Volume Enclosed by Example Subdivision Surfaces with Sharp Creases

by Jan Hakenberg - May 27th, 2014

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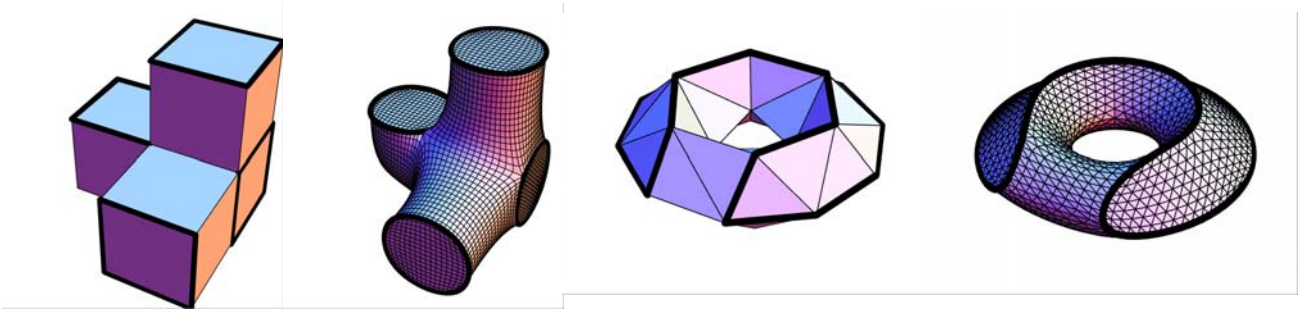


Figure: a) Several rounds of Catmull-Clark subdivision with sharp creases. Four unit cubes are glued together, and four cycles are defined as creases. The limit surface encloses a volume of 3.0782373525914434605842... The exact value in symbolic form is stated in a section below. b) Several iterations of Loop subdivision with sharp creases. In the article, we define the torus mesh and obtain the volume enclosed by the limit surface. ■

Abstract

The formula for the volume enclosed by subdivision surfaces has been identified only recently. We present example meshes with cycles of edges defined as sharp creases, and state the volume enclosed by their limit surface defined by Catmull-Clark, and Loop subdivision. The article can serve as a reference for future implementations of the volume formula.

Introduction

Surface subdivision schemes are designed to produce surfaces that appear smooth everywhere. Subdivision with creases is a simple extension to the algorithm that provides the option to model sharp features in the surface. Across the crease, the surface normal is generally not continuous, see the illustration above.

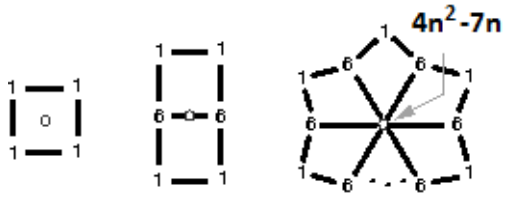
An early use of creases in subdivision surfaces, was to model the fingernails of the character *Geri* in Pixar's 1997 short film *Geri's game*. Subdivision surfaces with creases are part of modern surface modeling tools [Autodesk 2013].

[Nasri 1987] introduces subdivision rules along the boundary of open meshes. [Hoppe et al. 1994] generalizes the subdivision rules for the Loop scheme to handle creases in the interior of the mesh. [DeRose et al. 1998] presents the extension for the Catmull-Clark subdivision scheme. Along the edge cycle selected as crease, cubic B-spline subdivision rules for curves applies. In particular, points outside the crease do not affect the refinement of the curve. In the limit, the crease is identical to a cubic B-spline curve.

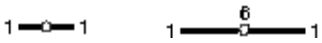
In this article, we are concerned with Catmull-Clark and Loop subdivision surfaces with one or more sharp creases. The volume enclosed by the limit surface is well defined for closed, orientable meshes. The general concept is derived in [Hakenberg et al. 2014]. We plan a publication in the near future, that elaborates on the derivation of the trilinear forms that are required when computing the volumes defined by meshes with creases.

Catmull-Clark with sharp creases

The Catmull-Clark subdivision scheme is published as [Catmull/Clark 1978]. The algorithm applies to meshes with quads. Weights are specified for the insertion of a face midpoint, and edge midpoint, as well as the repositioning of a vertex.



Subdivision rules along the creases are identical to cubic B-spline subdivision for curves:



The weights are subject to normalization so that their sum adds up to 1.

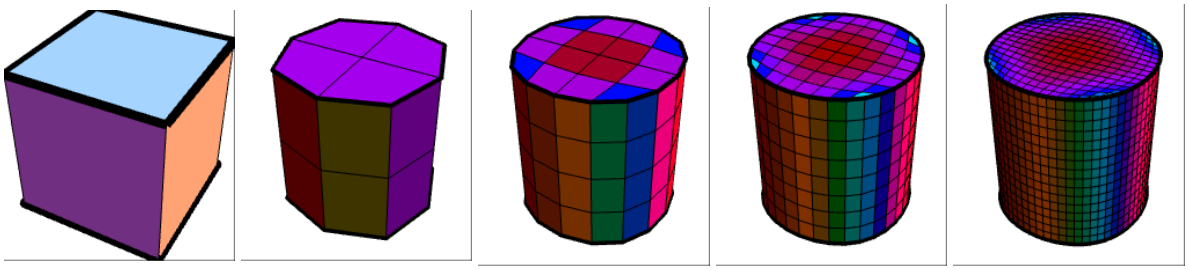
Subdivision rules along the creases have precedence over rules for the suface, [DeRose et al. 1998].

Cylinder

```
Vertices ↓
{{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {1, 1, 0}, {0, 0, 1}, {1, 0, 1}, {0, 1, 1}, {1, 1, 1}}

Faces ↓
{{1, 2, 6, 5}, {3, 4, 2, 1}, {4, 8, 6, 2}, {5, 6, 8, 7}, {7, 8, 4, 3}, {7, 3, 1, 5}}

Cycles ↓
{{1, 2, 4, 3}, {5, 6, 8, 7}}
```

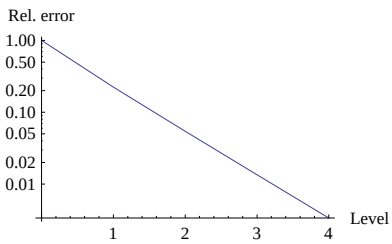


```
Required valences τ(f) ∈ {3}

Limit volume ↓ (≈0.677777777777777777777777777778)
```

$$\frac{61}{90}$$

Level	Volume	Delta to ∞
0	1	0.322222
1	$\frac{3}{4}$	0.0722222
2	$\frac{89}{128}$	0.0175347
3	$\frac{1397}{2048}$	0.00435113
4	$\frac{22\,245}{32\,768}$	0.00108575

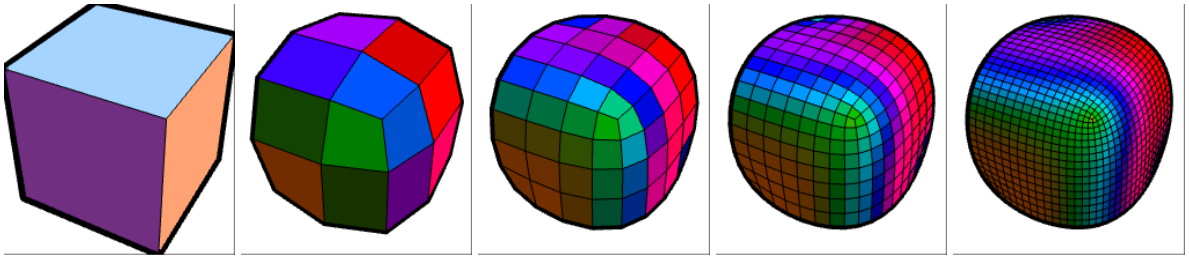


Turtle

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

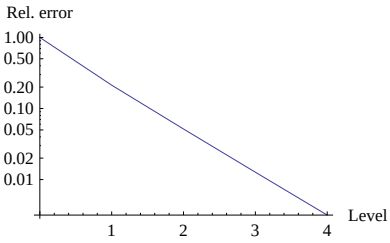
{{1, 2, 4, 8, 7, 5}}



Limit volume ↓ (≈0.57947172465866032756882175641 2)

1132154292988905962687822319635655294248426411657595675604634439231767415911402174231221505199986433239557895200733 /
1953769691965220683285616338283141695785866973316757373073867704148625084032621163935197511801774547016289468416000

Level	Volume	Delta to ∞
0	1	0.420528
1	$\frac{257}{384}$	0.0897991
2	$\frac{709\,069}{1\,179\,648}$	0.0216135
3	$\frac{8\,680\,738\,420\,613}{14\,843\,406\,974\,976}$	0.00534943
4	$\frac{423\,746\,016\,064\,084\,729}{729\,583\,139\,634\,020\,352}$	0.00133393

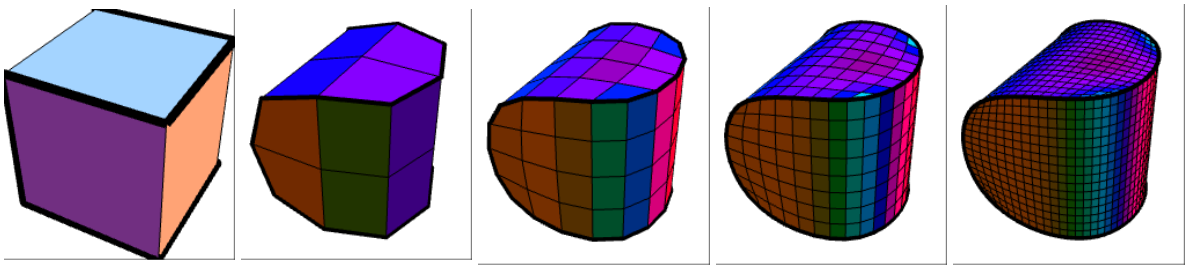


Tennis ball

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

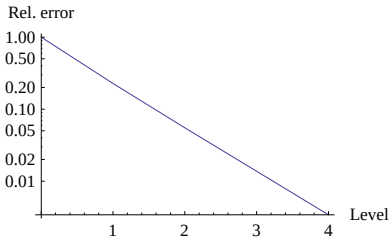
{{1, 2, 4, 3, 7, 8, 6, 5}}



Limit volume ↓ (≈0.67617246228116918094670794749 7)

$$\frac{997\,309\,875\,392\,669}{1\,474\,934\,178\,816\,000}$$

Level	Volume	Delta to ∞
0	1	0.323828
1	$\frac{3}{4}$	0.0738275
2	$\frac{45\,487}{65\,536}$	0.0179041
3	$\frac{91\,350\,169}{134\,217\,728}$	0.00443933
4	$\frac{744\,677\,191\,601}{1\,099\,511\,627\,776}$	0.0011075

