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# The Schwarzschild Solution and its Implications for Gravitational Waves: Part I

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**Abstract.** The so-called 'Schwarzschild solution' is not Schwarzschild's solution, but a corruption, due to David Hilbert (December 1916), of the Schwarzschild/Droste solution, wherein $m$ is allegedly the mass of the source of an associated gravitational field and the quantity $r$ is said to be able to go down to zero (although no proof of this claim has ever been advanced), so that there are two alleged 'singularities', one at $r = 2m$ and another at $r = 0$. It is routinely claimed that $r = 2m$ is a 'coordinate' or 'removable' singularity which denotes the so-called 'Schwarzschild radius' (event horizon) and that a 'physical' singularity is at $r = 0$. The quantity $r$ in the so-called 'Schwarzschild solution' has never been rightly identified by the physicists, who, although proposing many and varied concepts for what $r$ denotes, effectively treat it as a radial distance from a source of the gravitational field at the origin of coordinates. The consequence of this is that the intrinsic geometry of the metric manifold has been violated. It is easily proven that the said quantity $r$ is in fact the inverse square root of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section of the 'Schwarzschild solution' and so does not in itself define any distance whatsoever in the that manifold. With the correct identification of the associated Gaussian curvature it is also easily proven that there is only one singularity associated with all Schwarzschild metrics, of which there is an infinite number that are equivalent. Thus, the standard removal of the singularity at $r = 2m$ is, in a very real sense, removal of the *wrong* singularity, very simply demonstrated herein. In addition, the 'field equations' $R_{\mu\nu} = 0$ define a spacetime that contains no matter, and since the 'Principle of Superposition' does not apply in General Relativity, it is impossible for Schwarzschild black holes to persist and mutually interact in a mutual spacetime that by construction contains no matter. Consequently, there are no black holes associated with the equations $R_{\mu\nu} = 0$ and therefore no related gravitational waves.

**Keywords:** Schwarzschild Solution, Gravitational Waves, Gaussian Curvature, Proper Radius, Singularity
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## INTRODUCTION

It is reported almost invariably in the literature that Schwarzschild's solution for $R_{\mu\nu} = 0$ is (using $c = 1$, $G = 1$),

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

where it is asserted by inspection that $r$ can go down to zero in some way, producing an infinitely dense point-mass singularity there, with an event horizon at the 'Schwarzschild radius' at $r = 2m$ a black hole. Contrast this metric with that actually obtained by K. Schwarzschild in 1915 (published January 1916),

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2}$$

$$R = R(r) = \left(r^3 + \alpha^3\right)^{1/3}, \qquad 0 < r < \infty,$$

where $\alpha$ is an undetermined constant. There is only *one* singularity in Schwarzschild's solution, at $r = 0$, to which his solution is constructed. Contrary to the usual claims Schwarzschild did not set $\alpha = 2m$ where $m$ is mass; he did not breathe a single word about the bizarre object that is called a black hole; he did not allege the so-called 'Schwarzschild radius'; he did not claim that there is an 'event horizon' (by any other name); and his solution clearly forbids the black hole because when Schwarzschild's $r = 0$, his $R = \alpha$, and so there is no possibility for his $R$ to be less than $\alpha$ let alone take the value $R = 0$. All this can be easily verified by simply reading Schwarzschild's original paper (Schwarzschild, 1916a), in which he *constructs* his solution so that the singularity occurs at the *"origin"* of coordinates. Thus, eq. (1) for $0 \le r < 2m$ is *inconsistent* with Schwarzschild's true solution, eq. (2). It is also inconsistent with the intrinsic geometry of the line-element, whereas eq. (2) is geometrically consistent, as demonstrated below. Thus, eq. (1) for $0 \le r < 2m$ is meaningless.

