Greenberg Parafermions and a Microscopic Model of the Fractional Quantum Hall Effect

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Abstract

So far all theoretical models claiming to explain the Fractional Quantum Hall Effect are macroscopic in nature. In this paper we suggest a truly microscopic structure of this phenomenon. At the base is how electron charge is defined in the group SU(N) for arbitrary values of integer N. It is shown how all discovered charges in the Fractional Quantum Hall Effect are accounted for in this model. We show how Greenberg Parafermions, obeying parastatistics, are fundamentally required within this picture to explain the Fractional Quantum Hall Effect. We also show how both the Fractional Quantum Hall Effect and the Integral Quantum Hall Effect are explained in a common unified description in this microscopic model.

Keywords: Fractional Quantum Hall Effect, Integral Quantum Hall Effect, Greenberg Parastatistics, Unitary Groups, Electric Charge

Completely unexpected effects in the transport properties of two dimensional electron gas, subjected to low temperatures and strong magnetic fields, the so called Fractional Quantum Hall Effect (FQHE) and Integral Quantum Hall Effect (IQHE), have brought forth issues related to macroscopic and microscopic quantum mechanics. All kind of fractional charges with odd denominators like, 1/2, 2/3, 3/5, 4/5, 2/7, 3/11, 6/23, 2/9, 10/21 plus many others with even-denominators like 5/2, have been identified experimentally.

Earlier one knew of fractional quark charges of magnitude 2/3 and -1/3. Now we are being confronted with all kind of new fractional charges in FQHE. This is certainly one of the most puzzling issues in physics today. It is popularly felt that FQHE implies existence of quasiparticle collective states within the broad framework of macroscopic quantum mechanics. For example in Laughlin's model [1], FQHE arises within a gedanken experiment approach to the gauge transformation of wave functions in a magnetic field. His approach emphasizes a kind of macroscopic quantum phenomenon which however says nothing about the actual quantization process.

We'd like to refer to Laughlin's quotation in ref.[2], "And I see the fractional quantum Hall effect as a deep and important procedure for our guidance. Its fractionally charged excitations are, I believe, related to the fractionally charged quarks of the standard model of particle physics." Indeed, below we make this "guess" into a concrete and specific mathematical structure which actually relates these so called "quark" fractional charges to the fractional charges of the FQHE. An additional feature of our model here, is that it accounts for all the fractional charges already identified experimentally and makes predictions for many more to be discovered in future. It also explains the anomalous fractional charge of 5/2 well. It also it unifies the FQHE with the IQHE as well. Interestingly it is Greenberg parastatistics [3] which is fundamental for our model.

Let us go back to our paper of 1983, "What Leptons Really are!" [4]. Here we quote from the Abstract, "It is shown that leptons in reality are free quarks in disguise. These quarks acquire zero or unit charges as they leak out weakly from 'baryons' and mesons to exist singly as leptons".

But, first the issue of electric charge quantization in the Standard Model (SM) of particle physics based on the group structure $SU(N)_c \otimes SU(2)_L \otimes U(1)_Y$ with the number of colours N_c=3. It was shown by the author that the electric charge is actually fully and consistently quantized in the SM with these given as [5,6]:

$$Q(u) = \frac{1}{2}(1 + \frac{1}{N_c})$$

$$Q(d) = \frac{1}{2}(-1 + \frac{1}{N_c}) \tag{1}$$

For N_c=3 it gives the correct electric charge of u- and d-quarks. It is surprising that electric charge has contributions for the colour which arises from the orthogonal group $SU(N)_c$ in the SM group of $SU(N)_c \otimes SU(2)_L \otimes U(1)_Y$ with respect to the electroweak group $SU(2)_L \otimes U(1)_Y$. QCD does not know of the electric charge but the electric charge knows of QCD (actually through anomaly in SM). Note that the baryon number in the SM is $\frac{1}{N_c}$.

