On the First Quantization Theory of Photon

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Abstract: Geometrical optics has high similarity with classical particle mechanics . After first quantization program classical particle mechanics can obtain non-relativistic quantum mechanics that suitable for micro , but quantum mechanics does not describe the photon that its static mass is zero . So whether there is a kind of "quantization" approach for geometrical optics , by which non-relativistic quantum mechanics that can be used to describe the photon can be obtained . It's quantum mechanics of photon similar to the Schrödinger wave mechanics under the neglection of the formation and annihilation of photon . In this paper , several methods that can be used to find the first quantization theory of photon are given .

Key words: photon , quantization , photon wave function , wave equation , path integrals

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1. Introduction

Throughout the quantum world, the theory of micro particle with mass has experienced a development process from classical mechanics to quantum mechanics, and then experienced a development process from classical field theory to quantum field theory, but for massless photon, lack of the first quantization process from "classical optics" to "quantum optics", just second quantization process from electromagnetic field to quantum field directly. The concept of light quantum was first proposed by Einstein in 1905. By a comparative analysis of the similarity between geometrical optics and classical particle mechanics, de Broglie generalize the concept of light quantum into material particles, and puts forward the concept of "matter waves". After that, Schrödinger found out a wave equation for matter waves, thus established the wave mechanics, and promote the development of quantum mechanics, but quantum mechanics only applicable to the particle that its static mass is not zero, and does not describe the photon that its static mass is zero. The establishment of quantum mechanics can be said to be building on the classical particle mechanics, on the other hand, classical particle mechanics and geometrical optics are similar in many ways. Comprehensive above description, it is indeed possible to established a quantum mechanics or first quantization theory of photon that similar to the Schrödinger wave mechanics under the neglection of the formation and annihilation of photon.

2. Modified Geometrical Optics

In geometrical optics, the optical path length along a curve $\mathcal{C}$ from A to B is defined to be the line integral

$$\mathcal{S}_O = \int_A^B n(x_1, x_2, x_3) ds \tag{1}$$

Here $n(x_1, x_2, x_3)$ is the index of refraction of the media, ds is a line element. One sees from the optical path length that the integrand, i.e. the index of refraction of the media $n(x_1, x_2, x_3)$ is a function of position $\boldsymbol{x}$. But we all know that for different wavelengths of light, the index of refraction of the media have different values. In electromagnetism n is directly related to the electric and magnetic parameterrs ε and μ relates optics to electricity and magnetism[1]:

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

Here ε_0 and μ_0 are the permittivity and the permeability of the vacuum, respectively. ε and μ are the permittivity and the permeability of the material, respectively. Due to ε and μ are may depend on frequency of light, namely $\varepsilon = \varepsilon(\omega)$ and $\mu = \mu(\omega)$, and the frequency ω is associated with the wavelength λ , so the index of refraction of the material are also depend on wavelength. In conclusion, the index of refraction of the material are depend on not only the material itself but also the light itself.

This forces us to correct for difinition of optical path length of formula (1). In consideration of the analytical mechanics, the Lagrangian function L of action $S[\boldsymbol{x}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{x}, \dot{\boldsymbol{x}},) dt$ is a function of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$. In analogy to action, we can redefine the optical path length as follows

$$\mathcal{S}[\boldsymbol{x}(s)] = \int_A^B m(\boldsymbol{x}, \dot{\boldsymbol{x}}) ds$$

Here $m(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is a new definition of Lagrangian function of geometrical optics, called the general index of refraction of material. $\dot{\boldsymbol{x}}$ defined as $\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds}$.

Now therefore, Fermat's principle rewritten as follows

$$\delta S[\boldsymbol{x}(s)] = \delta \int_A^B m(\boldsymbol{x}, \dot{\boldsymbol{x}}) ds = 0$$

then reduces to the Euler-Lagrange equation

$$\frac{\partial m}{\partial x_i} - \frac{d}{ds} \frac{\partial m}{\partial \dot{x}_i} = 0 \tag{2}$$

Define the generaized momentum of modified geometrical optics as follows

$$p_i = \frac{\partial m}{\partial \dot{x}_i}$$

written in vector form as

$$\boldsymbol{p} = \dot{\nabla} m \tag{3}$$

where $\dot{\nabla} = \sum_i \frac{\partial}{\partial \dot{x}_i} \boldsymbol{i}$.

From Einstein's hypothesis of photon, the momentum of the photon follows as

$$\boldsymbol{p} = \hbar \boldsymbol{k}$$

Here $\boldsymbol{k}(|\boldsymbol{k}| = \frac{2\pi}{\lambda})$ is the wave vector and λ is the wavelength. Substituting this into Eq.(3), we obtain

$$\dot{\nabla} m = \hbar \boldsymbol{k}$$

Assuming that $\boldsymbol{k}$ is not an explicit function of $\dot{\boldsymbol{x}}$, we obtain

$$m(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \sum_i \hbar k_i \dot{x}_i - n(\boldsymbol{x}) = \hbar \boldsymbol{k} \cdot \dot{\boldsymbol{x}} - n(\boldsymbol{x}) \tag{4}$$

Let $\xi(\dot{\boldsymbol{x}})$ be a function of $\dot{\boldsymbol{x}}$ defined by the equation

$$\xi(\dot{\boldsymbol{x}}) = \hbar \boldsymbol{k} \cdot \dot{\boldsymbol{x}} \tag{5}$$

then given as

$$m(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \xi(\dot{\boldsymbol{x}}) - n(\boldsymbol{x}) \tag{6}$$

In analytical mechanics the Lagrangian L under the conservative field $V(\boldsymbol{x})$ has the form $L(\boldsymbol{x},\dot{\boldsymbol{x}}) = T(\dot{\boldsymbol{x}}) - V(\boldsymbol{x})$,where $T(\dot{\boldsymbol{x}})$ is kinetic energy of system. This shows that the Lagrangian L depend on the system itself $T(\dot{\boldsymbol{x}})$ and the environment $V(\boldsymbol{x})$. In analogy to Lagrangian L, $n(\boldsymbol{x})$ is a function that depend on the environment, i.e.material itself, it's similar to the index of refraction of material in unmodified geometrical optics, and $\xi(\dot{\boldsymbol{x}})$ is a physical quantity that used to describe the characteristics of photon itself, it's similar to the kinetic energy of particle. If take into account the photons with different wavelength, then ξ is a function of $\boldsymbol{k}$ and $\dot{\boldsymbol{x}}$, namely $\xi = \xi(\dot{\boldsymbol{x}}, \boldsymbol{k})$, therefore the general index of refraction of material $\mathcal{n}$ is a function of $\boldsymbol{k}$, $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}$, namely $\mathcal{n} = \mathcal{n}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{k})$, this is also a good illustration for why the index of refraction of material depend on the wavelength of light. We're only considering the situation that the light wave or photon has certain wavelength here. i.e.only considering the situation $\mathcal{n} = \mathcal{n}(\boldsymbol{x}, \dot{\boldsymbol{x}})$.