In the usual interpretation of Hilbert's (Abrams, 1989; Antoci, 2001; Loinger, 2002) version, eq. (1), of Schwarzschild's solution, the quantity $r$ therein has ***never*** been properly identified. It has been variously and vaguely called "*the radius*" of a sphere (Mould, 1994; Dodson and Poston, 1991; Carroll, 1997), the "*radius of a 2-sphere*" (Bruhn, 2008), the "*coordinate radius*" (Wald, 1984), the "*radial coordinate*" (Carroll and Ostile, 1996; Misner, Thorne and Wheeler, 1970), the "*radial space coordinate*" (Zel'dovich and Novikov, 1996), the "*areal radius*" (Wald, 1984; Ludvigsen, 1999), the "*reduced circumference*" (Taylor and Wheeler, 2000), and even "*a gauge choice: it defines the coordinate r*" ('t Hooft, 2008). In the particular case of $r = 2m = 2GM/c^2$ it is almost invariably referred to as the "*Schwarzschild radius*" or the "*gravitational radius*". However, none of these various and vague concepts of what $r$ is are correct because the irrefutable geometrical fact is that $r$, in the spatial section of Hilbert's version of the Schwarzschild/Droste line-element, is the inverse square root of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section (Levi-Civita, 1977; Schwarzschild, 1916b; Crothers, 2005), and as such it ***does not*** of itself determine the geodesic radial distance from the centre of spherical symmetry located at an arbitrary point in the related pseudo-Riemannian metric manifold. It does not of itself determine any distance at all in the spherically symmetric metric manifold. It is the radius of Gaussian curvature merely by virtue of its formal geometric relationship to the Gaussian curvature. It must also be emphasized that a geometry is completely determined by the ***form*** of its line-element (Tolman, 1987).

Since $r$ in eq. (1) can be replaced by *any* analytic function $R_c(r)$ (Abrams, 1989; Loinger, 2002; Levi-Civita, 1977; Crothers, 2005; Eddington, 1960) without disturbing spherical symmetry and without violation of the field equations $R_{\mu\nu} = 0$, which is very easily verified, satisfaction of the field equations is a necessary but insufficient condition for a solution for Einstein's static vacuum 'gravitational' field. Moreover, the admissible *form* of $R_c(r)$ must be determined in such a way that an infinite number of equivalent metrics is generated thereby (Crothers, 2005; Eddington, 1960). In addition, the identification of the origin of coordinates and the properties of points must also be clarified in relation to the non-Euclidean geometry of Einstein's gravitational field. In relation to eq. (1) it has been routinely presumed that geometric points in the spatial section (which is non-Euclidean) have the very same properties of points in the spatial section (Euclidean) of Minkowski spacetime. However, it is easily proven that the non-Euclidean geometric points in the spatial section of Einstein's non-Euclidean gravitational field do not possess the same characteristics of the Euclidean geometric points in the spatial section of Minkowski spacetime (Crothers, 2005; Brillouin, 1923). This should not be surprising, since the indefinite metric of Einstein's Theory of Relativity admits of other geometrical oddities, such as null vectors, i.e. non-zero vectors that have zero magnitude and which are orthogonal to themselves (Foster and Nightingale, 1993).

## 3-D SPHERICALLY SYMMETRIC METRIC MANIFOLDS

A line-element or squared differential element of arc-length, in spherical coordinates, for 3-dimensional Euclidean space is,

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{3}$$

$$0 \le r < \infty$$

The scalar $r$ can be construed, by calculation, as the magnitude of the radius vector **r** from the origin of the coordinate system, the said origin coincident with the centre of the associated sphere. All the components of the metric tensor are well-defined, and related geometrical quantities are fixed by the line-element. Indeed, the radius $R_p(r)$ of the associated sphere ($\theta = const.$, $\varphi = const.$) is given by,

$$R_p = \int_0^r dr = r,$$

the circumference $C_p$ of a great circle ($\theta = \pi/2$, $r = const.$) is,

$$C_p = r\int_0^{2\pi} d\varphi = 2\pi r,$$

the area $A_p$ of the spherically symmetric surface ($r = const.$) is,

$$A_p = r^2\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi r^2,$$

and the volume $V_p$ of the sphere is,

$$V_p = \int_0^r r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{4}{3}\pi r^3.$$