Now QCD for arbitrary number of colours in the group $SU(N_c)$ is of great interest for a proper understanding of the hadronic structure [7]. In all such studies (as for example manifested in the work in ref.[7]), Until our work [5,6], people always took electric charge of u- and d-quarks to be always 2/3 and -1/3 respectively for any number of colours N_c; that is, these charges are static and independent of colour. The author showed [5,6] that this was not correct and that the correct electric charges of u- and d- quarks are colour dependent as given in the above expression.

Now as a function of N_c the electric charges are not static. These are;
$N_c = 2$, Q(u)=3/4 and Q(d)=-1/4
$N_c = 3$, Q(u)=2/3 and Q(d)=-1/3
$N_c = 5$, Q(u)=3/5 and Q(d)=-2/5
$N_c = 7$, Q(u)=4/7 and Q(d)=-3/7
$N_c = 9$, Q(u)=5/9 and Q(d)=-4/9
$N_c = \infty$, Q(u)=1/2 and Q(d)=-1/2

It is intriguing that some of these charges have values which match the experimental values found in FQHE. But not to forget, that these are for quarks with colour degree of freedom. Within the same SM picture we also found that for the electron and the neutrino the charges in the SM were always -1 and 0 respectively [5,6] (and of course independent of colour).

Note that the SM $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ charges are exactly the same as those found as per the well known Gell-Nishijima expression for the group $SU_F(3)$ for the three flavour quark models [8];

$$Q = T_3 + \frac{Y}{2} = T_3 + \frac{(B+S)}{2} \tag{2}$$

Note that here the quark charges are defined in terms of the two diagonal generators of the flavour group $SU(3)_F$. This is completely different from the above SM definition of the electric charge. There (in the SM), the baryon number was arising from the colour degree of freedom, while here (in the group $SU_F(3)$ the baryon number as B=1/3 for S=0, is defined by the second diagonal generator of the SU(3) group.

We know that there is another flavour of quark, the c-quark which requires that the above group be expanded to a bigger flavour group $SU(4)_F$. Now this group has three diagonal generators [8]. The third new diagonal generator, now defines the new C quantum number and does nothing to the already defined baryon number for the three flavours. So the revised Gell-Mann-Nishijima expression of electric charge for 4-flavours is different in nature with respect to the original definition. We now know of six flavours of quarks, and then the Gell-Mann-Nishijima definition for the electric charge is generalized for the group $SU(6)_F$ as

$$Q = T_3 + \frac{(B + S + C + b + t)}{2} \tag{3}$$

where the quantum numbers S,C,b,t are -1, 1, -1, and 1 for the strange,charm, beauty and top quarks respectively. As for the case of the charm quark, every time, the new diagonal generator arising in the larger flavour group, defines the corresponding new quantum number as above. Note, once again, as to how the $SU(3)_F$ is distinct from the other flavour groups due to the reason of it defining the baryon number and the strangeness S at the same time, which the other higher flavours groups do not do. The full significance of this asymmetry between $SU_F(3)$ and flavour group for 4,5 and 6 flavours is not fully understood yet.

The next question is that we have a general expression of electric charge for 2,3,4,5,6 flavours which matches the experimental value of 2/3 and -1/3. IN contrast We got a unique value of the electric charges for different colours in the SM for arbitrary number of colours in the group $SU(N)_c \otimes SU(2)_L \otimes U(1)_Y$ and which matches with the above only for N_c=3.

Is the above expression of electric charge eqn. (3) for six flavours, the most general and intrinsic definition within the group structure $SU(6)_F$? The answer right away is in the negative, as we already saw that the second diagonal generator of $SU(3)_F$ defined both the baryon number and the new quark quantum number (Strangeness) while in the higher flavour groups the new generator only defined the quantum number associated with the new flavour quarks coming in. Thus the 3-flavour group is distinct from the 4-,5-, and 6-flavour groups.

