x$) iff there exists an (O) -sequence $(\sigma_p)_p$ in Y with $\{\lambda \in \Lambda : \rho(x_\lambda, x) \le \sigma_p\} \in \mathcal{F}$ for each $p \in \mathbb{N}$. A net $(x_\lambda)_{\lambda \in \Lambda}$ in R is $(O\mathcal{F})$ *-Cauchy* iff there is an (O) -sequence $(\tau_p)_p$ in Y such that for every $p \in \mathbb{N}$ there is $F_0 \in \mathcal{F}$ with $\rho(x_\lambda, x_\xi) \le \tau_p$ for each λ, $\xi \in F_0$. Note that, since R is Dedekind complete, a net $(f_\lambda)_\lambda$ in R is $(O\mathcal{F})$ -convergent if and only if it is $(O\mathcal{F})$ -Cauchy.

Let Ξ be any nonempty set. We say that a family $\{(x_{\lambda,\xi})_\lambda : \xi \in \Xi\}$ in R $(O\mathcal{F})$ *-converges to* $x_\xi \in R$ *uniformly with respect to* $\xi \in \Xi$ (shortly, $(UO\mathcal{F})$ *-converges to* x_ξ) as λ varies in Λ, iff there is an (O) -sequence

$(v_p)_p$ in Y with

$$\{\lambda \in \Lambda : \bigvee_{\xi\in\Xi} \rho(x_{\lambda,\xi}, x_\xi) \le v_p\} \in \mathcal{F} \quad \text{for every } p \in \mathbb{N}.$$

A family $\{(x_{\lambda,\xi})_\lambda : \xi \in \Xi\}$ $(RO\mathcal{F})$*-converges* to $x_\xi \in R$ (as λ varies in Λ) iff there exists an (O)-sequence $(\sigma_p)_p$ in Y such that for each $p \in \mathbb{N}$ and $\xi \in \Xi$ we get $\{\lambda \in \Lambda : \rho(x_{\lambda,\xi}, x_\xi) \le \sigma_p\} \in \mathcal{F}$. By (RO)*-convergence* we will denote the $(RO\mathcal{F}_\Lambda)$-convergence. Observe that, when $R = Y = \mathbb{R}$, $(RO\mathcal{F})$-convergence coincides with usual filter convergence.

Let $x \in X$. A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be $\mathcal{F}$*-exhaustive at* x iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there exist a neighborhood U of x and a set $F \in \mathcal{F}$ such that for each $\lambda \in F$ and $z \in U$ we have $\rho(f_\lambda(z), f_\lambda(x)) \le \sigma_p$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *weakly* $\mathcal{F}$*-exhaustive at* x iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is a neighborhood U of x such that for every $z \in U$ there is $F_z \in \mathcal{F}$ with $\rho(f_\lambda(z), f_\lambda(x)) \le \sigma_p$ whenever $\lambda \in F_z$.

We say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *(weakly)* $\mathcal{F}$*-exhaustive on* X iff it is (weakly) $\mathcal{F}$-exhaustive at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$.

Similarly as above it is possible to formulate the notions of $(RO\mathcal{F})$- and $(UO\mathcal{F})$-forward (backward) convergence and the concepts of (weak) $\mathcal{F}$-forward (backward) exhaustiveness.

Of course the concepts of (weak, forward, backward) filter exhaustiveness can be given also analogously for sequences of functions, by taking $\Lambda = \mathbb{N}$ with the usual order.

In general, the notion of weak $\mathcal{F}$-exhaustiveness is strictly weaker than that of $\mathcal{F}$-exhaustiveness, even when $\Lambda = \mathbb{N}$ and $R = Y = \mathbb{R}$.

3 The main results

We now give, in the context of filter convergence and lattice groups, a necessary and sufficient condition under which the limit of a pointwise convergent net $(f_\lambda)_\lambda$ is (globally) continuous.