Substituting Eq.(6) into Eq.(2), we obtain

$$\frac{d}{ds}\frac{\partial \xi(\dot{\boldsymbol{x}})}{\partial \dot{x}_i} = -\frac{\partial n(\boldsymbol{x})}{\partial x_i}$$

written in vector form as

$$\frac{d}{ds}\dot{\nabla}\xi(\dot{\boldsymbol{x}}) = -\nabla n(\boldsymbol{x})$$

It is similar to eikonal $\mathcal{S}_O(\boldsymbol{x})$, define the modified eikonal $\mathcal{S}[\boldsymbol{x}(s), s]$ as follows

$$\mathcal{S}[\boldsymbol{x}(s), s] = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathcal{n}(\boldsymbol{x}, \dot{\boldsymbol{x}}) ds$$

As shown in Figure 1, it's analogy to unmodified geometrical optics[2], as $\boldsymbol{x}$ moves a distance $d\boldsymbol{x}$ away from P to P_1, the change in $\mathcal{S}[\boldsymbol{x}(s)]$ is

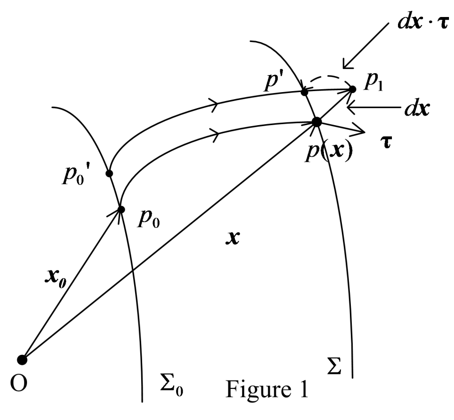

Figure 1

$$d\mathcal{S} = \mathcal{n}|P'P_1| = \mathcal{n}(d\boldsymbol{x}\cdot\boldsymbol{\tau}) \tag{7}$$

where $\boldsymbol{\tau}=\dfrac{d\boldsymbol{x}}{ds}$ is a unit vector at P tangent to the original ray P_0P. Since the modified eikonal $\mathcal{S}[\boldsymbol{x}(s),s]$ is a function of $\boldsymbol{x}$ and s, a total differential of $\mathcal{S}[\boldsymbol{x}(s),s]$ is then given as

$$d\mathcal{S}[\boldsymbol{x}(s),s]=\sum_i \frac{\partial \mathcal{S}}{\partial x_i}dx_i+\frac{\partial \mathcal{S}}{\partial s}ds=\nabla\mathcal{S}\cdot d\boldsymbol{x}+\frac{\partial \mathcal{S}}{\partial s}ds \tag{8}$$

Comparison of Eqs.(7) and (8) finally given as

$$\mathcal{n}(d\boldsymbol{x}\cdot\boldsymbol{\tau})=\nabla\mathcal{S}\cdot d\boldsymbol{x}+\frac{\partial \mathcal{S}}{\partial s}ds \tag{8.1}$$

since $\boldsymbol{\tau}=\dfrac{d\boldsymbol{x}}{ds}\Rightarrow d\boldsymbol{x}=ds\boldsymbol{\tau}\Rightarrow d\boldsymbol{x}\cdot\boldsymbol{\tau}=ds\boldsymbol{\tau}\cdot\boldsymbol{\tau}=ds$, Eq. (8.1) can be rewritten as

$$\mathcal{n}\boldsymbol{\tau}\cdot d\boldsymbol{x}=\nabla\mathcal{S}\cdot d\boldsymbol{x}+\frac{\partial \mathcal{S}}{\partial s}\boldsymbol{\tau}\cdot d\boldsymbol{x}$$

thus

$$\mathcal{n}\boldsymbol{\tau}=\nabla\mathcal{S}+\frac{\partial \mathcal{S}}{\partial s}\boldsymbol{\tau}$$

i.e.

$$\nabla\mathcal{S}=(\mathcal{n}-\frac{\partial \mathcal{S}}{\partial s})\boldsymbol{\tau} \tag{9}$$

The absolute value gives a result called as the modified eikonal equation

$$|\nabla\mathcal{S}|=\mathcal{n}-\frac{\partial \mathcal{S}}{\partial s}$$

Eq.(9) also can be reduced from the modified eikonal $\mathcal{S}[\boldsymbol{x}(s),s]$ by derivation

$$\sum_i \frac{\partial \mathcal{S}}{\partial x_i}\frac{dx_i}{ds}+\frac{\partial \mathcal{S}}{\partial s}=\frac{d}{ds}\mathcal{S}[\boldsymbol{x}(s),s]=\frac{d}{ds}\int_{x_0}^{x}\mathcal{n}(\boldsymbol{x},\dot{\boldsymbol{x}})ds=\mathcal{n}(\boldsymbol{x},\dot{\boldsymbol{x}})$$

$$\Rightarrow\nabla\mathcal{S}\cdot\boldsymbol{\tau}+\frac{\partial \mathcal{S}}{\partial s}=\mathcal{n}$$

$$\Rightarrow\nabla\mathcal{S}\cdot d\boldsymbol{x}=(\mathcal{n}-\frac{\partial \mathcal{S}}{\partial s})ds$$

$$\Rightarrow \nabla \mathcal{S} \cdot d\boldsymbol{x} = (n - \frac{\partial \mathcal{S}}{\partial s})\boldsymbol{\tau} \cdot d\boldsymbol{x}$$

$$\Rightarrow \nabla \mathcal{S} = (n - \frac{\partial \mathcal{S}}{\partial s})\boldsymbol{\tau}$$