So what is the most general intrinsic group theoretical definition of the electric charge for the group $SU(N)_F$ for an arbitrary number of flavours N_F ? This is what was done by the author in 1983 in ref.[4]

Let us see what the group theoretically consistent definition of the electric charge is for the flavour group $SU(N)_F$ for arbitrary number of flavours N_F. First let us rewrite the Gell-Mann-Nishijima definition of electric charge for the group $SU(3)_F$ as

$$Q_3 = T_3 + \frac{Y_3}{2} = T_3 + \frac{(B_3 + S_3)}{2} \tag{4}$$

where Q_3=Q, Y_3=Y, B_3=B and S_3=S as per the standard notation in eqn. (2). The subscript 3 here is used to indicate that it holds for the group $SU(3)_F$. Here the electric charge is defined in terms of the two diagonal generators of the group $SU(3)_F$. We take this as a fundamental property, which defines the electric charge consistently within the particular group under consideration, and use it to define electric charge for any arbitrary value of N_F for the group $SU(N)_F$. It is that, we should be able to define electric charge in terms (N_F-1) diagonal generators of the group $SU(N)_F$. So for the group $SU(4)_F$ we define:

$$Q_4 = T_3 + \frac{Y_3}{2} + \frac{Y_4}{3} \tag{5}$$

where Y_4 is to $SU(4)_F$ what Y_3 (or hypercharge) is to $SU(3)_F$. Its values for the quarks u-, d-, s- and c- are 1/4, 1/4, 1/4, -3/4 respectively (see Table 1). Note that in ref.[9] Y_4 is referred to as supercharge (Z). In fact such an expression had been suggested for the quark charge in ref.[10], though their interest was more in generalizing the triality concept of SU(3) to the group SU(N). Here we will go beyond this and generalize the concept to the other quantum numbers for any group [4]. For example note that $Y_3 = B_3 + S_3$ where B_3 (=B) is the SU(3) baryon number and S_3 (=S) is a new quantum number (strangeness) which distinguished it from the other two, u- and d-quarks. Similarly in $SU(4)_F$ let us define $Y_4 = B_4 + S_4$ where we identify B_4 with $SU(4)_F$ baryon number and S_4 with a new quantum number which will distinguish the fourth quark from the others. This, in the standard notation is the charm quantum number C. So it is S_4 (=-C)=-1 for the c-quark and zero for the others. So $Y_4 = B_4 + S_4$ looks like this:

$$\begin{pmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & -3/4 \end{pmatrix} = \begin{pmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 1/4 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \tag{6}$$

That is, the baryon number B_4 is 1/4 for all the quarks u,d,s,c. Using eqn (5), the electric charge Q_4 for the same quarks is 3/4, -1/4,-1/4,-1/4 respectively. These are displayed in Table 1.

We extend this approach to SU(5) right away.

$$Q_5 = T_3 + \frac{Y_3}{2} + \frac{Y_4}{3} + \frac{Y_5}{4} \tag{7}$$

Here Y_5 is to $SU(5)_F$ what Y_4 and Y_3 are to $SU(4)_F$ and $SU(3)_F$ respectively. Also $Y_5 = B_5 + S_5$ where B_5 is the baryon number and S_5 is the quantum number which distinguishes b-quark from all the others. Now the charges for u,d,s,c,b quarks in $SU(5)_F$ are 4/5, -1/5, -1/5, -1/5, -1/5 respectively (see Table 1).

These arguments are generalized to any $SU(N)_F$ for arbitrary flavours. Clearly therein the charge Q_{N_F} of the u-quark would be $\frac{(N_F-1)}{N_F}$ and that of all the others -$\frac{1}{N_F}$. Clearly same holds for two flavours also (see Table 1).

Note that as the experimental quark charges are always 2/3 or -1/3, and this holds only for 3-flavours group and not for a general flavour group. Then the question arises, as to what are all these above charges given in Tble 1? Although these are not experimental quarks, we continue to use the generic name "quarks" for them, in as much as these are fractionally charged.