Theorem 3.1 *Under the same above notations and assumptions, let* $\mathcal{F}$ *be a* (Λ)*-free filter of* Λ*, fix* $x \in X$*, and suppose that* $f_\lambda : X \to R$*,* $\lambda \in \Lambda$*,* $(RO\mathcal{F})$*-converges to* $f : X \to R$ *on* X *with respect to a single* (O)*-sequence* $(\sigma_p^*)_p$ *in* Y*. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ *is weakly* $\mathcal{F}$*-exhaustive at* x*;*

(ii) f *is continuous at* x*.*

Theorem 3.2 *Under the same notations and assumptions as in Theorem 3.1, suppose that* $f_\lambda : X \to R$*,* $\lambda \in \Lambda$*,* $(RO\mathcal{F})$*-converges to* $f : X \to R$ *on* X *with respect to a single* (O)*-sequence* $(\sigma_p^*)_p$ *in* Y*. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ *is weakly* $\mathcal{F}$*-exhaustive on* X*;*

(ii) f *is globally continuous on* X*.*

Proposition 3.3 *Let* $\mathcal{F}$*,* X*,* R *be as in Theorem 3.2,* $f_\lambda : X \to R$*,* $\lambda \in \Lambda$*, be a function net,* $\mathcal{F}$*-exhaustive* *on X and $(RO\mathcal{F})$-convergent to $f \in R^X$ on X.*

Then f is globally continuous on X, and the net $(f_\lambda)_\lambda$ $(UO\mathcal{F})$-converges on every compact subset $C \subset X$ with respect to a single (O)-sequence, independent of C.

Proposition 3.4 *Let $\mathcal{F}$, X, R be as above. If $f_\lambda : X \to R$, $\lambda \in \Lambda$, is a net of functions, globally continuous with respect to a single (O)-sequence independent of λ and $(UO\mathcal{F})$-convergent to $f \in R^X$ on X, then f is globally continuous and $(f_\lambda)_\lambda$ is $\mathcal{F}$-exhaustive on X.*

Remark 3.5 Proceeding analogously as above, it is possible to see that Theorems 3.1, 3.2 and Propositions 3.3, 3.4 hold even when the distance function ρ does not satisfy necessarily symmetric property, and $(RO\mathcal{F})$- ($(UO\mathcal{F})$-) convergence, (weak) $\mathcal{F}$-exhaustiveness and continuity are replaced by $(RO\mathcal{F})$- ($(UO\mathcal{F})$-) forward (backward) convergence, (weak) $\mathcal{F}$-forward (backward) exhaustiveness and forward (backward) continuity respectively, under the hypothesis that the forward and backward convergences are equivalent.

We now give some versions of Ascoli-type theorems in the context of lattice groups and filter exhaustive nets. Note that in our context, since we deal with abstract structures which are not necessarily by a topology, it will be advisable to deal with suitable notions of "filter closedness" and "filter compactness" in relation with convergences, which are not necessarily generated by a Hausdorff topology. For example, note that in the space $L^0([0,1],\Sigma,\nu)$ of all measurable functions on $[0,1]$ with respect to the σ-algebra Σ of all Borel subsets of $[0,1]$ and the Lebesgue measure ν, with identification up to ν-null sets, order convergence coincides with almost everywhere convergence, which does not have a topological nature. Moreover there exist Dedekind complete vector lattices which do not have any Hausdorff compatible vector topology, for which every bounded monotone increasing sequence converges to its supremum.

Given a directed set Λ, a (Λ)-free filter $\mathcal{F}$ of Λ, a topological space X, a cone metric space R and a nonempty set $\Phi \subset R^X$, we say that Φ is $(RO\mathcal{F})$*-compact* (resp. $(c\mathcal{F})$*-compact*) iff every net $(f_\lambda)_{\lambda\in\Lambda}$ in Φ admits a subnet $(f_{\lambda_\kappa})_{\kappa\in\Lambda}$, $(RO\mathcal{F})$-convergent to an element $f \in \Phi$ (resp. $(UO\mathcal{F})$-convergent to an element $f \in \Phi$ on every compact subset $C \subset X$ with respect to a single (O)-sequence independent of C). We say that Φ is $(RO\mathcal{F})$-*closed* iff $f \in \Phi$ whenever $(f_\lambda)_\lambda$ is a net in Φ, $(RO\mathcal{F})$-convergent to $f \in R^X$. The $(RO\mathcal{F})$*-closure* of Φ is the set of the functions $f \in R^X$, having a net $(f_\lambda)_\lambda$ in Φ, $(RO\mathcal{F})$-convergent to f. Analogously as above, it is possible to formulate the notions of $(c\mathcal{F})$*-closedness* and of $(c\mathcal{F})$*-closure*. Note that Φ is $(RO\mathcal{F})$-closed (resp. $(c\mathcal{F})$-closed) if and only if it coincides with its $(RO\mathcal{F})$-closure (resp. $(c\mathcal{F})$-closure).