Consider the generalisation of eq. (3) to a non-Euclidean 3-dimensional spherically symmetric metric manifold by the line-element,

$$ds^2 = dR_p^2 + R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = \Psi(R_c)dR_c^2 + R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{4}$$

$$R_c = R_c(r), \qquad R_c(0) \le R_c(r) < \infty,$$

where both $\Psi(R_c(r))$ and $R_c(r)$ are *a priori* unknown analytic functions. Since neither $\Psi(R_c(r))$ nor $R_c(r)$ are known, eq. (4) may or may not be well-defined at $R_c(0)$ one cannot know until $\Psi(R_c(r))$ and $R_c(r)$ are somehow specified. With this proviso, there is a one-to-one point-wise correspondence between the manifolds described by eqs. (3) and (4), i.e. a mapping, as the differential geometers have explained (Levi-Civita, 1977). If $R_c(r)$ is constant, eq. (4) reduces to a 2-dimensional spherically symmetric geodesic surface described by,

$$ds^2 = R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5}$$

If $r$ is constant, eq. (3) reduces to the 2-dimensional spherically symmetric surface described by,

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{6}$$

A surface is a manifold in its own right. It need not be considered in relation to an embedding space. Therefore, quantities appearing in its line-element must be identified in relation to the surface, not to any embedding space it might be in:

> ``*And in any case, if the metric form of a surface is known for a certain system of intrinsic coordinates, then all the results concerning the intrinsic geometry of this surface can be obtained without appealing to the embedding space."* (Efimov, 1980)

Note that eqs. (3) and (4) have the same metrical form and that eqs. (5) and (6) have the same metrical form. Metrics of the same form share the same fundamental relations between the components of their respective metric tensors. For example, consider eq. (4) in relation to eq. (3). For eq. (4), the radial geodesic distance (i.e. the proper radius) from the point at the centre of spherical symmetry ($\theta = const.$, $\varphi = const.$) is,

$$R_p = \int_0^{R_p} dR_p = \int_{R_c(0)}^{R_c(r)} \sqrt{\Psi(R_c(r))}\, dR_c(r) = \int_0^r \sqrt{\Psi(R_c(r))}\frac{dR_c(r)}{dr}dr,$$

the circumference $C_p$ of a great circle ($\theta = \pi/2$, $R_c(r) = const.$) is,

$$C_p = R_c(r)\int_0^{2\pi} d\varphi = 2\pi R_c(r),$$

the area $A_p$ of the spherically symmetric geodesic surface ($R_c(r) = const.$) is,

$$A_p = R_c^2(r)\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi R_c^2(r),$$

and the volume $V_p$ of the geodesic sphere is,

$$V_p = \int_0^{R_p} R_c^2(r)\,dR_p \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\varphi = 4\pi \int_{R_c(0)}^{R_c(r)} \sqrt{\Psi(R_c(r))}\,R_c^2(r)\,dR_c(r)$$

$$= 4\pi \int_0^{r} \sqrt{\Psi(R_c(r))}\,R_c^2(r)\frac{dR_c(r)}{dr}\,dr$$

In the case of the 2-dimensional metric manifold given by eq. (5) the Riemannian curvature associated with eq. (4) (which depends upon both position and direction) reduces to the Gaussian curvature $K$ (which depends only upon position), and is given by (Levi-Civita, 1997; Crothers, 2007; Crothers, 2008; Kay, 1998; Kreyszig, 1991; McConnell, 1957; Pauli, 1981; Struik, 1988),

$$K = \frac{R_{1212}}{g}, \tag{7}$$

where $R_{1212}$ is a component of the Riemann tensor of the 1st kind and $g = g_{11}g_{22} = g_{\theta\theta}g_{\varphi\varphi}$ (because the metric tensor of eq. (5) is diagonal). Now recall from elementary differential geometry and tensor analysis that

$$R_{\mu\nu\rho\sigma} = g_{\mu\gamma}R^{\gamma}_{.\nu\rho\sigma} \qquad R^{1}_{.212} = \frac{\partial\Gamma^1_{22}}{\partial x^1} - \frac{\partial\Gamma^1_{21}}{\partial x^2} + \Gamma^k_{22}\Gamma^1_{k1} - \Gamma^k_{21}\Gamma^1_{k2}$$