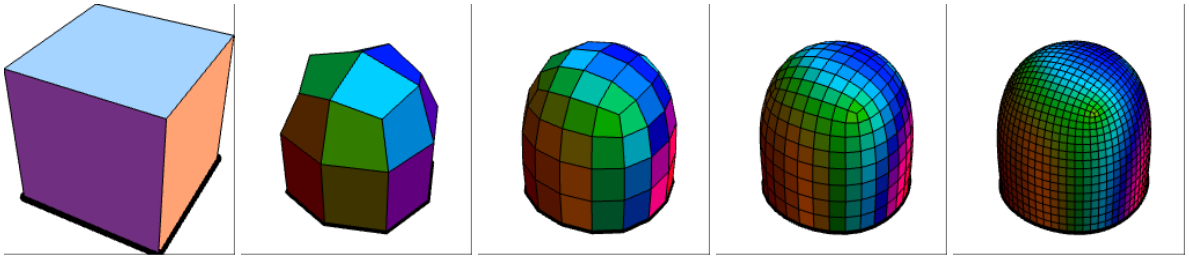


Zuckerhut

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

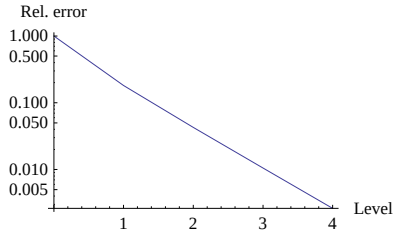
$$\begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix}$$



Limit volume ↓ $(\approx 0.49731213753788524000940902731 7)$

$$\frac{7689526519121125559036887642490636525275343056784148302536575374141986015343991700279478229073628113693043100761}{1546217342932905474803467742070683706484174155726989225393095105383865479302807174193952109975024311756907151360 0}$$

Level	Volume	Delta to ∞
0	1	0.502688
1	$\frac{113}{192}$	0.0912295
2	$\frac{5\,508\,035}{10\,616\,832}$	0.02149
3	$\frac{25\,178\,791\,892\,939}{50\,096\,498\,540\,544}$	0.00529369
4	$\frac{3\,683\,399\,265\,599\,706\,331}{7\,387\,029\,288\,794\,456\,064}$	0.00131852



Corner Tube

Vertices ↓

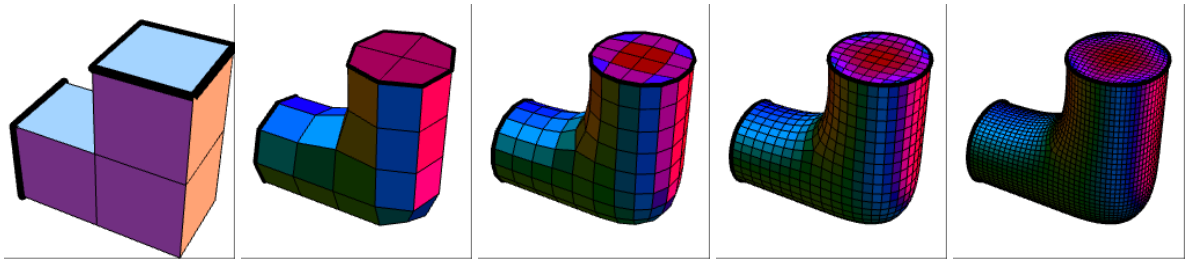
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 4 & 2 & 1 \\ 5 & 6 & 8 & 7 \\ 7 & 8 & 4 & 3 \\ 7 & 3 & 1 & 5 \\ 2 & 11 & 13 & 6 \\ 6 & 13 & 15 & 9 \\ 4 & 12 & 11 & 2 \\ 12 & 14 & 13 & 11 \\ 14 & 16 & 15 & 13 \\ 9 & 15 & 16 & 10 \\ 6 & 9 & 10 & 8 \\ 4 & 8 & 14 & 12 \\ 8 & 10 & 16 & 14 \end{pmatrix}$$

Cycles ↓

$$\begin{pmatrix} 1 & 5 & 7 & 3 \\ 9 & 10 & 16 & 15 \end{pmatrix}$$

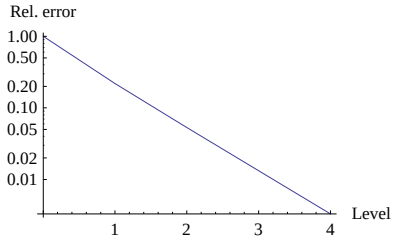


Required valences $\tau(f) \in \{3, 4, 5\}$

Limit volume ↓ $(\approx 2.093102901747952121847120876$

653943157064618658630222204296662874762028527065312848770446741445617046546552317399817604380961350951745012352498697464416439212627471993075577310229237168700371842308878769366167476855484976847385490084318523874402433496961665764191429448984778388655119062616737296198944836838846318507183829548080536247146265835534351600526448884360455002220017784873984134482457822050678097153 /
312427619549191842462961233606806117810975531600110184429159201604673131169732873155252557751325447357072111100575774207474914703352062520322321995476903637927753317184861172764371975433111200004753888314382155304647237844173311474661967792210407945938440106045054904777178727588896341480707155138016118109130888469049096272144957257791687356078934062172396501583369608234188800000

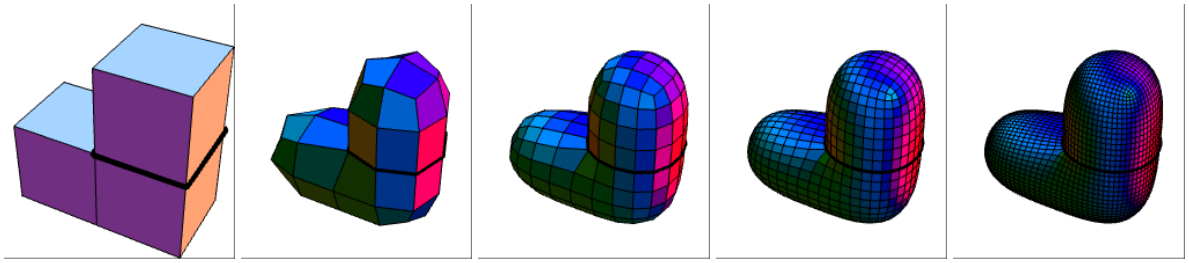
Level	Volume	Delta to ∞
0	3	0.906897
1	$\frac{22\,001}{9600}$	0.198668
2	$\frac{17\,051\,108\,689}{7\,962\,624\,000}$	0.0482903
3	$\frac{131\,822\,400\,326\,669\,681}{62\,620\,623\,175\,680\,000}$	0.0119927
4	$\frac{12\,903\,272\,369\,191\,902\,384\,803}{6\,155\,857\,740\,662\,046\,720\,000}$	0.00299368



Corner Ring

The vertex coordinates and topology of the mesh are specified in a section above.

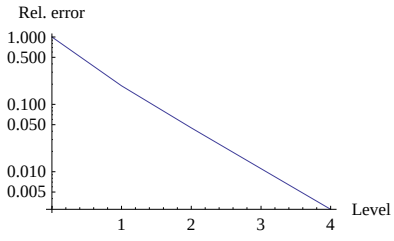
Cycles ↓
{6 13 14 8}



Limit volume ↓ $(\approx 1.647908813910382310252064213$

116993697797859002487652750894776076001262873714353314630960588875362971920090296897903751538539373406848198636070073798505305279862346885339929632578762623884755927875975548745845402000151540357882719940479342748791084719386009704027333463 51 /
709952497433643999872014759981048097656976673759372951174754774049802927585341345728287926089048805477100580493491701455175963430395253078033758924830514714197575658693337075642299303769981560959065822447822287965086016093904393995360665600

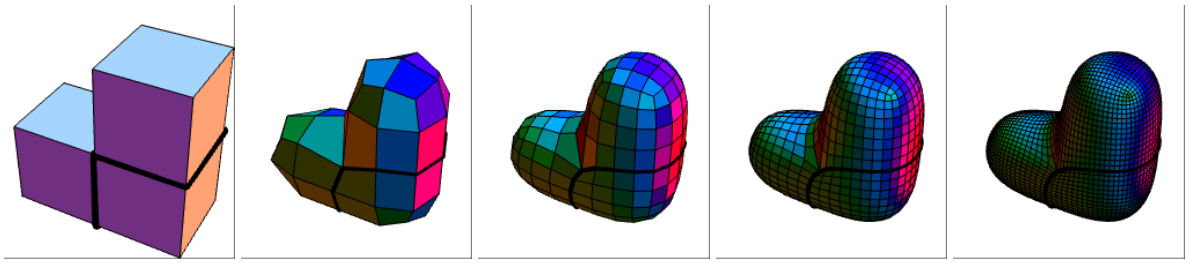
Level	Volume	Delta to ∞
0	3	1.35209
1	$\frac{731}{384}$	0.255737
2	$\frac{870\,716\,083}{509\,607\,936}$	0.0606911
3	$\frac{666\,438\,252\,865\,261}{400\,771\,988\,324\,352}$	0.0149775
4	$\frac{97\,605\,765\,716\,000\,959\,141}{59\,096\,234\,310\,355\,648\,512}$	0.00373222



Corner Brace

The vertex coordinates and topology of the mesh are specified in a section above.

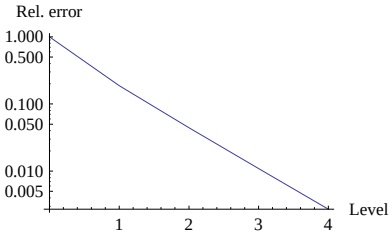
Cycles ↓
{2 6 13 14 8 4}



Limit volume ↓ (≈1.72153905085557105565906380580)

1745277357801287534995100519251326336842139532491585318236737633654902834708932375428577012912689659307207444997143533932409309229039
1124237310504600974624164435169421787258950617363762391707746149619444429246727174541169840265959184143440341313994158900970562946213
6514504792403702507457840675517 /
1013789002888966711256324730898792058924211076008360685519544203771842435929836168979986948912559249892748702074047235076111750844310
3212877748016297380225221513791929428695285001025564002697793476845187649195805083451540979742547294222255661071334747187955565081223
7993645336217165699940149248000

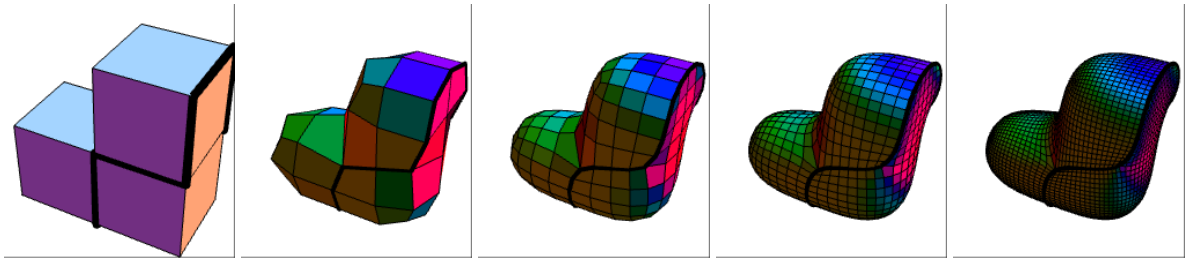
Level	Volume	Delta to ∞
0	3	1.27846
1	$\frac{251}{128}$	0.239398
2	$\frac{453\,069\,443}{254\,803\,968}$	0.0565708
3	$\frac{695\,528\,303\,052\,109}{400\,771\,988\,324\,352}$	0.0139323
4	$\frac{101\,941\,446\,489\,560\,392\,993}{59\,096\,234\,310\,355\,648\,512}$	0.00346843



Corner Shoe

The vertex coordinates and topology of the mesh are specified in a section above.