Here, consider first a total differential of general function of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$, namely $g(\boldsymbol{x}, \dot{\boldsymbol{x}})$, we have

$$dg\,(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \sum_i \frac{\partial g}{\partial x_i} dx_i + \sum_i \frac{\partial g}{\partial \dot{x}_i} d\dot{x}_i$$

$$= \nabla g \cdot d\boldsymbol{x} + \dot{\nabla} g \cdot d\dot{\boldsymbol{x}}$$

used

$$\boldsymbol{\tau} = \frac{d\boldsymbol{x}}{ds}, \dot{\boldsymbol{\tau}} = \frac{d\boldsymbol{\tau}}{ds} = \frac{d}{ds}\frac{d\boldsymbol{x}}{ds} = \frac{d\dot{\boldsymbol{x}}}{ds}$$

we have

$$\frac{dg}{ds} = \nabla g \cdot \boldsymbol{\tau} + \dot{\nabla} g \cdot \dot{\boldsymbol{\tau}} \tag{9.1}$$

Substituting Eq.(9) into Eq.(9.1), we have

$$\frac{dg}{ds} = \frac{1}{(n - \frac{\partial \mathcal{S}}{\partial s})} \nabla \mathcal{S} \cdot \nabla g + \dot{\nabla} g \cdot \dot{\boldsymbol{\tau}} \tag{9.2}$$

We now make a special choice of g ,namely, that $g = n\tau_i = \frac{\partial \mathcal{S}}{\partial x_i} - \frac{\partial \mathcal{S}}{\partial s}\tau_i$, then Eq.(9.2) gives

$$\frac{d}{ds}(n\tau_i) = \frac{1}{(n - \frac{\partial \mathcal{S}}{\partial s})} \nabla \mathcal{S} \cdot \nabla (\frac{\partial \mathcal{S}}{\partial x_i} - \frac{\partial \mathcal{S}}{\partial s}\tau_i) + \dot{\boldsymbol{\tau}} \cdot \dot{\nabla} (\frac{\partial \mathcal{S}}{\partial x_i} - \frac{\partial \mathcal{S}}{\partial s}\tau_i)$$

$$= \frac{1}{(n - \frac{\partial \mathcal{S}}{\partial s})} \nabla \mathcal{S} \cdot (\frac{\partial \nabla \mathcal{S}}{\partial x_i} - \frac{\partial \nabla \mathcal{S}}{\partial s}\tau_i - \frac{\partial \mathcal{S}}{\partial s}\nabla \tau_i) + \dot{\boldsymbol{\tau}} \cdot (\frac{\partial \dot{\nabla} \mathcal{S}}{\partial x_i} - \frac{\partial \dot{\nabla} \mathcal{S}}{\partial s}\tau_i - \frac{\partial \mathcal{S}}{\partial s}\dot{\nabla} \tau_i)$$

from

$$\dot{\nabla}\mathcal{S}[\boldsymbol{x}(s),s]=\sum_i\frac{\partial}{\partial\dot{x}_i}\mathcal{S}[\boldsymbol{x}(s),s]=0,\dot{\boldsymbol{\tau}}\cdot\dot{\nabla}\tau_i=\dot{\tau}_i,\nabla\tau_i=\nabla\dot{x}_i=0$$

we have

$$\frac{d}{ds}(m\tau_i)=\frac{1}{(m-\frac{\partial\mathcal{S}}{\partial s})}(\nabla\mathcal{S}\cdot\frac{\partial\nabla\mathcal{S}}{\partial x_i}-\tau_i\nabla\mathcal{S}\cdot\frac{\partial\nabla\mathcal{S}}{\partial s})-\frac{\partial\mathcal{S}}{\partial s}\dot{\tau}_i$$

$$=\frac{1}{2(m-\frac{\partial\mathcal{S}}{\partial s})}(\frac{\partial(\nabla\mathcal{S})^2}{\partial x_i}-\tau_i\frac{\partial(\nabla\mathcal{S})^2}{\partial s})-\frac{\partial\mathcal{S}}{\partial s}\dot{\tau}_i \qquad (9.3)$$

Substituting Eq.(9) into Eq.(9.3), we have

$$\frac{d}{ds}(m\tau_i)=\frac{1}{2(m-\frac{\partial\mathcal{S}}{\partial s})}(\frac{\partial}{\partial x_i}(m-\frac{\partial\mathcal{S}}{\partial s})^2-\tau_i\frac{\partial}{\partial s}(m-\frac{\partial\mathcal{S}}{\partial s})^2)-\frac{\partial\mathcal{S}}{\partial s}\dot{\tau}_i$$

$$=\frac{\partial}{\partial x_i}(m-\frac{\partial\mathcal{S}}{\partial s})-\tau_i\frac{\partial}{\partial s}(m-\frac{\partial\mathcal{S}}{\partial s})-\frac{\partial\mathcal{S}}{\partial s}\dot{\tau}_i$$

Written in vector form as

$$\frac{d}{ds}(m\boldsymbol{\tau})=\nabla(m-\frac{\partial\mathcal{S}}{\partial s})-[\frac{\partial}{\partial s}(m-\frac{\partial\mathcal{S}}{\partial s})]\boldsymbol{\tau}-\frac{\partial\mathcal{S}}{\partial s}\dot{\boldsymbol{\tau}}$$

This is the partial differential equation satisfied by tangential unit vector of light ray $\boldsymbol{\tau}$, called the general light ray equation.

$$(m+\frac{\partial\mathcal{S}}{\partial s})\frac{d\boldsymbol{\tau}}{ds}+[\frac{dm}{ds}+\frac{\partial}{\partial s}(m-\frac{\partial\mathcal{S}}{\partial s})]\boldsymbol{\tau}-\nabla(m-\frac{\partial\mathcal{S}}{\partial s})=0 \qquad (10)$$

When $m(\boldsymbol{x},\dot{\boldsymbol{x}})=m$ is a constant, we have

$$\nabla m=0,\frac{dm}{ds}=0$$

$$\mathcal{S}=\int_{x_0}^{x}m(\boldsymbol{x},\dot{\boldsymbol{x}})ds=m\int_{x_0}^{x}ds=m(s-s_0)$$

$$\frac{\partial\mathcal{S}}{\partial s}=\frac{\partial}{\partial s}[m(s-s_0)]=m$$

Substituting those equations into Eq.(10), lead to

$$\frac{d\boldsymbol{\tau}}{ds}=0 \tag{10.1}$$

Equation (10.1) then give for a straight ray equation

$$\boldsymbol{\tau}=\mathbf{a}s+\mathbf{b}$$

where $\mathbf{a}$、$\mathbf{b}$ are constant vectors. This shows that the light travels through material in a straight line when the general index of refraction of material is constant.