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial\left(\frac{1}{2}\ln|g_{ii}|\right)}{\partial x^j} \qquad \Gamma^i_{jj} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i}, \quad (i \neq j) \tag{8}$$

and all other $\Gamma^i_{jk}$ vanish. In the above, $i,j,k = 1,2$; $x^1 = \theta$, $x^2 = \varphi$. Applying expressions (7) and (8) to expression (5) gives,

$$K = \frac{1}{R_c^2} \tag{9}$$

so that $R_c(r)$ is the inverse square root of the Gaussian curvature, i.e. the radius of Gaussian curvature, and hence, in eq. (6) the quantity $r$ therein is the radius of Gaussian curvature because this Gaussian curvature is intrinsic to all geometric surfaces having the form of eq. (5) (Levi-Civita, 1977), and a geometry is completely determined by the *form* of its line-element (Tolman, 1987). Indeed, any 2-dimensional surface has an intrinsic Gaussian curvature. Note that according to eqs. (3), (6) and (7), the radius calculated for (3) gives the same value as the associated radius of Gaussian curvature of a spherically symmetric surface in the space of eq. (3). Thus, the Gaussian curvature (and hence the radius of Gaussian curvature) of the spherically symmetric surface in the space of (3) can be associated with the calculated radius, from eq. (3). This is a consequence of the Euclidean nature of the space of eq. (3), which describes the spatial section of Minkowski spacetime. However, this is *not* a general relationship. The radius of Gaussian curvature does not directly determine any distance at all in Einstein's gravitational manifold but in fact determines the Gaussian curvature of the spherically symmetric geodesic surface through any point in the spatial section of the gravitational manifold, as proven by expression (9). Thus, the quantity $r$ in eq. (1) is the inverse square root of the Gaussian curvature (i.e. the radius of Gaussian curvature) of a spherically symmetric geodesic surface in the spatial section, not the radial geodesic distance from the centre of spherical symmetry of the spatial section or any other distance, of itself, in the manifold. This simple geometric fact subverts most of the usual claims made for eq. (1).

## THE STANDARD DERIVATION

The usual derivation[a] begins with the following metric for Minkowski spacetime (using $c = 1$),

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{10}$$

and proposes a generalisation thereof as or equivalent to,

[a] (See references marked with *.)

$$ds^2 = e^{\lambda}dt^2 - e^{\beta}dr^2 - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{11}$$

where $\lambda$, $\beta$ and $R$ are all unknown functions of only $r$, to be determined, and so that the signature of (10) is maintained. The form of $R(r)$ is then *assumed* so that $R(r) = r$, to get,

$$ds^2 = e^{\lambda}dt^2 - e^{\beta}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{12}$$

It is then required that $e^{\lambda}$ and $e^{\beta}$ be determined[b] so as to satisfy $R_{\mu\nu} = 0$. Now note that eq. (12) not only retains the signature -2, but also retains the signature (+,–,–,–), because $e^{\lambda} > 0$ and $e^{\beta} > 0$. Thus, neither $e^{\lambda}$ nor $e^{\beta}$ can change sign.

The Standard analysis then obtains the solution given by eq. (1), wherein the constant $m$ is claimed to be the mass generating the alleged gravitational field. By inspection of (1) the Standard analysis asserts that there are two singularities, one at $r = 2m$ and one at $r = 0$. It is claimed that $r = 2m$ is a removable coordinate singularity, and that $r = 0$ a physical singularity. It is also asserted that $r = 2m$ gives the event horizon (the 'Schwarzschild radius') of a black hole and that $r = 0$ is the position of the infinitely dense point-mass singularity of the black hole, produced by irresistible gravitational collapse.