As such, let us continue as we did in ref. [4]. In $SU(N)_F$ the charge Q_{N_F} for the u-quark is $\frac{(N_F-1)}{N_F}$ and that of the d-quark is -$\frac{1}{N_F}$ and the baryon number B_{N_F} for both is $\frac{1}{N_F}$. Let us go down in the sequence $N_F \to (N_F - 1) \to N_F - 2 \to$ ... In Table 1, we may go right up to $N_F = 2$. Let us extrapolate to $N_F = 1$ and then we find that baryon number goes to 1 and the u-quark and the d- quark charges go over to 0 and -1 respectively. Note the uncanny similarity to the leptons, wherein the lepton number is identified with the baryon number 1 above and the charges 0 and -1 with the neutrino and the electron charges respectively. And this led to the earlier conclusion, that actually the "leptons are quarks in disguise" [4]. One may assume here some kind of a tumbling mechanism through the above flavours, to achieve this correspondence.

Now let us assume that the above $SU(N)_F$ model charge arises in an independent model structure for what we now call "general-leptons". So waht we are saying is that the structure of these "general-leptons" is determined by the above group, which we now relabel as $SU(N)_{lep}$.

Then this "general-electron", is having a lepton number $\frac{1}{N_F}$ and an electric charge -$\frac{1}{N_F}$. Hence, for example, for N_F=5, as given in Table 1, the general-electron has a lepton number 1/5 and an electric charge -1/5. We therefore have unique correspondence with the charges in the FQHE. Hence the FQHE of charge 4/5 (as quoted in FQHE literature), would require to put four of these general-electrons together as a composite whole. Also the FQHE of 5/2 is simple composite of five charge 1/2 general-electrons of the group $SU(2)_{lep}$. Note that for N_F=4, two charges 1/4+1/4=1/2 and hence, as all the charges should be independent, this is disallowed. Note that this picture is general enough to account for any observed fractional charges. And of course, this model predicts many more to be discovered in the future.

Now remember that the Gell-Mann-Nishijima charge for $SU(3)_F$ quark model works for a symmetric state with the three quark flavours, two spins and the orbital degrees of freedom. This was a crisis, as to the fermionic character of the quarks [8].

In 1964, Greenberg suggested [3] that to resolve the above issue of the symmetry of the quarks, one can assume that quarks are parafermions of order three. This corresponds to a three-valued hidden colour degree of freedom, which was introduced by Han and Nambu in 1965.

Thus for $SU(3)_F$ quarks, the proper understanding of the symmetry puzzle was the introduction of a three valued colour degree of freedom. Now here we see that a general-electron too, as per Table 1, has a charge of 1/3 for an independent group $SU(3)_{lep}$. What is the difference between the quarks of the group $SU(3)_F$ and the general-electrons of the group $SU(3)_{lep}$?

Here we suggest that both of the above correspond to being Greenberg parafermions of order three. Then what distinguishes them?

Note that in the Han-Nambu picture of the quark model, one can build a gauge theory of Quantum Chromodynamics. In contrast, the Greenberg parafermion formalism is not directly amenable to gauging. This is true, as the parastatistics of H. S. Green cannot be gauged. This is because, the commutation rules for the Green components with equal index, are not the same as the Green components with unequal Green number values commutator rules. Greenberg and Macrae showed [11] how to modify the Green parameters so that it can be gauged by reformulating parastatistics with Grassmann numbers.

Observable quantities are intimately connected with the gauge symmetry of the theory. Within the framework of parastatistics if only currents such as

$$[\overline{\Psi}(x), \gamma^{\mu}\Psi(x)] \tag{8}$$

are observed then the gauge symmetry is SU(3). However if the observables are given by the current

$$[\Psi(x), \gamma^{\mu}\Psi(x)] \tag{9}$$

then the gauge symmetry would be SO(3).