3. Quantization of Modified Geometrical Optics

The definition of Hamiltonian in modified geometrical optics can't imitate the definition of Hamiltonian in analytical mechanics because of this is unreasonable, as shown below, if we defining the Hamiltonian in modified geometrical optics as follows

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{p})=\sum_i p_i\dot{x}_i-m(\boldsymbol{x},\dot{\boldsymbol{x}}) \tag{11}$$

where $\dot{x}_i=\frac{dx_i}{ds}$、 $p_i=\frac{\partial m}{\partial \dot{x}_i}$, after substituting Eqs.(4) and (6) into Eq.(11) we have

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{p})=\boldsymbol{p}\cdot\dot{\boldsymbol{x}}-\hbar\boldsymbol{k}\cdot\dot{\boldsymbol{x}}+n(\boldsymbol{x})=n(\boldsymbol{x})$$

It is shown that Hamiltonian $\mathcal{H}$ only depend on the function of position that reflect the environment, but the Hamiltonian depend on both the system itself and the environment, in addition, it represents the energy of system in analytical mechanics. But $\mathcal{H}$ can't represents the energy of system. Therefore, the Hamiltonian in modified geometrical optics can't be defined by Eq.(11).

Compute the variation of the modified eikonal $S[\boldsymbol{x}(s),s]$, we obtain

$$\delta S=\delta\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} m(\boldsymbol{x},\dot{\boldsymbol{x}})ds=\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \delta m(\boldsymbol{x},\dot{\boldsymbol{x}})ds$$

$$=\int_{x_0}^{x}(\sum_i \frac{\partial n}{\partial x_i}\delta x_i+\sum_i \frac{\partial n}{\partial \dot{x}_i}\delta \dot{x}_i)ds$$

$$=\sum_i \int_{x_0}^{x}(\frac{\partial n}{\partial x_i}\delta x_i+\frac{\partial n}{\partial \dot{x}_i}\delta \dot{x}_i)ds$$

$$=\sum_i \int_{x_0}^{x}(\frac{\partial n}{\partial x_i}\delta x_i ds+\frac{\partial n}{\partial \dot{x}_i}d\delta x_i)$$

$$=\sum_i \frac{\partial n}{\partial \dot{x}_i}\delta x_i\Big|_{x_0}^{x}+\sum_i \int_{x_0}^{x}(\frac{\partial n}{\partial x_i}-\frac{d}{ds}\frac{\partial n}{\partial \dot{x}_i})\delta x_i ds$$

by Eq.(2) and $\delta \boldsymbol{x}_0=0$, we have

$$\delta S=\sum_i \frac{\partial n}{\partial \dot{x}_i}\delta x_i=\sum_i p_i \delta x_i$$

since

$$\delta S=\sum_i \frac{\partial S}{\partial x_i}\delta x_i$$

we have

$$p_i=\frac{\partial S}{\partial x_i}$$

By the modified eikonal equation (9), we have

$$\frac{\partial S}{\partial s}=n-\sum_i \frac{\partial S}{\partial x_i}\dot{x}_i=n-\sum_i p_i \dot{x}_i=-n(\boldsymbol{x})$$

If we substitute this back into the square modulus of the modified eikonal equation (9), we obtain

$$(\nabla S)^2=(n-\frac{\partial S}{\partial s})^2=\xi^2(\dot{\boldsymbol{x}}) \tag{12}$$

It is Similar to the first quantization for quantum mechanics from analytical mechanics[3], let ψ_γ be a wave function of photon, and assuming that the relationship between ψ_γ and S is

$$\psi_\gamma = e^{\frac{i}{\hbar}S}$$

Then, we have

$$\frac{\partial S}{\partial x_i} = \frac{\hbar}{i}\frac{\partial \ln\psi_\gamma}{\partial x_i} = -\frac{i\hbar}{\psi_\gamma}\frac{\partial \psi_\gamma}{\partial x_i} \tag{13}$$

substitute this back into Eq.(12), we obtain

$$\hbar^2\sum_i(\frac{\partial \psi_\gamma}{\partial x_i})(\frac{\partial \psi_\gamma}{\partial x_i})^* - \xi^2\psi_\gamma\psi_\gamma^* = 0 \tag{13.1}$$

The triple integral over entire space of Eq.(13.1) is

$$\iiint \hbar^2\sum_i(\frac{\partial \psi_\gamma}{\partial x_i})(\frac{\partial \psi_\gamma}{\partial x_i})^* - \xi^2\psi_\gamma\psi_\gamma^* d\boldsymbol{x} = 0 \tag{13.2}$$

Compute the variation of Eq.(13.2) ,we have

$$\delta\iiint \Psi_\gamma[\psi,\psi^*,\frac{\partial \psi}{\partial x_i},(\frac{\partial \psi}{\partial x_i})^*,x_i]d\boldsymbol{x} = \delta\iiint \hbar^2\sum_i(\frac{\partial \psi_\gamma}{\partial x_i})(\frac{\partial \psi_\gamma}{\partial x_i})^* - \xi^2\psi_\gamma\psi_\gamma^* d\boldsymbol{x} = 0 \tag{13.3}$$

Therefor, the corresponding Euler-Lagrange equation system are

$$\begin{cases} \dfrac{\partial \Psi_\gamma}{\partial \psi_\gamma} - \sum_i \dfrac{\partial}{\partial x_i}\dfrac{\partial \Psi_\gamma}{\partial(\dfrac{\partial \psi_\gamma}{\partial x_i})} = 0 \\ \dfrac{\partial \Psi_\gamma}{\partial \psi_\gamma^*} - \sum_i \dfrac{\partial}{\partial x_i}\dfrac{\partial \Psi_\gamma}{\partial(\dfrac{\partial \psi_\gamma}{\partial x_i})^*} = 0 \end{cases} \tag{14}$$

By Eq.(14) we obtain the time independent Schrödinger type equation of photon:

$$\hbar^2\sum_i \frac{\partial^2 \psi_\gamma}{\partial x_i^{\,2}} + \xi^2\psi_\gamma = 0 \tag{15}$$

It’s similar to the wave equation in quantized unmodified geometrical optics[4]

$$(\frac{\partial^2}{\partial x_1^{\ 2}}+\frac{\partial^2}{\partial x_2^{\ 2}}+\frac{\partial^2}{\partial x_3^{\ 2}})\psi=-\frac{n^2}{\kappa^2}\psi \tag{15.1}$$

(here $\kappa=\frac{\lambda_0}{2\pi}$ is constant similar to $\hbar$, λ_0 is wavelength of light in free space, and we obtain the geometrical optics when $\kappa\to 0$). Moreover, the corresponding quantities $\frac{\xi}{\hbar}=\frac{\boldsymbol{p}\cdot\dot{\boldsymbol{x}}}{\hbar}$ and $\frac{n}{\kappa}=\frac{n\boldsymbol{p}_0}{\hbar}$ ($\boldsymbol{p}_0$ is momentum of photon in free space) both depend on the momentum of photon. As we can see, Eq.(15.1) become Eq.(15) by defining $\kappa=\frac{n\hbar}{\xi}$.