However, these claims cannot be true. First, the construction of eq. (12) to obtain eq. (1) in satisfaction of $R_{\mu\nu} = 0$ is such that neither $e^{\lambda}$ nor $e^{\beta}$ can change sign, because $e^{\lambda} > 0$ and $e^{\beta} > 0$. Therefore the claim that $r$ can take values less than $2m$ is false; a contradiction by the very construction of the metric (12) leading to metric eq. (1). Furthermore, since neither $e^{\lambda}$ nor $e^{\beta}$ can ever be zero, the claim that $r = 2m$ is a removable coordinate singularity is also false. In addition, the true nature of $r$ in both eqs. (12) and (1) is entirely overlooked, and the geometric relations between the components of the metric tensor, fixed by the *form* of the line-element, are not applied, in consequence of which the Standard analysis fatally falters. To highlight further, rewrite eq. (11) as,

$$ds^2 = A(R_c)dt^2 - B(R_c)dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{13}$$

where $A(R_c)$, $B(R_c)$, $R_c(r) > 0$. The solution for $R_{\mu\nu} = 0$ then takes the form,

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1}dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{14}$$

$$R_c = R_c(r),$$

where $\alpha$ is a constant. It remains to determine the admissible form of $R_c(r)$, which, from the foregoing, is the inverse square root of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section of the manifold associated with eq. (14), owing to the metrical *form* of eq. (14). From above, the proper radius for a metric of the *form* of eq. (14) is,

$$R_p = \int \frac{dR_c}{\sqrt{1 - \frac{\alpha}{R_c}}} = \sqrt{R_c(R_c - \alpha)} + \alpha \ln\left(\sqrt{R_c} + \sqrt{R_c - \alpha}\right) + k \tag{15}$$

where $k$ is a constant. Now for some $r_o$, $R_p(r_o) = 0$. Then by (15) it is required that $R_c(r_o) = \alpha$ and $k = -\alpha\ln\sqrt{\alpha}$ so

$$R_p(r) = \sqrt{R_c(R_c - \alpha)} + \alpha \ln\left(\frac{\sqrt{R_c} + \sqrt{R_c - \alpha}}{\sqrt{\alpha}}\right) \tag{16}$$

$$R_c = R_c(r),$$

It is thus also determined that the Gaussian curvature of the spherically symmetric geodesic surface of the spatial section ranges not from $\infty$ to 0, as it does for Euclidean 3-space, but from $\alpha^{-2}$ to 0. This is an inevitable consequence of the non-Euclidean geometry described by eq. (14).

Schwarzschild's true solution, eq. (2), must be a particular case of the general expression sought for $R_c(r)$. Brillouin's solution (Abrams, 1989; Brillouin, 1923) must also be a particular case, viz.,

$$ds^2 = \left(1 - \frac{\alpha}{r + \alpha}\right)dt^2 - \left(1 - \frac{\alpha}{r + \alpha}\right)^{-1}dr^2 - (r + \alpha)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad 0 < r < \infty, \tag{17}$$

[b] ($\lambda$ and $\beta$ are taken as real valued functions of the real variable $r$.)

and Droste's solution (Droste, 1917) must as well be a particular solution, viz.,

$$ds^2 = \left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad \alpha < r < \infty, \tag{18}$$

These particular solutions must all be particular cases in an infinite set of equivalent metrics (Eddington, 1923). The only admissible form for $R_c(r)$ is (Crothers, 2005; Crothers, 2005b; Crothers, 2005c),

$$R_c(r) = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n} = \frac{1}{\sqrt{K(r)}}, \qquad r \in \mathbf{R}, \quad n \in \mathbf{R}^+, \quad r \neq r_o. \tag{19}$$

where $r_o$ and $n$ are entirely arbitrary constants. So the solution for $R_{\mu\nu} = 0$ is,

$$ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{20}$$

$$R_c(r) = \left(\left|r - r_o\right|^n + \alpha^n\right)^{1/n} = \frac{1}{\sqrt{K(r)}}, \qquad r \in \mathbf{R}, \quad n \in \mathbf{R}^+, \quad r \neq r_o.$$