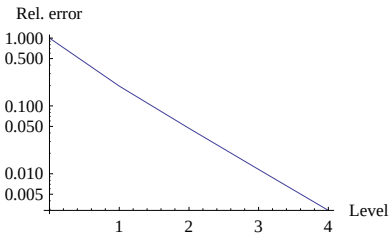
Cycles ↓
(2 6 13 15 16 14 8 4)



Limit volume ↓ (≈1.87822860728054740794676838889)

4188136528598074026103468734503564087625163561618390252464436570841538316039721031152648928038493370311804745130925188780120272218564
2024106582174562302830467511558657864270375309106277926292545778875111561729224015225825145612555839709234584991929907716218725562724
1125520666708237202243929170226562340305936841902928307547024273964401842463613049813595104142313159384067891074341182979003328618953
25428770748597 /
2229833212189223193037100715687882810508066800230477598019012287087330089449178223994914593062665670147831572882713386600234744389635
7019440434628091249160091507489874036109561900055566673675595556537698325973157994163890020531612862756264718408795836261258223708410
5741302693088548775328036463600213491738769515833868236921111950763715519594038135782647896864730073939383776304991311577898727582976
22903318118400

Level	Volume	Delta to ∞
0	3	1.12177
1	$\frac{403}{192}$	0.22073
2	$\frac{983\,934\,521}{509\,607\,936}$	0.0525391
3	$\frac{757\,935\,358\,140\,383}{400\,771\,988\,324\,352}$	0.0129598
4	$\frac{111\,186\,970\,467\,454\,112\,633}{59\,096\,234\,310\,355\,648\,512}$	0.00322749

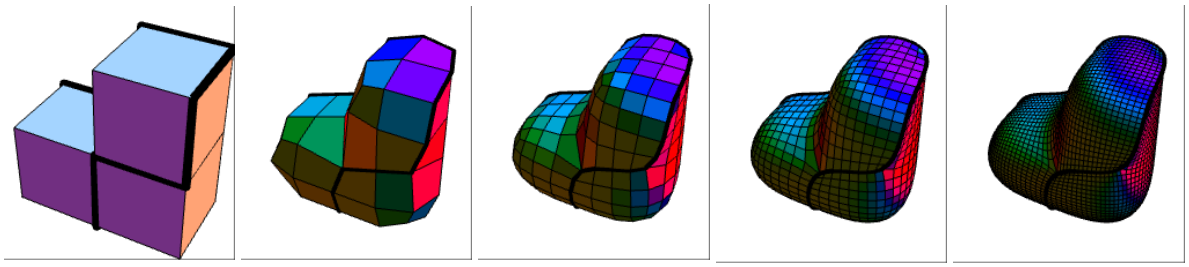


Corner Circuit

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

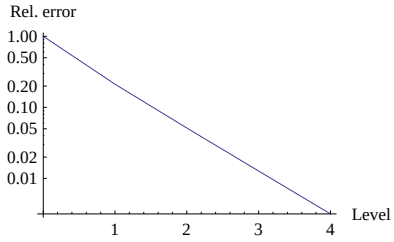
```
{ 2  6  13  15  16  10  8  7  3  4 }
```



Limit volume ↓ (≈2.095659190108903422065202968 89)

508122116017444472266017537187460219349254440341413134574449887615773782406844514219104351712976032806325738268017531438959975435493680344719766011669840583591784654583290492539760338583538222284769263310452629731767868932506809518927565351437930092352123855007529283425005736730833046073481936797429097057290768380356929183977307827002477908266527144486904100541814111824098159611559765291232233056846940510653088476 89 /
242464098368513490128007131729902446674697936573401851266536814671103029544897634679316796780840126711170578071134289732262796437576279107270774314936281042247686424422038875810838777707192906671866214115989868512383177022612849557610529893417062107094967439166001211012973749043249104007683310181678730203889244805385794986532414867275160783121106409668347722293864130087142398759369796232352164773018943918810726400 0

Level	Volume	Delta to ∞
0	3	0.904341
1	$\frac{1757}{768}$	0.192101
2	$\frac{1\,091\,603\,531}{509\,607\,936}$	0.0463866
3	$\frac{211\,121\,355\,820\,861}{100\,192\,997\,081\,088}$	0.0114876
4	$\frac{82\,676\,557\,220\,075\,493\,373}{39\,397\,489\,540\,237\,099\,008}$	0.0028643

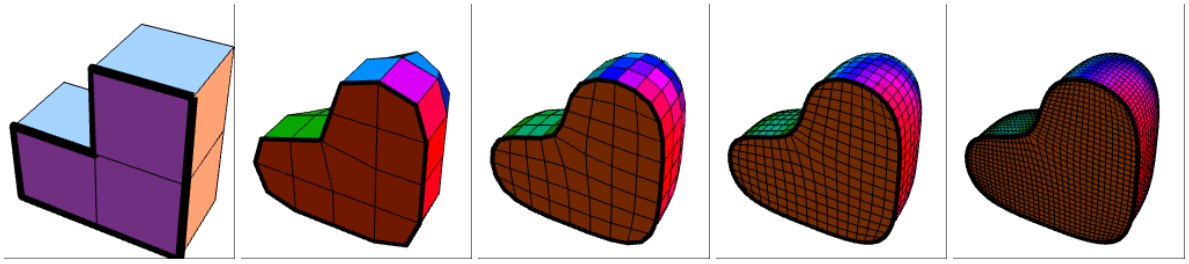


Flat-X Corner

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

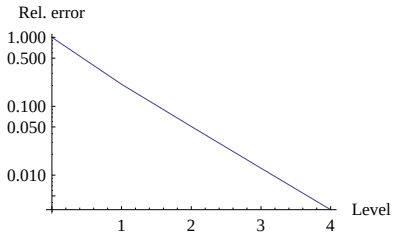
```
{ 1  2  11  13  15  9  6  5 }
```



Limit volume ↓ (≈2.189954886691970404583635984 83)

502888758244429957405910692476896096023850159621449137586343372781351720302385103640699205577802072991115952001705894143778677338937033676474574919765401551708663929848603656019420933607729201543749355511519131120329294081765224677016583577508099695098372923746584065606252040566470165569270163015922265728847396254712773448404382744239974910555948938481057785122367088439246589561 / 229634300368656004210276506701002496591067015726080985555432013179434751409753661769110629947224203807448001658923194042494062306963765952436906666675524173876898688130872961981813401943336732003494107911070884148915719815467383933876546327274649840264754779431153550112263647778388109883197590264418468102112030247510857600265435844768902067180165356967114286637766620521287680000

Level	Volume	Delta to ∞
0	3	0.810045
1	$\frac{45\,301}{19\,200}$	0.169472
2	$\frac{142\,121\,426\,551}{63\,700\,992\,000}$	0.041116
3	$\frac{551\,101\,913\,735\,327\,647}{250\,482\,492\,702\,720\,000}$	0.0102065
4	$\frac{26\,993\,463\,821\,766\,752\,751\,677}{12\,311\,715\,481\,324\,093\,440\,000}$	0.00254736

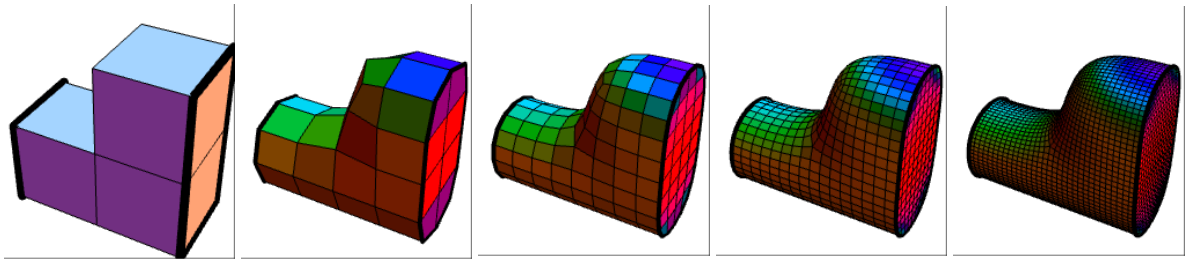


Flat-Y Corner

The vertex coordinates and topology of the mesh are specified in a section above.

Edges ↓

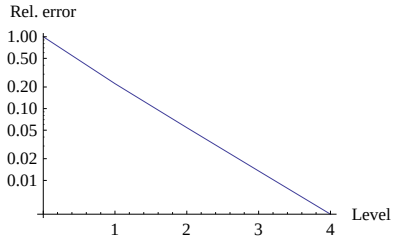
$$\begin{pmatrix} \{11, 12, 14, 16, 15, 13\} \\ \{1, 3, 7, 5\} \end{pmatrix}$$



Limit volume ↓ (≈2.250151274156192685098460623 03)

1497715691491296160889245950338853537208550878548404845223947069330513650846012254245647633322444542038681800226421023516594294903113
1490985714648055828344929007031679980481265638383269281645868830607835639668529210255433079936634943120511817249432447467283956481890
3462875389742539549285609067082350670322419253301605376336388571668532967001553039163339347177887120474396229237227 /
6656066677352347948123956715971086857712087412350173494360348208099558011876917732437989273832585617607188453881831711376639487158370
0276068668599036383818515043098008948684632409681722706299131447567510455328738816150671149966357645375747036130388482537239983511697
1047032941385040031850202200076643034232496240651452422492830561114425768715124685610715105590329787426238054400 00

Level	Volume	Delta to ∞
0	3	0.749849
1	$\frac{23\,201}{9600}$	0.16662
2	$\frac{72\,966\,026\,641}{31\,850\,496\,000}$	0.0407401
3	$\frac{141\,540\,499\,487\,418\,671}{62\,620\,623\,175\,680\,000}$	0.0101344
4	$\frac{27\,734\,382\,472\,486\,770\,138\,709}{12\,311\,715\,481\,324\,093\,440\,000}$	0.00253094

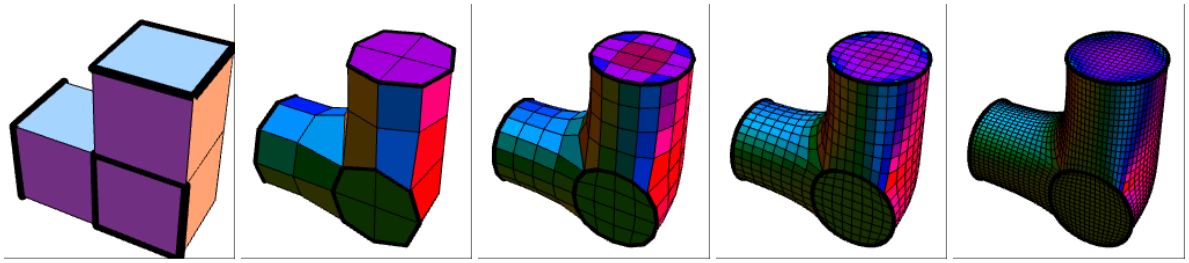


Corner Split

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

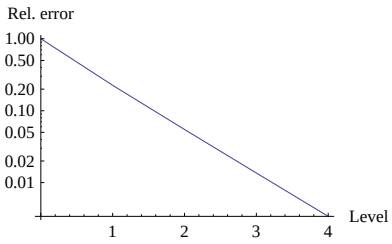
$$\begin{pmatrix} 1 & 5 & 7 & 3 \\ 9 & 10 & 16 & 15 \\ 2 & 6 & 13 & 11 \end{pmatrix}$$