If we consider the hypothesis about the microstructure of light ray that described in section 5, i.e. the motion of photon is restricted to the imaginary cylinder region, called the light-cylinder ray, as shown in Figure 4, then all of the geometric quantities that concerning light in this section, such as s and $\boldsymbol{\tau}$ are quantities that used to describe the light-cylinder ray, and for those quantities such as $\boldsymbol{p}$、 $\boldsymbol{k}$ are used to describe the photon in the light-cylinder ray. Therefor, the tangential unit vector of light ray $\boldsymbol{\tau}$ in Eq.(5) does not necessarily have to be parallel to $\boldsymbol{k}$, it should have an angle θ that changed by changing the photon position, from $|\boldsymbol{\tau}|=1$ we can rewritten the Eq.(5) as follows

$$\xi=\boldsymbol{p}\cdot\boldsymbol{\tau}=\vartheta|\boldsymbol{p}|$$

Where $\vartheta=cos\,\theta$ is a parameter.

From the relation between the energy and the momentum of the photon $E^2=c^2\boldsymbol{p}^2$ we obtain

$$\xi^2=\frac{\vartheta^2}{c^2}E^2$$

substitute this back into Eq.(15), we obtain

$$\hbar^2\nabla^2\psi_\gamma + \frac{\vartheta^2}{c^2}E^2\psi_\gamma = 0 \tag{16}$$

If similar to the quantum mechanics that substituting $E\psi_\gamma$ into the following equation[3]

$$\hat{E}\psi_\gamma = i\hbar\frac{\partial}{\partial t}\psi_\gamma = E\psi_\gamma$$

where $\hat{E}$ is energy operator, then Eq.(16) can be rewritten as follows

$$\nabla^2\psi_\gamma(\boldsymbol{x},t) - \frac{\vartheta^2}{c^2}\frac{\partial^2}{\partial^2 t}\psi_\gamma(\boldsymbol{x},t) = 0$$

It's called the quantized wave equation of photon. This equation reduces to the classic wave equation of light in vacuum for $\vartheta \equiv 1$,i.e. $\boldsymbol{\tau}$ parallel to $\boldsymbol{k}$

$$\nabla^2\psi_\gamma(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}\psi_\gamma(\boldsymbol{x},t) = 0$$

4. Path Integrals of Photon

Feynman's path integrals[5] provided another form of quantization theory for particle, which the Schrödinger wave equation can be deduced, moreover , the derivation of Schrödinger wave equation does have many similarities to the theory of light; for instance, in form, the action similar to the optical path; the formula

$$\psi(\boldsymbol{x}_b,t_b) = \int_{-\infty}^{\infty} K(\boldsymbol{x}_b,t_b;\boldsymbol{x}_a,t_a)\psi(\boldsymbol{x}_a,t_a)d\boldsymbol{x}_a$$

Similar to the integral formula of Huygens principle. Those similarity may provide a way to quantizing photon, i.e. the first quantization theory of photon can be obtained by the similarity method of path integrals.

As shown in Figure 2, assuming that the probability $P_\gamma(b,a)$ to go from a point $\boldsymbol{x}_a$ at the arc

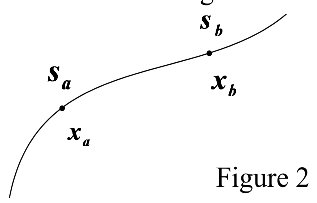

Figure 2

length of light-curve s_a to the point $\boldsymbol{x}_b$ at the arc length of light-curve s_b is

$$P_\gamma(b,a) = \left|K_\gamma(b,a)\right|^2$$

where $K_\gamma(b,a)$ is the sum of contribution $\phi_\gamma[\boldsymbol{x}(s)]$ from each path of photon

$$K_\gamma(b,a) = \sum_{\text{over } all\ paths\ from\ a\ to\ b} \phi_\gamma[\boldsymbol{x}(s)]$$

we also assuming that the contribution of a path has a phase proportional to the modified eikonal $\mathcal{S}[\boldsymbol{x}(s)]$

$$\phi_\gamma[\boldsymbol{x}(s)] = \text{const}\, e^{\frac{i}{\hbar}\mathcal{S}[\boldsymbol{x}(s)]}$$

where $\mathcal{S}[\boldsymbol{x}(s)]$ defined by the following

$$\mathcal{S}[\boldsymbol{x}(s)] = \int_{\boldsymbol{x}_a}^{\boldsymbol{x}_b} m(\boldsymbol{x}, \dot{\boldsymbol{x}})ds = \int_{\boldsymbol{x}_a}^{\boldsymbol{x}_b} [\hbar \boldsymbol{k}\cdot\dot{\boldsymbol{x}} - n(\boldsymbol{x})]ds$$

We also can define the $K_\gamma(b,a)$ by divide the interval which from a to b into the N subintervals

$$K_\gamma(b,a) = \int_{x_{N-1}} \cdots \int_{x_1} K_\gamma(b, N-1) \cdots K_\gamma(i+1, i) \cdots K_\gamma(1,a)dx_1 \cdots dx_{N-1}$$

In this definition the kernel for the photon to go between two points separated by an infinitesimal arc length of light-curve τ_γ is

$$K_\gamma(i+1,i) = C^{-1} e^{\frac{i}{\hbar}\tau_\gamma m\left(\frac{x_{i+1}-x_i}{\tau_\gamma}, \frac{x_{i+1}+x_i}{2}\right)}$$

which is correct to first order in τ_γ ,and C is a constant.