Then if $r_o = 0$, $r > r_o$, $n = 1$, Brillouin's solution eq. (17) results. If $r_o = 0$, $r > r_o$, $n = 3$, then Schwarzschild's actual solution eq. (2) results. If $r_o = \alpha$, $r > r_o$, $n = 1$, then Droste's solution eq. (18) results, which is the correct solution in the particular metric of eq. (1). In addition the required infinite set of equivalent metrics is thereby obtained, all of which are asymptotically Minkowski spacetime. Furthermore, if the constant $\alpha$ is set to zero, eq. (20) reduces to Minkowski spacetime, and if in addition $r_o$ is set to zero, that the usual Minkowski metric of eq. (10) is obtained.

It is clear from expression (20) that there is only ever one singularity, at the arbitrary constant $r_o$, where $R_c(r_o) = \alpha$ for all $r_o$ for all $n$ and $R_p(r_o) = 0$ for all $r_o$ for all $n$, and that all components of the metric tensor are affected by the constant $\alpha$. Hence, the "removal" of the singularity at $r = 2m$ in eq. (1) is fallacious, and, in a very real sense, is a removal of the *wrong* singularity, because it is clear from expression (20) and the *form* of the line-element at eq. (13), in accordance with the intrinsic geometry of the line-element as given in above and the generalisation at eq. (11), that there is no singularity at $r = 0$ in eq. (1) and that $0 \leq r \leq 2m$ therein is meaningless. The Standard claims for eq. (1) violate the geometry fixed by form of its line-element and contradict the generalisations at eqs. (11) and (12) from which it has been obtained by the Standard method. There is therefore no black hole associated with eq. (1) since there is no black hole associated with eq. (2) and none with eq. (20), of which Schwarzschild's actual solution, eq. (2), Brillouin's solution, eq. (17), and Droste's solution, eq. (18), are just particular equivalent cases. Consequently, there can be no gravitational waves generated by black holes since the latter are fictitious.

The usual form of eq. (1) in isotropic coordinates is,

$$ds^2 = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} dt^2 - \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

where it is again usually alleged that $r$ can go down to zero. This expression has the very same metrical form as eq. (13) and so shares the very same geometrical character. Now the coefficient of $dt^2$ is zero when $r = m/2$, which, according to the physicists marks the 'radius' or 'event horizon' of a black hole, and where $m$ is the alleged point-mass of the black hole singularity located at $r = 0$, just as in eq. (1). This further amplifies the fact that the quantity $r$ appearing in both eq. (1) and its isotropic coordinate form is not a distance in the manifold described by these line-elements. Applying the intrinsic geometric relations detailed above it is clear that for the isotropic coordinate metric,

$$R_c(r) = r\left(1 + \frac{m}{2r}\right)^2, \qquad R_p(r) = r + m\ln\left(\frac{2r}{m}\right) - \frac{m^2}{2r} + \frac{m}{2}.$$

Hence, $R_c(m/2) = 2m$, and $R_p(m/2) = 0$, which are scalar invariants necessarily consistent with eq. (20). Furthermore, applying the same geometrical analysis leading to eq. (20), the generalised solution in isotropic coordinates is (Crothers, 2006),

$$ds^2 = \frac{\left(1-\frac{\alpha}{4h}\right)^2}{\left(1+\frac{\alpha}{4h}\right)^2} dt^2 - \left(1+\frac{\alpha}{4h}\right)^4 \left[dh^2 + h^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

$$h = h(r) = \left[\left|r - r_o\right|^n + \left(\frac{\alpha}{4}\right)^n\right]^{1/n}, \qquad r \in \mathbf{R}, \quad n \in \mathbf{R}^+, \quad r \neq r_o.$$