We now are in position to give the following abstract Ascoli-type theorem.

Theorem 3.6 *Under the same notations and hypotheses as above, if $\Phi \subset \Psi \subset R^X$, where Φ is $(c\mathcal{F})$-closed and Ψ is $(RO\mathcal{F})$-compact, and*

H'). every $(RO\mathcal{F})$-convergent net $(h_\lambda)_{\lambda\in\Lambda}$ in Φ has a subnet $(h_{\lambda_\xi})_{\xi\in\Lambda}$, $(RO\mathcal{F})$-convergent (in R^X) and $\mathcal{F}$-exhaustive,

then Φ is $(c\mathcal{F})$-compact.

Moreover, if Φ is $(c\mathcal{F})$-compact, then Φ satisfies condition H').

Theorem 3.7 *Let (X,d_X) and (R,d_R) be asymmetric metric spaces, such that d_X and d_R are real-valued distance functions, $\overline{y}$ be a fixed element of R, and $\mathcal{F}$ be any free filter of $\mathbb{N}$. Suppose that each subset of R, $\mathcal{F}$-forward closed and $\mathcal{F}$-forward bounded with respect to $\overline{y}$, is $\mathcal{F}$-forward compact, and that forward and backward convergence in R are equivalent. Let $\Phi \subset R^X$ be such that*

3.7.1) *every sequence* $(f_n)_n$ *in* Φ*, pointwise forward convergent in* R^X*, has a* $\mathcal{F}_{\text{cofin}}$ *-exhaustive subsequence in* Φ*;*

3.7.2) *every sequence* $(f_n)_n$ *in* Φ *has a subsequence* $(f_{n_r})_r$*,* $\mathcal{F}$ *-pointwise forward bounded in* R *with respect to* $\overline{y}$*.*

Then every sequence $(f_n)_n$ *in* Φ *admits a subsequence, uniformly convergent on every compact subset* $C \subset X$ *in the usual sense.*

Corollary 3.8 *Under the same hypotheses and notations as in Theorem 3.8, let* $\mathcal{F}$ *be a* P *-filter of* $\mathbb{N}$*. Let* $\Phi \subset R^X$ *satisfy 3.7.2) and be such that*

3.8.1) *every sequence* $(f_n)_n$ *in* Φ*, pointwise forward convergent in* R^X*, has a* $\mathcal{F}$ *-exhaustive subsequence in* Φ.

Then $(f_n)_n$ *has a subsequence, uniformly convergent on* X *in the usual sense.*

We now give some versions of abstract Ascoli-type theorems with respect to Lipschitz-type metrics.

Let (X,d) be a metric space endowed with a real-valued distance function, R be a Dedekind complete lattice group, $Y = R$ endowed with the absolute value, and let us add to R an extra element $+\infty$, satisfying the properties analogous to those of the element $+\infty$ of the extended real line. We say that $f: X \to R$ is *Lipschitz* iff there is a positive element $M \in R$ with $|f(x_1)-f(x_2)| \le d(x_1,x_2)M$ whenever x_1, $x_2 \in X$, and in this case we set

$$\Pi(f) := \bigvee \left\{ \frac{|f(x_1)-f(x_2)|}{d(x_1,x_2)} : x_1, x_2 \in X : x_1 \neq x_2 \right\}. \tag{8}$$

If $f: X \to R$ is not Lipschitz, then we put $\Pi(f) := +\infty$. Note that, even if X is a compact metric space, $R = \mathbb{R}$ and $f: X \to \mathbb{R}$ is continuous, it may happen that $\Pi(f) = +\infty$: indeed, it is enough to take $f(x) = x^{1/2}$, $x \in [0,1]$.