Limit volume ↓ (≈2.20624380979957375919242727746)

1003590578952546710005936807324888585122482477441026192245464857144192840742308069186775794044502654554097350203347880297872280429987
2492889366585374980440146026913162961525489453914797363485729267902979935667413786900679080180304917188743920718282379020491527582440
6615386627089464782977879714121711013940177609779175963427715947889912769713416938707557562050031348635637694243333801765601385303463
1123933458383649222569027431689284593154522563092714660421357096870429705044231713352141949735970553790527910646155410773118451269083
9805869071142787599935960888568838873695006290272256685345994522249930079007629462433372851364448457265227 /
4548865245512996617158896118161787847484958760136216876167078027847294060907122369020481616504141179378639935664299051888391163023732
9208087336371977669730955744372376654427086692287630768756716807370526557846884059559151691581299829782971488411204581284349243933085
6812335196628199351531700627978202645789320169159236883759728959286824059952549492211382991779615295850275733377392304713618778078336
4122648368435508878536732519356439810565156317044859947177621633725261890878376399025173301688175544307771366979740523184650876046752
9925503597251093272627418797905265582330131336310406625435208113572758298396684395545591083764101120000

Level	Volume	Delta to ∞
0	3	0.793756
1	$\frac{15\,267}{6400}$	0.179225
2	$\frac{18\,430\,861}{8\,192\,000}$	0.0436172
3	$\frac{10\,284\,018\,897\,184\,763}{4\,638\,564\,679\,680\,000}$	0.0108254
4	$\frac{335\,751\,838\,109\,636\,656\,609}{151\,996\,487\,423\,754\,240\,000}$	0.00270091



Tripod Tube

Vertices ↓

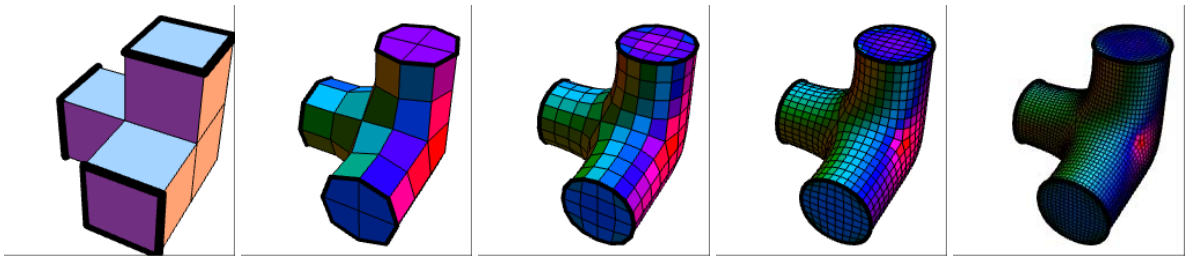
0	0	0
1	0	0
0	1	0
1	1	0
0	0	1
1	0	1
0	1	1
1	1	1
1	0	2
1	1	2
2	0	0
2	1	0
2	0	1
2	1	1
2	0	2
2	1	2
1	−1	0
2	−1	0
1	−1	1
2	−1	1

Faces ↓

1	2	6	5
3	4	2	1
5	6	8	7
7	8	4	3
7	3	1	5
6	13	15	9
4	12	11	2
12	14	13	11
14	16	15	13
9	15	16	10
6	9	10	8
4	8	14	12
8	10	16	14
17	18	20	19
19	20	13	6
11	13	20	18
11	18	17	2
17	19	6	2

Cycles ↓

1	5	7	3
9	10	16	15
17	18	20	19



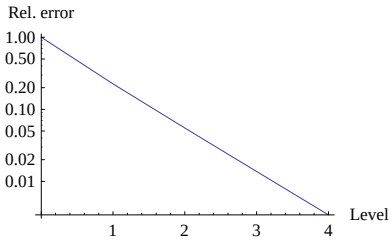
Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓ (≈3.05470967910788273403290950488)

2360071053974376239087836864127384328705651194984288871460757892568237913732025988168584095045998288070559665287463549829565215997990
8076431964844697355912509425219461313672590719378157042450428281259474143687706010439736133536776438428108791408386087330475289118710
7049662077372491197005291894976769034034402619855961045936710894373719820514764011155156724802564655838401183878752423957946433745578
3621936733341846417160870897569 /

772600771233889148773404741111123957538534364188783894555149279879735273707314476728604496559096979280431611698767301084043579264193552280887298563846148727399305553064826942610094308006791178675173515278915821693399374388464776072990163194305801386942630998683589481269429498286784360890529966870285443656397300655933729231132868504337820030532536875370502993824094092876940497059008626391555858836865632961642972833406484480000

Level	Volume	Delta to ∞
0	4	0.94529
1	$\frac{62\,753}{19\,200}$	0.213676
2	$\frac{3\,665\,094\,167}{1\,179\,648\,000}$	0.0522291
3	$\frac{14\,229\,729\,522\,299\,009}{4\,638\,564\,679\,680\,000}$	0.0129913
4	$\frac{1\,394\,394\,853\,301\,323\,455\,917}{455\,989\,462\,271\,262\,720\,000}$	0.00324444

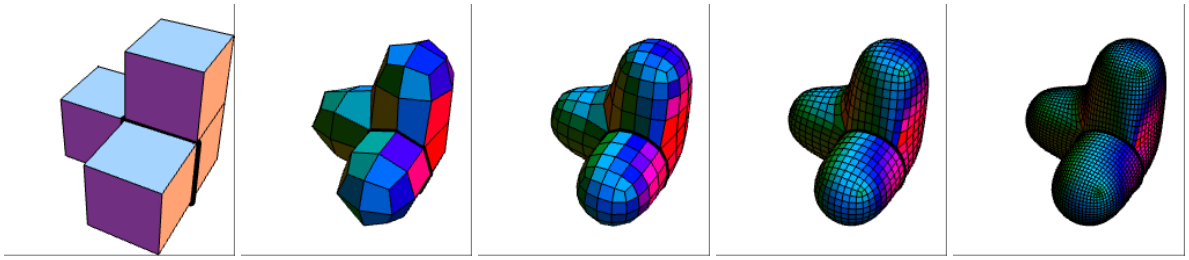


Tripod Ring

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

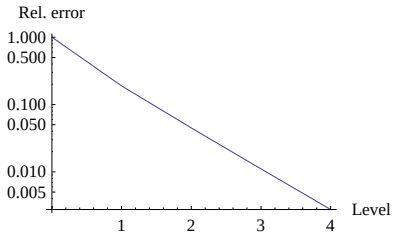
{2 11 13 6}



Limit volume ↓ (≈2.33666278608402541191533813335)

337676313427601628278223414584211711999769861119222883232937235142350793703534562311002798868211182707898835024637200185360169539474682568902186627600681812409162978641549505711648669406400250328225139815365389571632848593873494330879322609771707942065049582755369009195204245600649444606736768446131936228451039388118951609846543975214993970525629623965626321871237465667081055717181035136890287451915171577268804746662803572608416346940099014377632077807772829867568173742119013428327289827047772408554414558185287793251648701317518280261641419086576448172023744267906040442657965834074086697245154049501326373908401388083998738824471796732881347141181051101281467640667618419131 /
144512214359140707408817771320372924167793822293860830733300025491529368649187992067451366571461429944618757250691788874870130123557526762377663088442799488437891654410753633430947116437264380572210316737248807017161940958330237558704060946186928523861457822740527763843858358404428463539341772789821033278714050591573550358670995896562135330400278085490833435234653503186952623844708957304427030548382122701128295591049449689251029286947228361462481214562624171874990334466282264337724929786885498339725942622888759223724302028142985167117054434370693491744195031284540700351200398116264220642159865035215586322077249196093908350959470103631026937592488030596025934731114577920000

Level	Volume	Delta to ∞
0	4	1.66334
1	$\frac{16\,967}{6400}$	0.314431
2	$\frac{5\,689\,142\,783}{2\,359\,296\,000}$	0.0747103
3	$\frac{98\,318\,674\,937\,863\,297}{41\,747\,082\,117\,120\,000}$	0.0184401
4	$\frac{14\,412\,448\,319\,980\,303\,534\,231}{6\,155\,857\,740\,662\,046\,720\,000}$	0.00459475

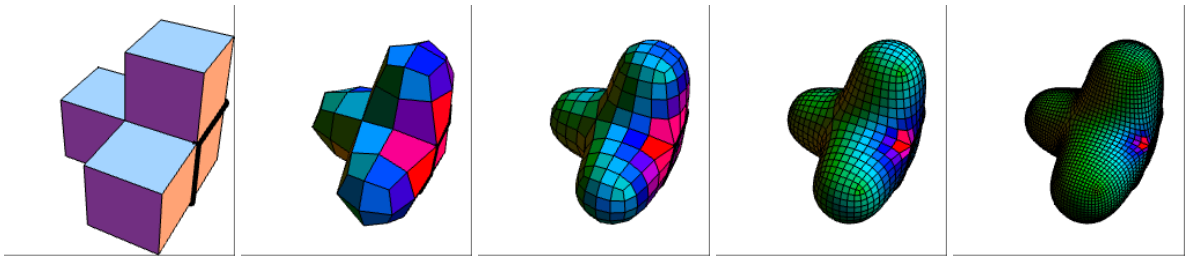


Tripod Brace

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

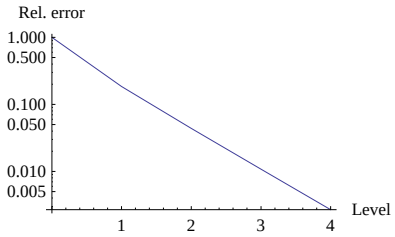
{11 13 14 8 4 2}



Limit volume ↓ (≈2.4932291029499505909204523129)

254175289525181517823162546323219493481316480397325216349774247061755490790050768756886086943294033198147023753651164827000978060012487307281046368078694810833429066080431537169800880625746352781666923245261527224063297010002355229338492125938748812322473914032420031341738487265605607724563973159794132940724179639202164139545623690542211692 3 /
101946222761656840038957507431125989789582064331649549637307263786471206367984914165644840932506739218936240306265255028565593409008290342774423824777439129400312131704394294707880870394991615875133637781340159912871153641117887601864417172065967237180770437883630618742989657372694294469480060498213364412993071222588608638837720133063147520 0

Level	Volume	Delta to ∞
0	4	1.50677
1	$\frac{2129}{768}$	0.278906
2	$\frac{144\,905\,555}{56\,623\,104}$	0.0658951
3	$\frac{670\,479\,511\,091\,597}{267\,181\,325\,549\,568}$	0.0162259
4	$\frac{98\,386\,084\,346\,636\,723\,647}{39\,397\,489\,540\,237\,099\,008}$	0.00403876