wherein $r_o$ and n are entirely arbitrary constants. Then,

$$R_c(r) = h(r)\left(1+\frac{\alpha}{4h(r)}\right)^2 = \frac{1}{\sqrt{K(r)}}, \qquad R_p(r) = h(r) + \frac{\alpha}{2}\ln\left(\frac{4h(r)}{\alpha}\right) - \frac{\alpha^2}{8h(r)} + \frac{\alpha}{4}.$$

and so

$$R_c(r_o) = \alpha, \quad R_p(r_o) = 0, \quad \forall r_o \ \forall n$$

which are scalar invariants, in accordance with eq. (20). Clearly in these isotropic coordinate expressions $r$ does not in itself denote any distance in the manifold, just as it does not in itself denote any distance in eq. (20) of which eqs. (1) and (2) are particular cases. It is a parameter for the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section and for the proper radius (i.e. the radial geodesic distance from the point at the centre of spherical symmetry of the spatial section).

Doughty (1981) has shown that the radial geodesic acceleration $a$ of a point in a manifold described by a line-element with the *form* of eq. (14) is given by

$$a = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)\left|g_{00,1}\right|}{2g_{00}}$$

Applying this to eq. (1) gives,

$$a = \frac{2m}{r^{3/2}\sqrt{r-2m}}$$

and so the radial geodesic acceleration is infinite at $r = 2m$, where, according to the usual interpretation of eq. (1), **there is no matter**! But it is plain from eq. (20) that the acceleration is given by,

$$a = \frac{\alpha}{R_c^{3/2}\sqrt{R_c - \alpha}}$$

and this is infinite at $R_c(r_o) = \alpha$, precisely where $R_p(r_o) = 0$, irrespective of the values of $r_o$ and n.

For eq. (1), when $2m < r < \infty$, the signature of eq. (1) is (+,–,–,–). But if $0 < r < 2m$ in eq. (1), then

$$g_{00} = \left(1-\frac{2m}{r}\right) \text{ is negative and } g_{11} = -\left(1-\frac{2m}{r}\right)^{-1} \text{ is positive.}$$

So the signature of eq. (1) changes to (–,+,–,–). Thus the rôles of t and r are interchanged. According to Misner, Thorne and Wheeler (1970), *"The most obvious pathology at r = 2M is the reversal there of the roles of t and r as timelike and spacelike coordinates. In the region r > 2M, the t direction, ∂=∂t, is timelike ($g_{tt}$ < 0) and the r direction, ∂=∂r, is spacelike ($g_{tt}$ > 0); but in the region r < 2M, ∂=∂t, is spacelike ($g_{tt}$ > 0) and ∂=∂r, is timelike ($g_{tt}$ < 0).*

*"What does it mean for r to `change in character from a spacelike coordinate to a timelike one'? The explorer in his jet-powered spaceship prior to arrival at r = 2M always has the option to turn on his jets and change his motion from decreasing r (infall) to increasing r (escape). Quite the contrary in the situation when he has once allowed himself to fall inside r = 2M. Then the further decrease of r represents the passage of time. No command that the traveler can give to his jet engine will turn back time. That unseen power of the world which drags everyone forward willy-nilly from age twenty to forty and from forty to eighty also drags the rocket in from time coordinate r = 2M to the later time coordinate r = 0. No human act of will, no engine, no rocket, no force (see exercise 31.3) can make time stand still. As surely as cells die, as surely as the traveler's watch ticks away 'the unforgiving minutes,' with equal certainty, and with never one halt along the way, r drops from 2M to 0.*

*"At r = 2M, where r and t exchange roles as space and time coordinates, $g_{tt}$ vanishes while $g_{rr}$ is infnite."*

To amplify this, set t = -r* and r = -t*, so that for 0 < r < 2m, eq. (1) becomes,

$$ds^2 = -\left(1+\frac{2m}{t*}\right)^{-1} dt*^2 + \left(1+\frac{2m}{t*}\right)dr*^2 - t*^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad -2m < t* < 0.$$

But this is now a *time-dependent* metric since all the components of the metric tensor are functions of the time *t**, and so this metric bears *no relationship* to the original *time-independent problem* to be solved. In other words, this metric is a *non-static solution to a static problem:- contra-hype!* Thus, in eq. (1), $0 \le r \le 2m$ is meaningless. Both Droste (1917) and Brillouin (1923) drew particular attention to this consequence.