We now fix a point $x_0 \in X$ and consider the following extended metric:

$$d_L(f_1,f_2) := |f_1(x_0) - f_2(x_0)| \vee \Pi(f_1 - f_2), f_1, f_2 : X \to R. \tag{9}$$

Given a directed set $(\Lambda, \ge)$ and a (Λ) -free filter $\mathcal{F}$ of Λ, we say that

$$(\mathcal{F}) \lim_{\lambda \in \Lambda} d_L(f_\lambda, f) = 0 \quad \text{or} \quad (\mathcal{F} d_L) \lim_{\lambda \in \Lambda} f_\lambda = f$$

iff there is an (O) -sequence $(\sigma_p)_p$, with

$$\{\lambda \in \Lambda : d_L(f_\lambda, f) \le \sigma_p\} \in \mathcal{F} \quad \text{for every } p \in \mathbb{N}.$$

In this case, we say that the net $(f_\lambda)_\lambda$ $(\mathcal{F} d_L)$ *-converges* to f. The net $(f_\lambda)_\lambda$ is $(\mathcal{F} d_L)$ *-Cauchy* iff there is an (O) -sequence $(\sigma_p)_p$ with the property that for every $p \in \mathbb{N}$ there is $F \in \mathcal{F}$ with $d_L(f_\xi, f_\zeta) \le \sigma_p$ whenever ξ, $\zeta \in F$.

Proposition 3.9 *Let* $f_\lambda : X \to R$*,* $\lambda \in \Lambda$*, be a function net,* $(\mathcal{F} d_L)$ *-convergent to* f*, and* x_0 *be related with* d_L*. Then, for every* $k > 0$*,* $(f_\lambda)_\lambda$ $(UO\mathcal{F})$ *-converges to* f *on the set* $S(x_0,k) := \{x \in X : d(x,x_0) < k\}$.

As a consequence of Proposition 3.10, we state the following completeness result.

Proposition 3.10 *Under the same above notations and hypotheses, let* $f_\lambda : X \to R$*,* $\lambda \in \Lambda$*, be an* $(\mathcal{F} d_L)$ *-Cauchy net of functions, globally continuous with respect to a single* (O) *-sequence. Then* $(f_\lambda)_\lambda$ $(\mathcal{F} d_L)$ *-converges to a globally continuous function* $f: X \to R$.

The next step is to give an Ascoli-type theorem involving d_L. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is $\mathcal{F}$ *-finitely* d_L *-bounded* iff there exists a finite number q of globally continuous functions $h_1, \ldots, h_q \in R^X$, of elements $r_1, \ldots, r_q \in R$, of sets $E_1, \ldots, E_q$ with $\Lambda = \bigcup_{j=1}^q E_j$, and a set $F \in \mathcal{F}$ such that $d_L(f_\lambda, h_j) \le r_j$ for every $j \in [1,q]$

and whenever $\lambda \in F \cap E_j$.

Theorem 3.11 *Let* X *be a metric space. If* $\Phi \subset \Psi \subset R^X$ *, where* Φ *is* $(c\mathcal{F})$ *-closed and* Ψ *is* $(RO\mathcal{F})$ *-compact, and if we assume that*

3.11.1) *every* $(RO\mathcal{F})$ *-convergent net in* Φ *is* $\mathcal{F}$ *-finitely* d_L *-bounded,*

then Φ *is* $(c\mathcal{F})$ *-compact.*

Theorem 3.12 *Let* $\mathcal{F}$ *be any free filter of* $\mathbb{N}$*,* X *be a separable metric space,* $\Phi \subset \mathbb{R}^X$ *be* $(c\mathcal{F}_{\mathrm{cofin}})$ *-closed, and such that every sequence* $f_n : X \to \mathbb{R}$*,* $n \in \mathbb{N}$*, in* Φ *is* $\mathcal{F}$ *-finitely* d_L *-bounded. Then* Φ *is* $(c\mathcal{F}_{\mathrm{cofin}})$ *-compact.*