Here, we assuming that the wave function of photon $\psi_\gamma(\boldsymbol{x}, s)$ is a function of position $\boldsymbol{x}$ and arc length of light-curve s , the wave function $\psi_\gamma(\boldsymbol{x}_b, s_b)$ and $\psi_\gamma(\boldsymbol{x}_a, s_a)$ represents the total amplitude in the point $(\boldsymbol{x}_b, s_b)$ and $(\boldsymbol{x}_a, s_a)$, respectively; and the amplitude to go from a point a to a point b is represented by

$K_\gamma(\boldsymbol{x}_b, s_b; \boldsymbol{x}_a, s_a)$; therefore, similarly we have

$$\psi_\gamma(\boldsymbol{x}_b, s_b) = \int_{-\infty}^{\infty} K_\gamma(\boldsymbol{x}_b, s_b; \boldsymbol{x}_a, s_a) \psi_\gamma(\boldsymbol{x}_a, s_a) d\boldsymbol{x}_a$$

We shall now consider the case of a photon moving in a material which the general index of refraction is $m(x, \dot{x})$ in one dimension, and the arc length of light-curve is $s_b = s_a + \tau_\gamma$, similarly we have

$$\psi_\gamma(x, s+\tau_\gamma) = C^{-1} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\tau_\gamma m(\frac{x-y}{\tau_\gamma}, \frac{x+y}{2})} \psi_\gamma(y, s) dy$$

In this case

$$m = \hbar k \frac{x-y}{\tau_\gamma} - n(\frac{x+y}{2})$$

we obtain

$$\psi_\gamma(x, s+\tau_\gamma) = C^{-1} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\hbar k(x-y)} e^{-\frac{i}{\hbar}\tau_\gamma n(\frac{x+y}{2})} \psi_\gamma(y, s) dy$$

by $k = \dfrac{2\pi}{\lambda}$ we have

$$\psi_\gamma(x, s+\tau_\gamma) = C^{-1} \int_{-\infty}^{\infty} e^{i\frac{2\pi}{\lambda}(x-y)} e^{-\frac{i}{\hbar}\tau_\gamma n(\frac{x+y}{2})} \psi_\gamma(y, s) dy$$

due to the photon considered as the light ray in this case, λ should be an infinitesimal quantity, in the same way, $y - x$ is an infinitesimal quantity; therefore, defining $y = x + \eta_\gamma$,where η_γ is a very small value, we obtain

$$\psi_\gamma(x, s+\tau_\gamma) = C^{-1} \int_{-\infty}^{\infty} e^{-i\frac{2\pi}{\lambda}\eta_\gamma} e^{-\frac{i}{\hbar}\tau_\gamma n(x+\frac{\eta_\gamma}{2})} \psi_\gamma(x+\eta_\gamma, s) d\eta_\gamma$$

Assuming $\tau_\gamma \sim \eta_\gamma{}^n$,where n is an integer, then $\tau_\gamma n(x+\frac{\eta_\gamma}{2})$ can be replaced by $\tau_\gamma n(x)$. Expanding the ψ_γ on the left-hand side to first order in τ_γ and the right-hand side to m th order in η_γ , we obtain

$$\psi_\gamma(x,s)+\tau_\gamma\frac{\partial\psi_\gamma}{\partial s}=C^{-1}\int_{-\infty}^{\infty}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}[1-\frac{i}{\hbar}\tau_\gamma n(x)][\psi_\gamma(x,s)+\sum_{m=1}^{n}\frac{\eta_\gamma{}^m}{m!}\frac{\partial^m\psi_\gamma}{\partial x^m}]d\eta_\gamma \quad (17)$$

If we are taking the limit τ_γ、 $\eta_\gamma\to 0$,we obtain

$$\psi_\gamma(x,s)=C^{-1}\int_{-\infty}^{\infty}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\psi_\gamma(x,s)d\eta_\gamma \quad (18)$$

but

$$\int_{-\infty}^{\infty}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}d\eta_\gamma=0$$

and $\psi_\gamma(x,s)$ need not be all vanish, therefore, the lower limit and the upper limit of the integration (18) should not be negative and positive infinite, respectively, but it should have a certain boundary; assuming that the lower limit and the upper limit of the integration (18) is $-d$ and $+d$, respectively, then the Eq.(18) can be rewritten as following

$$\psi_\gamma(x,s)=C^{-1}\int_{-d}^{d}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\psi_\gamma(x,s)d\eta_\gamma$$

thus

$$C=\int_{-d}^{d}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}d\eta_\gamma=\frac{\lambda}{\pi}sin\frac{2\pi}{\lambda}d$$

And Eq.(17) become

$$\psi_\gamma(x,s)+\tau_\gamma\frac{\partial\psi_\gamma}{\partial s}=C^{-1}\int_{-d}^{d}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}[1-\frac{i}{\hbar}\tau_\gamma n(x)][\psi_\gamma(x,s)+\sum_{m=1}^{n}\frac{\eta_\gamma{}^m}{m!}\frac{\partial^m\psi_\gamma}{\partial x^m}]d\eta_\gamma$$

$$=C^{-1}\int_{-d}^{d}\{[1-\frac{i}{\hbar}\tau_\gamma n(x)]e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\psi_\gamma(x,s)+[1-\frac{i}{\hbar}\tau_\gamma n(x)]e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\sum_{m=1}^{n}\frac{\eta_\gamma{}^m}{m!}\frac{\partial^m\psi_\gamma}{\partial x^m}\}d\eta_\gamma$$

$$=\psi_\gamma(x,s)-\frac{i}{\hbar}\tau_\gamma n(x)\psi_\gamma(x,s)+C^{-1}[1-\frac{i}{\hbar}\tau_\gamma n(x)]\sum_{m=1}^{n}\frac{\partial^m\psi_\gamma}{\partial x^m}\int_{-d}^{d}\frac{\eta_\gamma{}^m}{m!}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}d\eta_\gamma$$

since

$$\int_{-d}^{d}\eta_\gamma{}^m e^{-i\frac{2\pi}{\lambda}\eta_\gamma}d\eta_\gamma=e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\sum_{r=0}^{m}(-1)^r\frac{m\eta_\gamma{}^{m-r}}{(m-r)!(-i\frac{2\pi}{\lambda})^{r+1}}\Big|_{-d}^{d}$$

we have

$$\psi_\gamma(x,s)+\tau_\gamma\frac{\partial\psi_\gamma}{\partial s}=\psi_\gamma(x,s)-\frac{i}{\hbar}\tau_\gamma n(x)\psi_\gamma(x,s) \tag{19}$$

$$+\frac{\pi}{\lambda sin\frac{2\pi}{\lambda}d}[1-\frac{i}{\hbar}\tau_\gamma n(x)]\sum_{m=1}^{n}\frac{\partial^m\psi_\gamma}{\partial x^m}e^{-i\frac{2\pi}{\lambda}\eta_\gamma}\sum_{r=0}^{m}(-1)^r\frac{\eta_\gamma^{\ m-r}}{(m-r)!(-i\frac{2\pi}{\lambda})^{r+1}}\Big|_{-d}^{d}$$