It is also frequently claimed (Misner, Thorne and Wheeler, 1970; Kruskal, 1960; d'Inverno, 1992) in relation to eq. (1) that since the Riemann tensor scalar curvature invariant (the Kretschmann scalar) is finite at $r = 2m$, the latter is a 'coordinate singularity' or 'removable singularity'. But it has never been proven that Einstein's theory requires a singularity where the Kretschmann scalar is unbounded. In fact, it is not required. The Kretschmann scalar is not an independent curvature invariant. Although the Kretshmann scalar depends upon the components of the metric tensor, all the components of the metric tensor are functions of the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section, owing to the *form* of the line-element, in consequence of which the Kretschmann scalar is constrained by the intrinsic Gaussian curvature of the spherically symmetric geodesic surface in the spatial section. Recall that the Kretschmann scalar $f$ is,

$$f = R_{\alpha\beta\gamma\delta}\, R^{\alpha\beta\gamma\delta}.$$

Then by eq. (20),

$$f = 12\alpha^2 K^3 = \frac{12\alpha^2}{R_c^6} = \frac{12\alpha^2}{\left(\left|r - r_o\right|^n + \alpha^n\right)^{6/n}}$$

and so,

$$f\left(r_o\right) = \frac{12}{\alpha^4} \quad \forall r_o\ \forall n$$

Furthermore, Hagihara (1931) proved, in relation to eq. (1), that all geodesics that do not run into the boundary at $r = 2m$ are complete. The boundary at $r = 2m$ is the boundary marked by the point at the centre of spherical symmetry of the spatial section of the manifold, precisely where the radial geodesic distance is $R_p = 0$.

Einstein's field equations are non-linear, so the 'Principle of Superposition' does not apply. Before one can talk of relativistic binary systems it must first be proven that the two-body system is well-defined by General Relativity. This can be done in only two ways: (a) Derivation of an exact solution to Einstein's field equations for two bodies; or (b) Proof of an existence theorem. There are no known solutions to Einstein's field equations for the interaction of two (or more) masses. No existence theorem has ever been proven for latent solutions for such configurations of matter. The 'Schwarzschild' black hole is alleged from a line-element satisfying $R_{\mu\nu} = 0$. Since $R_{\mu\nu} = 0$ is by construction a Universe that contains no matter, a second black hole cannot simply be inserted into the spacetime of $R_{\mu\nu} = 0$ of a given black hole so that the resulting two black holes, each obtained *separately* from $R_{\mu\nu} = 0$, mutually persist and interact in a mutual spacetime that by construction ***contains no matter***! One cannot simply assert by an analogy with Newton's theory that two black holes can be components of binary systems, collide or merge (McVittie, 1978). Moreover, General Relativity has to date been unable to account for the simple experimental fact that two fixed bodies will attract one another upon release. The signatures of the black hole, an infinitely dense point-mass singularity and an event horizon, have never been identified anywhere, and so no black hole has ever been found. The Michell-Laplace dark body is not a black hole (McVittie, 1978).

## CONCLUSION

'Schwarzschild's solution' is not Schwarzschild's true solution. Black holes cannot be obtained from Schwarzschild's actual solution without violation of the intrinsic geometry of his solution. The quantity *r* appearing in 'Schwarzschild's solution' has never been correctly identified by the physicists. It is irrefutably the inverse square root of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section, not a distance of any kind in the manifold. The signatures of the black hole, an infinitely dense point-mass singularity and an event horizon, have never been identified anywhere, and so no black hole has ever been found. Since $R_{\mu\nu} = 0$ is by construction a spacetime devoid of all matter, the notion of associated black holes interacting is invalid. Neither Newton's theory nor Einstein's theory predicts black holes. The black hole is fictitious, and so no gravitational waves can be generated by black holes.

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