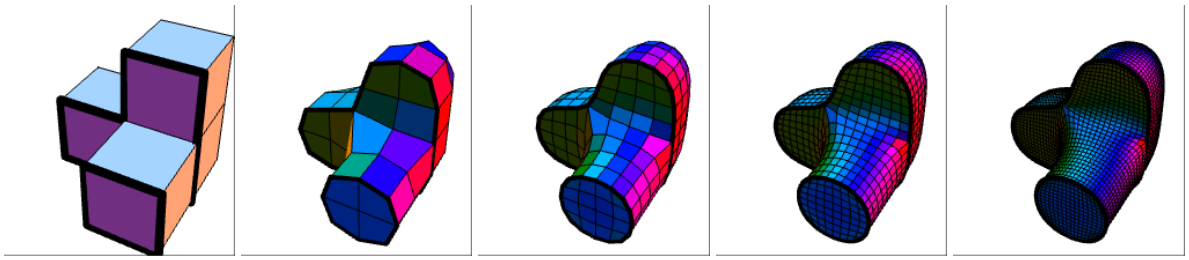


Tripod Bolt

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

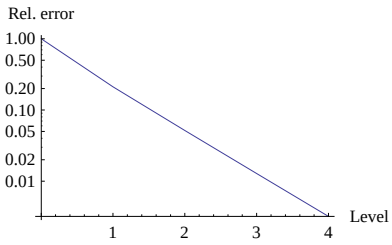
$$\begin{pmatrix} \{2, 11, 13, 15, 9, 6, 5, 1\} \\ \{17, 18, 20, 19\} \end{pmatrix}$$



Limit volume ↓ (≈3.035348521392355027369184535960)

1178382059493902145980831748268480200593984285690926043157299301067646304091007195271679444145722376079839299666509374403800614431378497787584594217135998376249352137383915792703091904972236017668805279702536927161045888156570435722294027824701715194527260607702802696078766291549157671029313844878011306782365758908452616464767443381163233892506825245417693480179577397815126543756342073061099026808028359356770776341201541634587007060942753347055320552045635793390445079993769929248351985144067670975903124527351967565970108724931059046988371898530982999732827113071567815429070966089510262627454902712498159117933154920819762676016847110845931 3 /
388219689168795189964860314687390292538392893197969575671635317393252791060046988714974060382212400220041363096944352826856190699493223285310522003574050845984894474041407052805934729644750532107876405266367182326598562988203661616928723400446197325483058123169309779803533252820125522441777195009745102516611997285044647801167741878263799675100028493240414503991334511326743253364977352326066160950451261934065999636062280386453412944910983721242095284352818828277482223260762047680347839149294100851641974618362423107814529486086539124371752539035284671926118605437408531877599755181612253261254967272230020599744496630760223346140018632294400 0

Level	Volume	Delta to ∞
0	4	0.964651
1	$\frac{62\,201}{19\,200}$	0.204287
2	$\frac{98\,261\,237\,213}{31\,850\,496\,000}$	0.0497286
3	$\frac{381\,699\,086\,808\,347\,261}{125\,241\,246\,351\,360\,000}$	0.0123622
4	$\frac{4\,676\,044\,545\,641\,311\,516\,069}{1\,538\,964\,435\,165\,511\,680\,000}$	0.00308722



Tripod Tube

The vertex coordinates and topology of the mesh are specified in a section above.

Vertices ↓

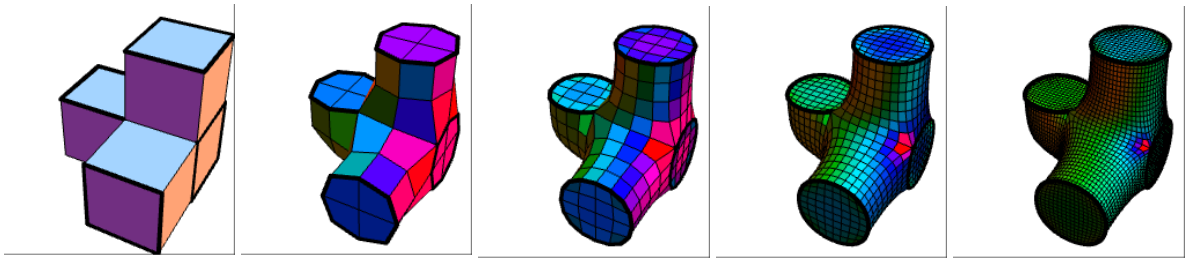
0	0	0
1	0	0
0	1	0
1	1	0
0	0	1
1	0	1
0	1	1
1	1	1
1	0	2
1	1	2
2	0	0
2	1	0
2	0	1
2	1	1
2	0	2
2	1	2
1	−1	0
2	−1	0
1	−1	1
2	−1	1

Faces ↓

1	2	6	5
3	4	2	1
5	6	8	7
7	8	4	3
7	3	1	5
6	13	15	9
4	12	11	2
12	14	13	11
14	16	15	13
9	15	16	10
6	9	10	8
4	8	14	12
8	10	16	14
17	18	20	19
19	20	13	6
11	13	20	18
11	18	17	2
17	19	6	2

Cycles ↓

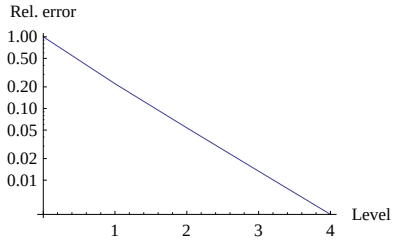
5	7	8	6
17	18	20	19
9	15	16	10
11	12	14	13



Limit volume ↓ (≈3.078237352591443460584219183 35)

3469341584350434074454856586941340668353801463824161306740196016927353011315805284166601604591139253682234057295029351682472946663846
1135472697847708620935435659797133780136323868515088405742691010520269919435740582093750068637355065893416272956750551479951520582642
6489992537815606778236372856690016893306819288312541246864897900101541012392339065496033659797753670682718438696980013979570541530441
5398118322701814044827420501068517762999198757761491496905952643868401489786067400595740152302860957735071326551308772436027472207550
8400829988921910514953440084860524812677558277306780727801846879337649908375411716228407111765965051700655022405643085632259186946051
5967349064243761204112696829211906763749873310835971988086829061746088894100454309858187210111582719724202011749087134576588927992241
9715348359993483251139415836572228834790697240859276609851042890184992473441683403161557968676767940441178767675466674662816924778953
7135567330288866862845997593 /
1127054605269387621428274066036397597248594048603634537141968144694503602203344880600687547803196866743858924871135983585064558495810
0592413568106513709684640947006899138804824576069655056234457957542534557674264818162357077819930759274073926965061028013866016482474
2818407968422669837804542934326915419155661749104683335684174536654843019407548298134701346002868298597268055939343784853577843630037
0424240954178891532480412637870797103278666721806501699536082121086487573110766137360101251130993839377214126451380136803649699429395
1708357529041324705431978320272807677488152614505992231990685004516689176790363472856498021964851516674090316545488943675266184247860
2271390533329460955249304145402454057336608198259207359912385363425512965303570846479311650921682030275601095181778526614882869197940
6312234431690111151963794718010220790942331811811798807507853088788749034272805407361844647614548023313056348260518808186035380918518 5
2442097435306921992192000000

Level	Volume	Delta to ∞
0	4	0.921763
1	$\frac{31\,513}{9600}$	0.204367
2	$\frac{199\,250\,633\,861}{63\,700\,992\,000}$	0.0496674
3	$\frac{96\,766\,414\,746\,064\,849}{31\,310\,311\,587\,840\,000}$	0.0123232
4	$\frac{18\,968\,116\,454\,531\,383\,089\,353}{6\,155\,857\,740\,662\,046\,720\,000}$	0.00307434

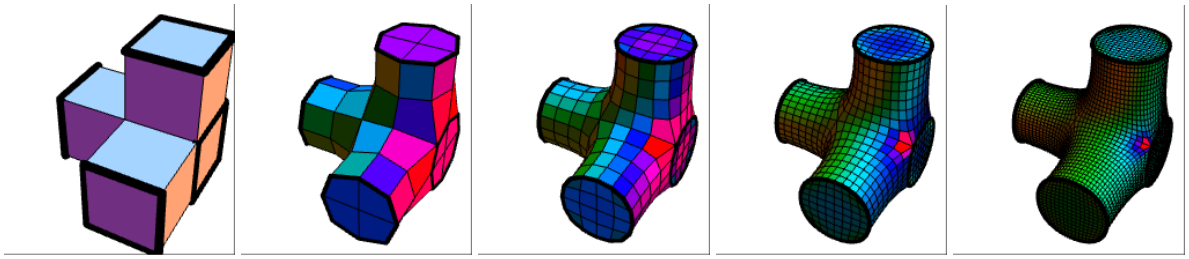


Tripod Split

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

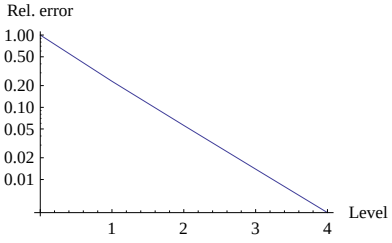
1	5	7	3
17	18	20	19
9	15	16	10
11	12	14	13



Limit volume ↓ (≈3.17188314820403482264337098604)

532589449055416020343787800435149661425011818244151537756663567348767155329201724904682449106959568769419596106981179258495993985335973591665236358341290734757930135210362124725913602119663352609080900796983133123979552923376591674331392214848192575700651899568467048635323231646850245675422762132393630789337173177553827195712163221588219856510982536694286961078647591547918553222802405773603473023145589546351226109258913201787424988166841137641629703667839297800884279954171510588253282725969553116933595409598279230542528387171472115282143124361115612167903839835277950530734545248108303112388695243165626567848694379171999414866872090291679095427449701960362659258376326082691438873996872922806582714527444646787277939214740556576091355499684477937934681694311152870208897325923601403372388371971252746464319428401567067175701807551284196218229166338285743196154876326640690343426046230930157765805773674724736788648224925497107630885745228374926818968951926748312129823622894673171245952438256286890081009653184498228634224000375529363801991585445004404326272875829976271496462558015078326191154884580486564155367953363202511808002257411954146895300817498213200573588519779232372824922745168082908034666307130731921641974343320412725251902725414826325693783677468961505280000 0

Level	Volume	Delta to ∞
0	4	0.828117
1	$\frac{8069}{2400}$	0.1902
2	$\frac{2\,531\,015\,087}{786\,432\,000}$	0.046469
3	$\frac{3\,281\,456\,117\,624\,203}{1\,030\,792\,151\,040\,000}$	0.0115483
4	$\frac{5\,361\,702\,516\,654\,204\,647}{1\,688\,849\,860\,263\,936\,000}$	0.0028825

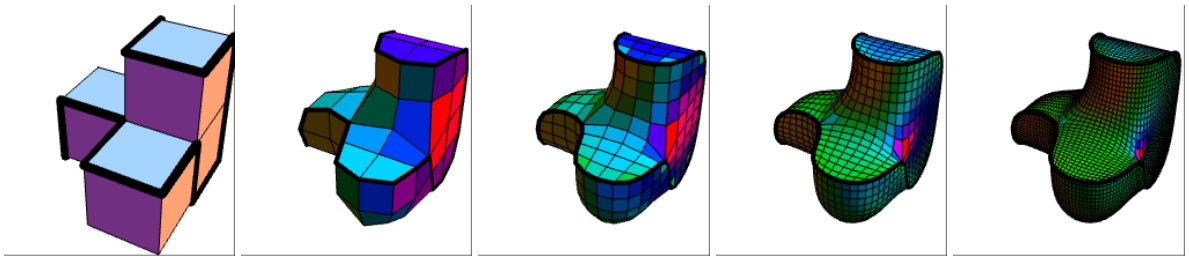


Tripod Circuit

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

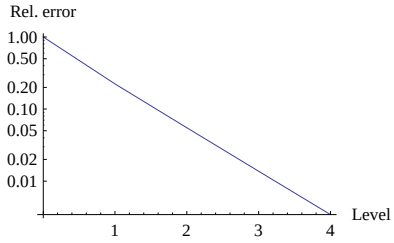
(6 19 20 13 11 12 14 16 15 9 10 8 4 3 1 5)