considering the special case of $n=2$, we have

$$\sum_{m=1}^{2}\frac{\partial^m\psi_\gamma}{\partial x^m}e^{-ik\eta_\gamma}\sum_{r=0}^{m}(-1)^r\frac{\eta_\gamma^{\ m-r}}{(m-r)!(-ik)^{r+1}}\Big|_{-d}^{d}$$

$$=\frac{\partial\psi_\gamma}{\partial x}e^{-ik\eta_\gamma}(\frac{\eta_\gamma}{-ik}-\frac{1}{(-ik)^2})\Big|_{-d}^{d}+\frac{\partial^2\psi_\gamma}{\partial x^2}e^{-ik\eta_\gamma}(\frac{\eta_\gamma^{\ 2}}{-2ik}-\frac{\eta_\gamma}{(-ik)^2}+\frac{1}{(-ik)^3})\Big|_{-d}^{d}$$

since λ is an infinitesimal quantity, so $k=\frac{2\pi}{\lambda}$ is an infinitely large quantity, we assuming that d and k are infinity of the same order, if we omit the higher order term of k, we obtain

$$\sum_{m=1}^{2}\frac{\partial^m\psi_\gamma}{\partial x^m}e^{-ik\eta_\gamma}\sum_{r=0}^{m}(-1)^r\frac{\eta_\gamma^{\ m-r}}{(m-r)!(-ik)^{r+1}}\Big|_{-d}^{d}$$

$$=i\frac{\partial\psi_\gamma}{\partial x}e^{-ik\eta_\gamma}\frac{\eta_\gamma}{k}\Big|_{-d}^{d}+i\frac{\partial^2\psi_\gamma}{\partial x^2}e^{-ik\eta_\gamma}\frac{\eta_\gamma^{\ 2}}{2k}\Big|_{-d}^{d}$$

$$=i\frac{2d}{k}coskd\frac{\partial\psi_\gamma}{\partial x}+\frac{d^2}{k}sinkd\frac{\partial^2\psi_\gamma}{\partial x^2}$$

Therefore, Eq.(19) rewritten as follows

$$\psi_\gamma(x,s)+\tau_\gamma\frac{\partial\psi_\gamma}{\partial s}=\psi_\gamma(x,s)-\frac{i}{\hbar}\tau_\gamma n(x)\psi_\gamma(x,s)$$

$$+\frac{k}{2sinkd}[1-\frac{i}{\hbar}\tau_\gamma n(x)](i\frac{2d}{k}coskd\frac{\partial\psi_\gamma}{\partial x}+\frac{d^2}{k}sinkd\frac{\partial^2\psi_\gamma}{\partial x^2})$$

i.e.

$$\frac{\partial\psi_\gamma}{\partial s}=-\frac{i}{\hbar}n(x)\psi_\gamma(x,s)+\frac{[1-\frac{i}{\hbar}\tau_\gamma n(x)]}{\tau_\gamma}(idcotkd\frac{\partial\psi_\gamma}{\partial x}+\frac{d^2}{2}\frac{\partial^2\psi_\gamma}{\partial x^2})$$

or

$$i\hbar\frac{\partial\psi_\gamma}{\partial s}=n(x)\psi_\gamma(x,s)+[\frac{i\hbar}{\tau_\gamma}+n(x)](idcotkd\frac{\partial\psi_\gamma}{\partial x}+\frac{d^2}{2}\frac{\partial^2\psi_\gamma}{\partial x^2})$$

Assuming that the value of τ_γ is greater than $\hbar$, we obtain

$$\frac{i\hbar}{n(x)}\frac{\partial\psi_\gamma}{\partial s}=\psi_\gamma(x,s)+idcotkd\frac{\partial\psi_\gamma}{\partial x}+\frac{d^2}{2}\frac{\partial^2\psi_\gamma}{\partial x^2}$$

as we know, $cotkd$ is an infinitely for $kd=m\pi,\ m=0,1,2\cdots\cdots$, in order that the equation can be used to describe all wave lengths of photon, we assuming that $cotkd=0$,i.e.

$$kd=\frac{1+2m}{2}\pi,\ m=0,1,2\cdots\cdots$$

therefore, we obtain

$$\frac{i\hbar}{n(x)}\frac{\partial\psi_\gamma}{\partial s}=\psi_\gamma(x,s)+\frac{(1+2m)^2\pi^2}{8k^2}\frac{\partial^2\psi_\gamma}{\partial x^2}$$

This is the wave equation of photon in one dimension by using the similarity method of path integrals. Corresponding equation in three dimension can be worked out in the same way

$$\frac{i\hbar}{n(\boldsymbol{x})}\frac{\partial}{\partial s}\psi_\gamma(\boldsymbol{x},s)=\psi_\gamma(\boldsymbol{x},s)+\frac{(1+2m)^2\pi^2}{8k^2}\nabla^2\psi_\gamma(\boldsymbol{x},s) \tag{20}$$

In quantum mechanics, we have the quantum continuity equation and the concept of the local conservation of probability, similarity, the quantum continuity equation of photon can be deduced by the following: from Eq.(20) we have

$$\psi_\gamma^*\frac{\partial\psi_\gamma}{\partial s}=-\frac{i}{\hbar}n(\boldsymbol{x})[\psi_\gamma^*\psi_\gamma+\frac{(1+2m)^2\pi^2}{8k^2}\psi_\gamma^*\nabla^2\psi_\gamma]$$

$$\psi_\gamma\frac{\partial\psi_\gamma^*}{\partial s}=\frac{i}{\hbar}n(\boldsymbol{x})[\psi_\gamma\psi_\gamma^*+\frac{(1+2m)^2\pi^2}{8k^2}\psi_\gamma\nabla^2\psi_\gamma^*]$$

add the two expressions, we obtain

$$\psi_\gamma \frac{\partial \psi_\gamma^*}{\partial s} + \psi_\gamma^* \frac{\partial \psi_\gamma}{\partial s} = \frac{i(1+2m)^2 \pi^2 n(\boldsymbol{x})}{8\hbar k^2} [\psi_\gamma \nabla^2 \psi_\gamma^* - \psi_\gamma^* \nabla^2 \psi_\gamma]$$

i.e.

$$\frac{\partial}{\partial s}(\psi_\gamma \psi_\gamma^*) = -\frac{i(1+2m)^2 \pi^2 n(\boldsymbol{x})}{8\hbar k^2} \nabla \cdot [\psi_\gamma^* \nabla \psi_\gamma - \psi_\gamma \nabla \psi_\gamma^*] \qquad (21)$$