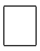
For quarks we know that currents of baryon number zero are relevant, and hence the currents in eqn. (8) are needed and hence the gauge group is SU(3) and around which the successful QCD gauge theory is built upon.

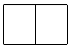
The current of eqn.(9) have non-zero baryon number. This is not true of quarks. But it seems to be appplcable to the FQHE. That is that in this form, the parafermions would reproduce FQHE structure. Hence the current in eqn. (9) should be useful to the general-electrons of the group $SU(3)_{lep}$. Hence we suggest that the Greenberg fermions hold for the general-leptons arisisng from the group structure $SU(3)_{lep}$ and leading to a SO(3) gauge structure.

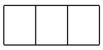
Hence the Greenberg parafermions hold both for the quarks with $SU(3)_c$ colour symmetry classification group and a $SU(3)_c$ gauge group for QCD; and for the general-electrons with $SU(3)_{lep}$ as the classifcation group (and giving its quantum numbers) and requiring an SO(3) gauge group to account for the FQHE and IQHE. For arbitrary N with Grassmann numbers for Greenberg fermions, we generalize the same to $SU(N)_{lep}$ as classification scheme for these and requiring an SO(N) gauge group struture for the same to model the FQHE and the IQHE, in an unified manner

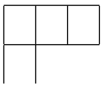
Below, we give few Young Diagrams of the representations of Greenberg parafermions of order three. Similarly we obtain what the representations of Greenberg parafermions of order N_F would appear as bosons and fermions, along with mixed symmetric states.

For Greenberg parafermions of order 3, the Young Diagrams displaying the symmetry e.g. are as given here (note: we quote charge in units of electronic charge - to conform to the standard FQHE literature) :

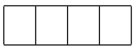
 **boson** ; charge 1/3 ; lepton number 1/3

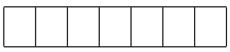
 **boson** ; charge 2/3 ; lepton number 2/3

 **fermion** ; charge 1 ; lepton number 1

 **mixed** ; charge 4/3 ; lepton number 4/3

And with Greenberg parafermions of order 7, we have e.g.

 **boson** ; charge 4/7 ; lepton number 4/7

 **fermion** ; charge 1 ; lepton number 1

In summary, we know that in $SU(3)_F$, we have quark charges, and for a proper understanding of hadron requires an independent three valued colour degree of freedom. In hadrons the three of colour, is sacred in as much as only three quark-kind are needed to make baryons with any kind, out of the six known flavours. We have shown that for SU(N), the group theoretically consistent definition of electric charge for any N are unique, and different from the corresponding charges of the quarks. These SU(N) charges for N flavours are actually the proper variable charges for leptons and as per the requirement of the Greenberg prastistics, reproduce an SO(N) gauge theoretical structure. Thus all the fractional charges of the FQHE are explained with these entities as being Greenberg parafermions. IQHE charges are also explained within the same structure. The whole picture is truly microscopic.

Table 1
Read it as if the SU(5) Table below is patched to the right of the main Table above it.

	I_2	B_2	S_2	Q_2	Y_3	B_3	S_3	Q_3	Y_4	B_4	S_4	Q_4
u	1/2	1/2	0	1/2	1/3	1/3	0	2/3	1/4	1/4	0	3/4
d	−1/2	1/2	−1	−1/2	1/3	1/3	0	−1/3	1/4	1/4	0	−1/4
s	0	0	0	0	−2/3	1/3	−1	−1/3	1/4	1/4	0	−1/4
c	0	0	0	0	0	0	0	0	−3/4	1/4	−1	−1/4
b	0	0	0	0	0	0	0	0	0	0	0	0

	Y_5	B_5	S_5	Q_5
u	1/5	1/5	0	4/5
d	1/5	1/5	0	−1/5
s	1/5	1/5	0	−1/5
c	1/5	1/5	0	−1/5
b	−4/5	1/5	−1	−1/5

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