Limit volume ↓ (≈3.27481658832701233335143933691

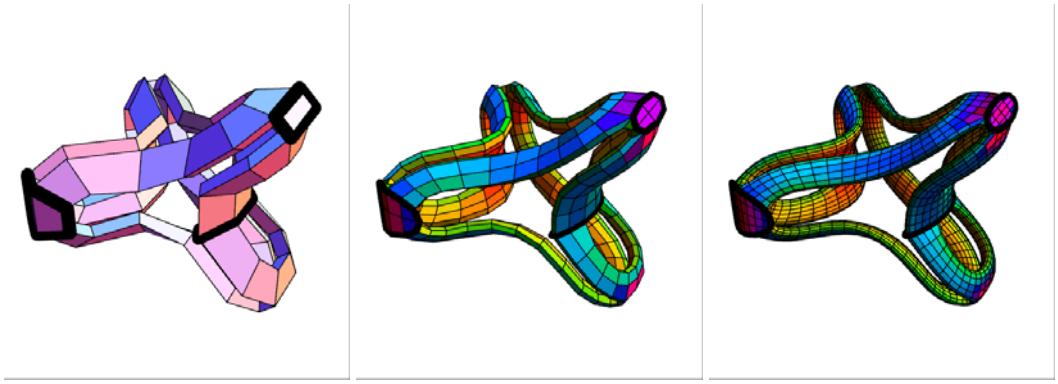
1186440248975551917540789004359931887574474508799765891293446722337396879692866030388136049342507269433353296212331341935347961864644
8963869722390185411387493261767474379475093558386752450994171545546953287347205977950258540100047812227775048553102375088815057595264
2070306626026659893977475798765079457684231016658942911439279534854178769271055191546257011811187220807638703217430426562995896268615
2064432344125226033381766340048255606847867057349042422255002799630409973854491012646992053886834820797331735207658631741687326425763
6026193464327968021919268292296955192891229179766743715837810380872143299787923950976071518804759994153534017653648951137156783355727
4504491812424911049 3
3622921213983657534739061527001749208886591124907091026483831639072641272035142961131005759946538048711592244274843147092994162197587
1959328080136272609831751379437760775935901138442090822180209453126406028275919202498598253390555967108079209062980932067476161050310
3956552904519902158093129824384081330429736124833074890749188186712681273273313497160325519422133276332489690227978685355962198565584
7939816117286370467609703709523304223767015021864404049084987988906007685069696366946763989757219443376929381555821193738768251845544
6381396245546696732858380269694343034353578045939807747440114989478164328547490944766373460742823585539154980298453254436749270423701
8370904246845440000

Level	Volume	Delta to ∞
0	4	0.725183
1	$\frac{87\,993}{25\,600}$	0.16241
2	$\frac{422\,300\,083\,507}{127\,401\,984\,000}$	0.0398891
3	$\frac{411\,387\,395\,873\,703\,661}{125\,241\,246\,351\,360\,000}$	0.00994309
4	$\frac{80\,698\,426\,770\,472\,127\,998\,361}{24\,623\,430\,962\,648\,186\,880\,000}$	0.00248571



Print11

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

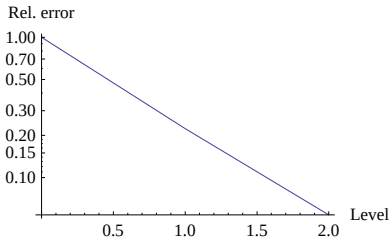


Required valences τ(f) ∈ {4, 5}

Limit volume ↓

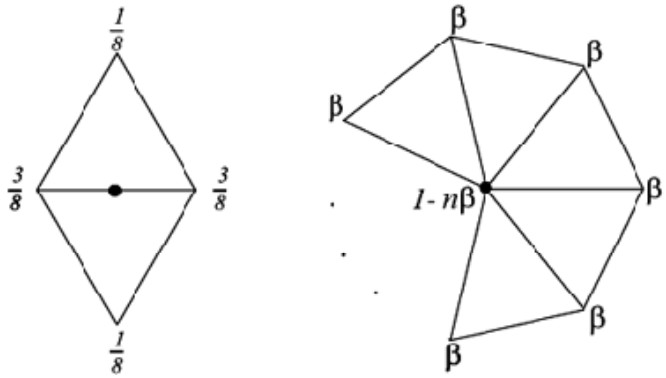
2.53504...

Level	Volume	Delta to ∞
0	3.63196	1.09692
1	2.77989	0.244848
2	2.59457	0.059529



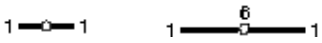
Loop with sharp creases

The Loop subdivision scheme is published as [Loop 1987]. The algorithm applies to meshes with triangles. The weights for the insertion of an edge midpoint, as well as the repositioning of a vertex that already existed in the input mesh are



where $\beta = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\,\text{Cos}\left[\frac{2\pi}{n}\right]\right)^2\right)$.

Subdivision rules along the creases are identical to cubic B-spline subdivision for curves:



The weights are subject to normalization so that their sum adds up to 1.

Subdivision rules along the creases have precedence over rules for the suface, [Hoppe et al. 1994].

Tetrahedron Flat

Vertices ↓

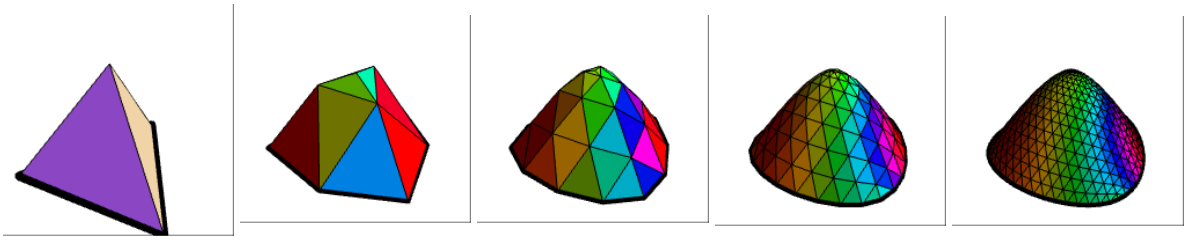
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Cycles ↓

$$(1\ 2\ 3)$$

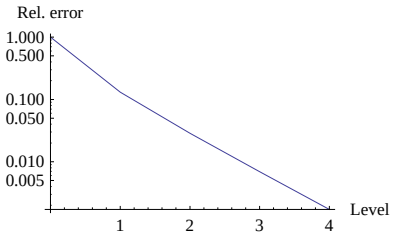


Required valences $\tau(f) \in \{3, 6\}$

Limit volume ↓ (≈0.030457099848192516255494592368 3)

$$\frac{9\,835\,279\,661\,079\,132\,863\,588\,159}{228\,340\,616\,075\,693\,288\,629\,862\,400\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.087394
1	$\frac{91}{1536\sqrt{2}}$	0.0114353
2	$\frac{146\,603}{3\,145\,728\sqrt{2}}$	0.00249679
3	$\frac{565\,996\,523}{12\,884\,901\,888\sqrt{2}}$	0.000604059
4	$\frac{2\,284\,416\,623\,621}{52\,776\,558\,133\,248\sqrt{2}}$	0.000149793



Tetrahedron Circuit

Vertices ↓

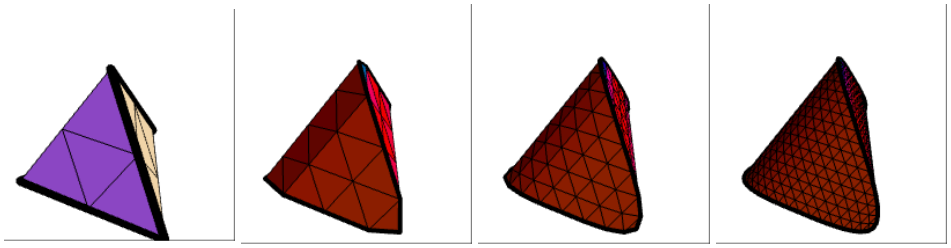
0	0	0
1	0	0
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0
$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\sqrt{\frac{2}{3}}$
$\frac{1}{2}$	0	0
$\frac{1}{4}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	0
$\frac{3}{4}$	$\frac{\sqrt{3}}{4}$	0
$\frac{3}{4}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$

Faces ↓

6	1	5
5	2	9
9	4	6
5	9	6
10	4	9
9	2	8
8	3	10
9	8	10
7	1	6
6	4	10
10	3	7
6	10	7
8	2	5
5	1	7
7	3	8
5	7	8

Cycles ↓

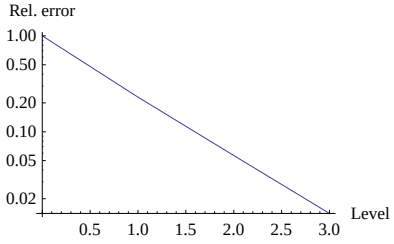
(1 5 2 9 4 10 3 7)



Limit volume ↓ (≈0.092737615548473792386698040 3475)

$$\frac{661}{5040\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.0251135
1	$\frac{107}{768\sqrt{2}}$	0.00577856
2	$\frac{6545}{49\,152\sqrt{2}}$	0.00141957
3	$\frac{414\,137}{3\,145\,728\sqrt{2}}$	0.000353421



Octahedron Side

Vertices ↓

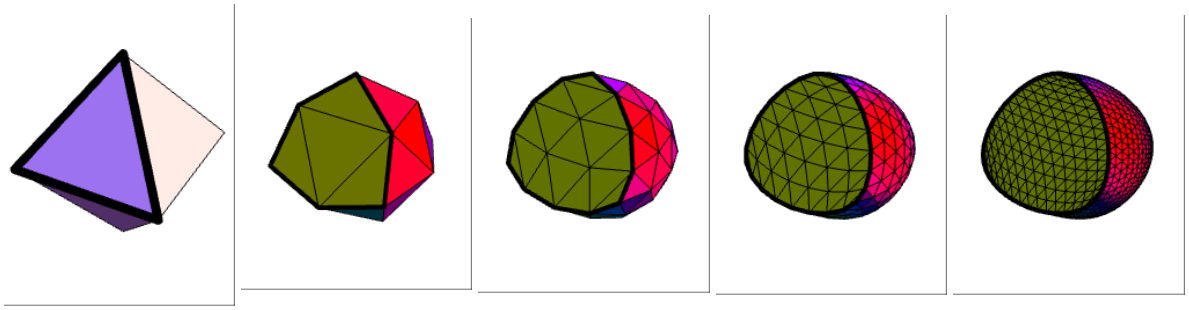
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 5 \\ 3 & 4 & 5 \\ 4 & 1 & 5 \\ 6 & 2 & 1 \\ 6 & 3 & 2 \\ 6 & 4 & 3 \\ 6 & 1 & 4 \end{pmatrix}$$