Defining the probability density of photon by $\rho_\gamma = \psi_\gamma(\boldsymbol{x}, s)\psi_\gamma^*(\boldsymbol{x}, s)$ and the probability current density of photon by $\boldsymbol{J}_\gamma$

$$\boldsymbol{J}_\gamma = -\frac{i(1+2m)^2 \pi^2}{8\hbar k^2} [\psi_\gamma^* \nabla \psi_\gamma - \psi_\gamma \nabla \psi_\gamma^*] \qquad (22)$$

Then Eq.(21) is the quantum continuity equation of photon

$$\frac{\partial}{\partial s} \rho_\gamma = n(\boldsymbol{x}) \nabla \cdot \boldsymbol{J}_\gamma$$

5. Light-ization of Photon

Now we are discussing the microstructure of light ray. First, we are discussing the microstructure of straight-light ray; then generalize to the curly-light ray. Suppose that a light ray traveling space from a point A to a point B in straight lines, we model light as beams of light, and assuming that the beam as cylinder with diameter of D, and the cylinder is composed of a large number of photons. Now we consider the motion of one of photons in the cylinder: assuming that the photon has certain momentum p, according to the Heisenberg uncertainty relation, the uncertainty in position is $\Delta x \geq \frac{\hbar}{2p}$, due to the motion of photon from a cross section of cylinder A to a cross section of cylinder B is restricted to the inside of the beam, we can assuming that the motion of photon is restricted to the cylinder with diameter of $l \sim k\dfrac{\hbar}{p}$, which included in the beam, where k is unknown quantity; due to $D \geq l \geq \Delta x$, the range of values of k should be $\dfrac{p}{\hbar}D \geq k \geq \dfrac{1}{2}$. Similarly, the motion of another photon is restricted to another similarly cylinder. The cylinder also called the light-cylinder, hence the beam of light is composed of a large number of light-cylinders, as shown in Figure 3.

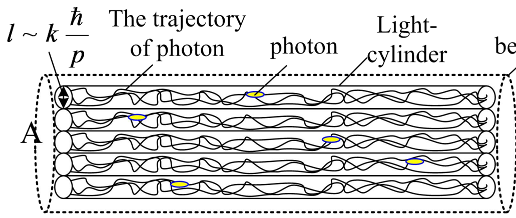

Figure 3

Now we generalize the hypothesis of microstructure of beam to the situation that the light traveling in inhomogeneous medium, in this case the beam is a curve. Suppose that a light ray traveling inhomogeneous medium from a point A to a point B , in a similar way the light beam composed of a large number of curved light-cylinders , and the curved light-cylinders parallel to the curved beam. Here we define the curved light-cylinder as the imaginary curved cylinder region $\mathcal{D}$ that include all possible paths of photon from A to B ; and define the curved light beam as the imaginary space region that composed of all the curved light-cylinders of photons in the beam, i.e. the macroscopic light ray, as shown in Figure 4

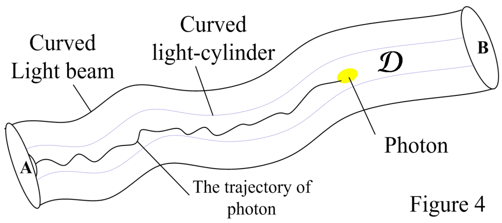

Figure 4

Now we consider the situation that the photon moving in one of curved light-cylinders, at this point, because the motion regions of photon are the regions of curved light-cylinder $\mathcal{D}$, by defining the wave function of photon as $\psi_\gamma(\boldsymbol{x}, s)$, we have the probability density of finding the photon in the infinitesimal volume $\boldsymbol{x}_0$

$$\rho_\gamma(\boldsymbol{x}_0) = \begin{cases} |\psi_\gamma(\boldsymbol{x}_0)|^2, & \boldsymbol{x}_0 \in \mathcal{D} \\ 0, & \boldsymbol{x}_0 \notin \mathcal{D} \end{cases}$$

This condition can be considered to be the characteristics which the wave function of photon $\psi_\gamma(\boldsymbol{x}, s)$ should be have.

Since we have supposed the curved light-cylinders parallel to the curved beam, the tangential unit vector of curved light-cylinder is same as the tangential unit vector of light ray $\boldsymbol{\tau} = \frac{d\boldsymbol{x}}{ds}$, moreover, $\boldsymbol{\tau}$ parallel to the probability current density of photon $\boldsymbol{J}_\gamma$, as demonstrated by the following:

In section 3, we assumed that the wave function of photon had the form $\psi_\gamma = e^{\frac{i}{\hbar}S}$, now we substitute this into the probability current density of photon $\boldsymbol{J}_\gamma$, i.e. Eq(22), we obtain

$$\begin{aligned}\boldsymbol{J}_\gamma &= -\frac{i(1+2m)^2\pi^2}{8\hbar k^2}[e^{-\frac{i}{\hbar}S}\nabla e^{\frac{i}{\hbar}S} - e^{\frac{i}{\hbar}S}\nabla e^{-\frac{i}{\hbar}S}] \\ &= \frac{(1+2m)^2\pi^2}{4\hbar^2 k^2}\nabla S\end{aligned}$$

then substituting the modified eikonal equation (9) into the above equation, we obtain

$$\boldsymbol{J}_\gamma = \frac{(1+2m)^2\pi^2}{4\hbar^2 k^2}(n - \frac{\partial S}{\partial s})\boldsymbol{\tau}$$

As can be seen from the above equation, the tangential unit vector of curved light-cylinder $\boldsymbol{\tau}$ parallel to the probability current density of photon $\boldsymbol{J}_\gamma$, and $\boldsymbol{J}_\gamma$ and $\boldsymbol{\tau}$ point in the same direction. Since $|\boldsymbol{\tau}| = 1$, we have

$$\boldsymbol{\tau} = \frac{\boldsymbol{J}_\gamma}{|\boldsymbol{J}_\gamma|}$$

Since the tangential unit vector of curved light-cylinder $\boldsymbol{\tau}$ belong to geometrical optics,and the probability current density of photon $\boldsymbol{J}_\gamma$ belong to quantization theory, the above equation which is to reflect the relationship between the tangential unit vector $\boldsymbol{\tau}$ and the probability current density $\boldsymbol{J}_{\gamma}$ provide the possibility of transforming the quantization theory of photon into the geometrical theory of light.

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