Cycles ↓

$$(1\ 2\ 5)$$

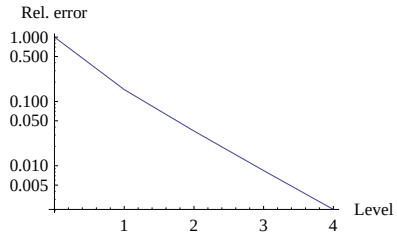


Required valences $\tau(f) \in \{4, 6\}$

Limit volume ↓ $(\approx 0.16582619515335146901889229 8809)$

$$\frac{63\,016\,739\,687\,919\,830\,173\,887\,904\,272\,662\,896\,458\,427\,151}{268\,712\,454\,750\,518\,172\,386\,780\,278\,357\,066\,776\,576\,000\,000\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.305578
1	$\frac{3691}{12\,288\,\sqrt{2}}$	0.0465705
2	$\frac{401\,790\,833}{1\,610\,612\,736\,\sqrt{2}}$	0.0105719
3	$\frac{6\,284\,764\,323\,923}{26\,388\,279\,066\,624\,\sqrt{2}}$	0.00258189
4	$\frac{101\,783\,282\,602\,417\,163}{432\,345\,564\,227\,567\,616\,\sqrt{2}}$	0.000641684



Octahedron Short Loop

Vertices ↓

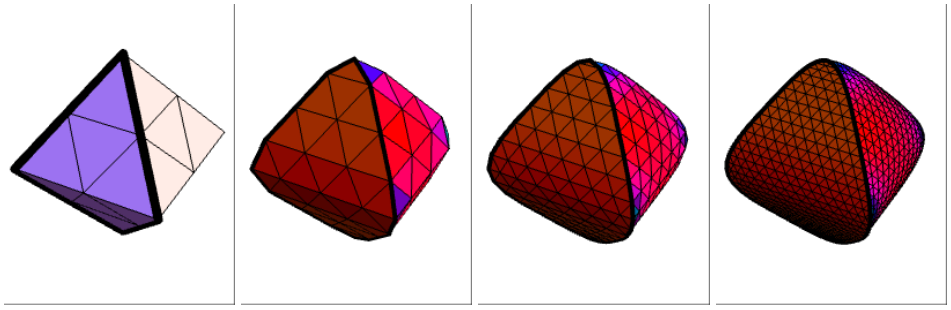
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} \\ 1 & \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{2\sqrt{2}} \\ \frac{3}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{2\sqrt{2}} \\ \frac{3}{4} & \frac{3}{4} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{2\sqrt{2}} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 8 & 1 & 7 \\ 7 & 2 & 12 \\ 12 & 5 & 8 \\ 7 & 12 & 8 \\ 12 & 2 & 11 \\ 11 & 3 & 15 \\ 15 & 5 & 12 \\ 11 & 15 & 12 \\ 15 & 3 & 14 \\ 14 & 4 & 17 \\ 17 & 5 & 15 \\ 14 & 17 & 15 \\ 17 & 4 & 9 \\ 9 & 1 & 8 \\ 8 & 5 & 17 \\ 9 & 8 & 17 \\ 10 & 6 & 13 \\ 13 & 2 & 7 \\ 7 & 1 & 10 \\ 13 & 7 & 10 \\ 13 & 6 & 16 \\ 16 & 3 & 11 \\ 11 & 2 & 13 \\ 16 & 11 & 13 \\ 16 & 6 & 18 \\ 18 & 4 & 14 \\ 14 & 3 & 16 \\ 18 & 14 & 16 \\ 18 & 6 & 10 \\ 10 & 1 & 9 \\ 9 & 4 & 18 \\ 10 & 9 & 18 \end{pmatrix}$$

Cycles ↓

$$(1\ 10\ 6\ 13\ 2\ 12\ 5\ 8)$$

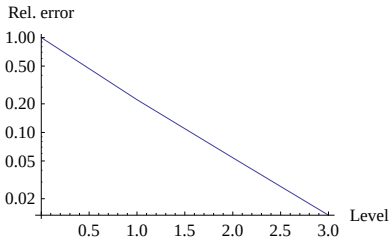


Required valences $\tau(f) \in \{4, 6\}$

Limit volume ↓ $(\approx 0.37633928823869976006650523662 88)$

$$\frac{274\,962\,183\,466\,592\,197\,331\,396\,286\,238\,960\,674\,452\,153}{516\,628\,559\,853\,596\,339\,732\,884\,446\,304\,175\,016\,000\,000\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.0950652
1	$\frac{27\,625}{49\,152\,\sqrt{2}}$	0.0210774
2	$\frac{108\,615\,577}{201\,326\,592\,\sqrt{2}}$	0.0051444
3	$\frac{440\,381\,182\,291}{824\,633\,720\,832\,\sqrt{2}}$	0.00127869

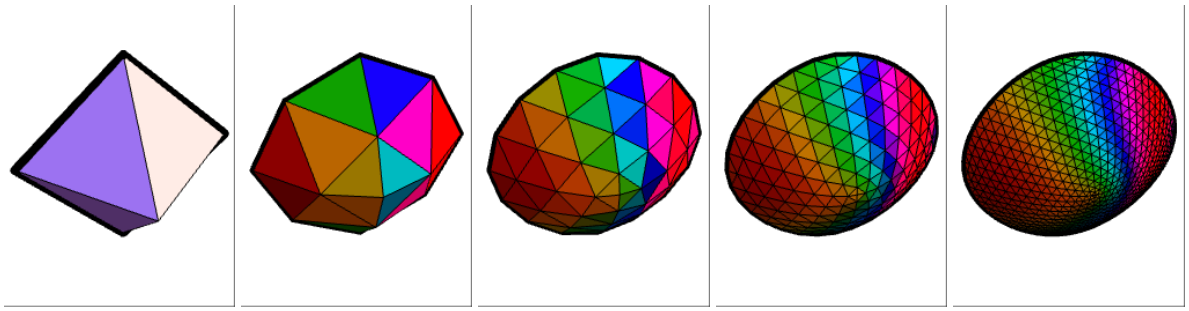


Octahedron Split

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

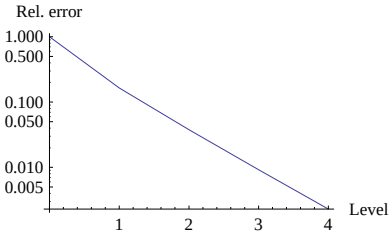
(1 5 3 6)



Limit volume ↓ (≈0.186117527071030358172669303980)

$$\frac{644\,654\,305\,568\,923\,817\,326\,730\,671\,735\,648\,899\,901\,901}{2\,449\,202\,061\,528\,160\,425\,400\,341\,078\,775\,348\,224\,000\,000\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.285287
1	$\frac{675}{2048\,\sqrt{2}}$	0.0469377
2	$\frac{2\,335\,659}{8\,388\,608\,\sqrt{2}}$	0.0107638
3	$\frac{9\,171\,883\,435}{34\,359\,738\,368\,\sqrt{2}}$	0.00263539
4	$\frac{37\,173\,951\,538\,611}{140\,737\,488\,355\,328\,\sqrt{2}}$	0.000655404



Octahedron ZigZag

Vertices ↓

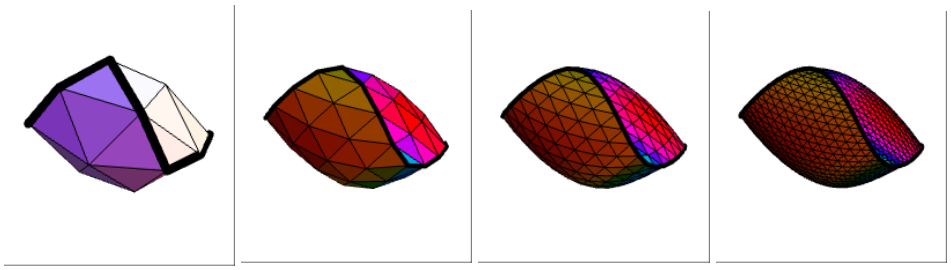
$$\begin{pmatrix} \frac{1}{16} & \frac{3}{16} & \frac{1}{8\sqrt{2}} \\ \frac{15}{16} & \frac{3}{16} & \frac{1}{8\sqrt{2}} \\ \frac{15}{16} & \frac{13}{16} & -\frac{1}{8\sqrt{2}} \\ \frac{1}{16} & \frac{13}{16} & -\frac{1}{8\sqrt{2}} \\ \frac{1}{2} & \frac{3}{8} & \frac{3}{4\sqrt{2}} \\ \frac{1}{2} & \frac{5}{8} & -\frac{3}{4\sqrt{2}} \\ \frac{1}{2} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2} & 0 \\ \frac{5}{16} & \frac{5}{16} & -\frac{3}{8\sqrt{2}} \\ 1 & \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{2\sqrt{2}} \\ \frac{11}{16} & \frac{5}{16} & -\frac{3}{8\sqrt{2}} \\ \frac{1}{2} & \frac{7}{8} & 0 \\ \frac{11}{16} & \frac{11}{16} & \frac{3}{8\sqrt{2}} \\ \frac{3}{4} & \frac{3}{4} & -\frac{1}{2\sqrt{2}} \\ \frac{5}{16} & \frac{11}{16} & \frac{3}{8\sqrt{2}} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 8 & 1 & 7 \\ 7 & 2 & 12 \\ 12 & 5 & 8 \\ 7 & 12 & 8 \\ 12 & 2 & 11 \\ 11 & 3 & 15 \\ 15 & 5 & 12 \\ 11 & 15 & 12 \\ 15 & 3 & 14 \\ 14 & 4 & 17 \\ 17 & 5 & 15 \\ 14 & 17 & 15 \\ 17 & 4 & 9 \\ 9 & 1 & 8 \\ 8 & 5 & 17 \\ 9 & 8 & 17 \\ 10 & 6 & 13 \\ 13 & 2 & 7 \\ 7 & 1 & 10 \\ 13 & 7 & 10 \\ 13 & 6 & 16 \\ 16 & 3 & 11 \\ 11 & 2 & 13 \\ 16 & 11 & 13 \\ 16 & 6 & 18 \\ 18 & 4 & 14 \\ 14 & 3 & 16 \\ 18 & 14 & 16 \\ 18 & 6 & 10 \\ 10 & 1 & 9 \\ 9 & 4 & 18 \\ 10 & 9 & 18 \end{pmatrix}$$

Cycles ↓

$$(1\ 8\ 5\ 12\ 2\ 11\ 3\ 16\ 6\ 18\ 4\ 9)$$

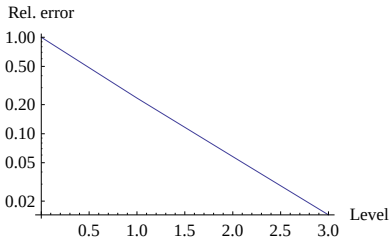


Required valences $\tau(f) \in \{4, 6\}$

Limit volume ↓ ($\approx$0.225920304813625220400776827994)

$$\frac{5797}{18\,144\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{197}{512\sqrt{2}}$	0.0461501
1	$\frac{65\,825}{196\,608\sqrt{2}}$	0.0108214
2	$\frac{16\,270\,553}{50\,331\,648\sqrt{2}}$	0.00266387
3	$\frac{1\,376\,269\,811}{4\,294\,967\,296\sqrt{2}}$	0.000663427

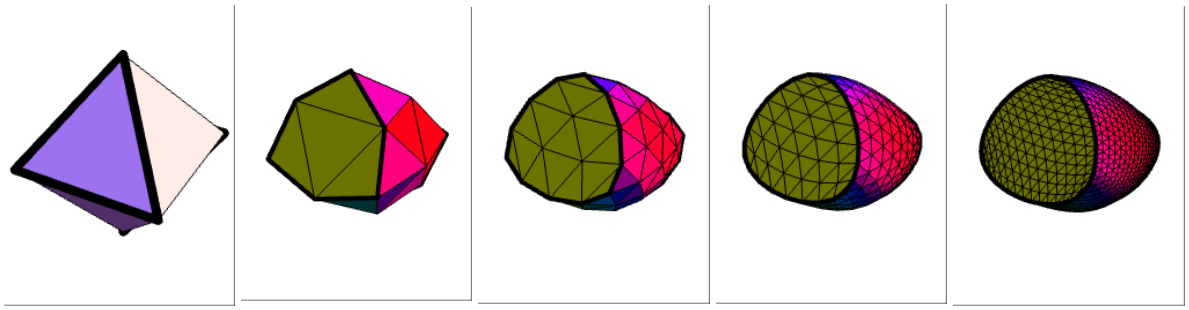


Octahedron Plates

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

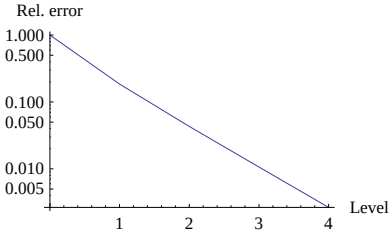
$$\begin{pmatrix} 1 & 5 & 2 \\ 6 & 3 & 4 \end{pmatrix}$$



Limit volume ↓ (≈0.22728432252424741855741425924 8)

$$\frac{9}{28\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.24412
1	$\frac{37}{96\sqrt{2}}$	0.0452464
2	$\frac{4133}{12\,288\sqrt{2}}$	0.0105471
3	$\frac{255\,665}{786\,432\sqrt{2}}$	0.00259246
4	$\frac{16\,223\,969}{50\,331\,648\sqrt{2}}$	0.000645399



Prism Drum

Vertices ↓

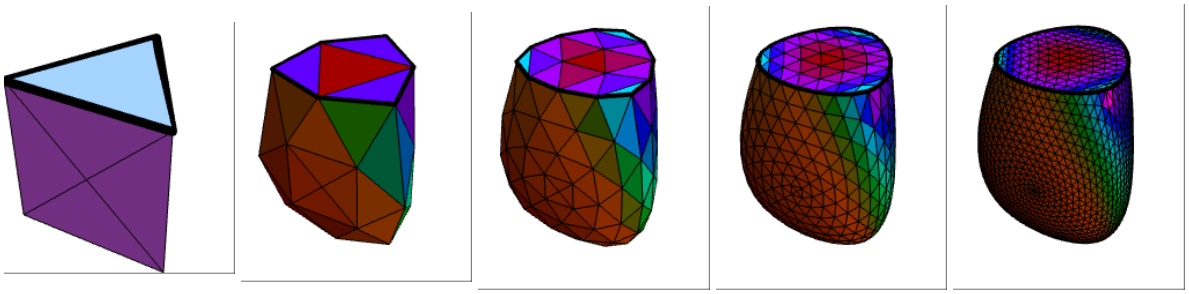
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 3 & 2 \\ 7 & 4 & 1 \\ 7 & 1 & 2 \\ 7 & 2 & 5 \\ 7 & 5 & 4 \\ 4 & 5 & 6 \\ 8 & 5 & 2 \\ 8 & 2 & 3 \\ 8 & 3 & 6 \\ 8 & 6 & 5 \\ 9 & 6 & 3 \\ 9 & 3 & 1 \\ 9 & 1 & 4 \\ 9 & 4 & 6 \end{pmatrix}$$

Cycles ↓

$$(4\ \ 5\ \ 6)$$

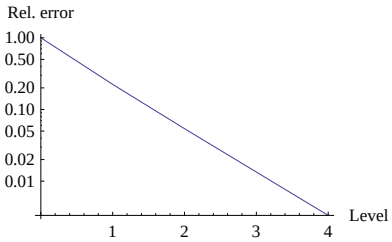


Required valences $\tau(f) \in \{4, 5, 6\}$

Limit volume ↓ $(\approx 0.259001674143006938229606944794)$

$$\left(2\,475\,127\,520\,074\,130\,635\,042\,526\,525\,648\,170\,787\,587\,610\,971\,841\,164\,961\,867\,261\,759\,908\,826\,701\,638\,882\,964\,258\,697\,279 - 690\,038\,962\,881\,214\,188\,256\,954\,998\,018\,842\,883\,399\,094\,620\,010\,352\,214\,129\,738\,035\,141 + 30\,320\,052\,482\,851\,631\,698\,356\,650\,363\,271\,519\,513\,947\,201\,984\,475\,344\,855\,812\,220\,893\,976\,748\,720\,547\,743\,867\,931\,418 - 893\,946\,368\,226\,832\,802\,073\,883\,760\,825\,190\,704\,394\,847\,695\,401\,546\,499\,693\,403\,747\,651\,\sqrt{5}\right) \Big/ \left(5\,668\,529\,225\,890\,450\,802\,551\,075\,312\,322\,624\,561\,263\,481\,125\,139\,134\,477\,943\,244\,232\,305\,177\,047\,323\,992\,798\,823\,222\,375 - 450\,991\,537\,102\,676\,617\,594\,682\,623\,204\,779\,780\,083\,289\,119\,801\,729\,713\,103\,175\,680\,000\,\sqrt{3}\right)$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.174011
1	$\frac{\sqrt{3}\,(21\,937+275\sqrt{5})}{131\,072}$	0.0390106
2	$\frac{\sqrt{3}\,(161\,949\,653+1\,989\,238\sqrt{5})}{1\,073\,741\,824}$	0.00941417
3	$\frac{\sqrt{3}\,(5\,167\,075\,550\,515+63\,321\,718\,441\sqrt{5})}{35\,184\,372\,088\,832}$	0.00233257
4	$\frac{7\sqrt{3}\,(12\,012\,993\,370\,388\,089+147\,170\,613\,873\,957\sqrt{5})}{576\,460\,752\,303\,423\,488}$	0.000581877

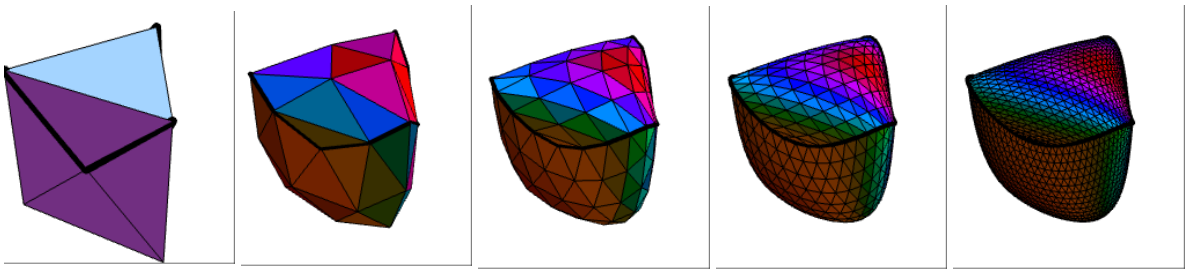


Prism ZigZag

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

$$(4\ \ 7\ \ 5\ \ 8\ \ 6\ \ 9)$$

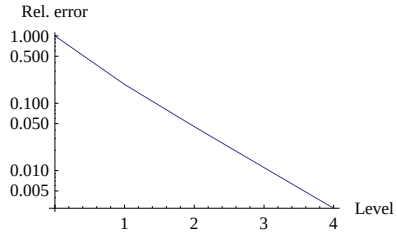


Required valences $\tau(f) \in \{4, 5, 6\}$

Limit volume ↓ $(\approx 0.240139559624273058574994148236)$

$$\left(151\,725\,977\,678\,673\,852\,030\,013\,438\,036\,633\,083\,453\,857\,787\,382\,423\,814\,834\,287\,323\,591\,314\,199\,601\,650\,121\,225\,000\,758 - 262\,155\,279\,680\,507\,473 + 2\,052\,306\,403\,079\,314\,090\,142\,667\,294\,316\,424\,839\,690\,517\,380\,494\,902\,496\,955\,290\,667\,284\,768\,439\,789\,893\,582\,975\,172 - 486\,824\,638\,604\,105\,475\,\sqrt{5}\right) \Big/ \left(375\,817\,088\,906\,660\,972\,864\,094\,276\,610\,628\,205\,301\,403\,995\,393\,174\,024\,628\,197\,304\,049\,563\,453\,643\,356\,669\,187\,943\,487 - 392\,608\,776\,355\,840\,000\,\sqrt{3}\right)$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.192873
1	$\frac{5\sqrt{3}\left(4069+55\sqrt{5}\right)}{131\,072}$	0.0368353
2	$\frac{\sqrt{3}\left(149\,726\,399+2\,020\,328\sqrt{5}\right)}{1\,073\,741\,824}$	0.00867111
3	$\frac{\sqrt{3}\left(4\,777\,170\,305\,611+64\,564\,244\,793\sqrt{5}\right)}{35\,184\,372\,088\,832}$	0.00213726
4	$\frac{\sqrt{3}\left(77\,749\,321\,038\,945\,491+1\,051\,437\,881\,349\,359\sqrt{5}\right)}{576\,460\,752\,303\,423\,488}$	0.000532464

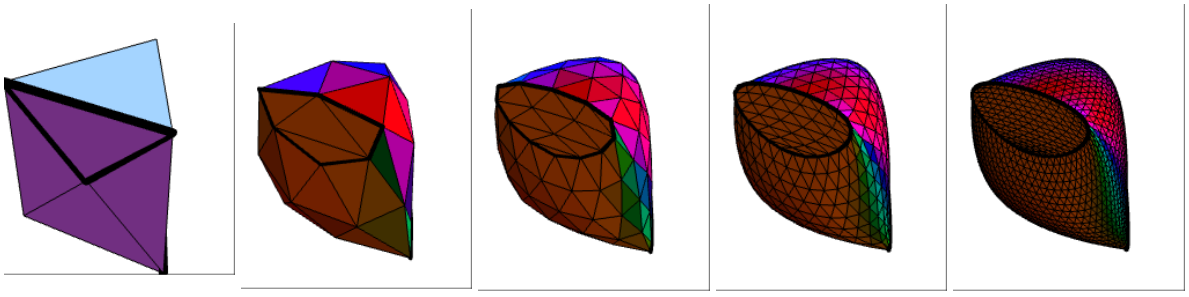


Prism Patched

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

$$\begin{pmatrix} 4 & 7 & 5 \\ 2 & 8 & 3 \end{pmatrix}$$

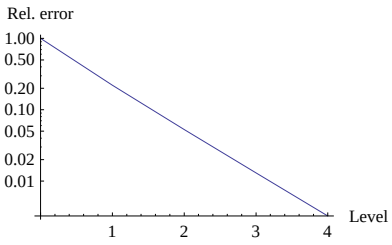


Limit volume ↓ $(\approx 0.252862398395537999582286071283)$

$$\left(852\,625\,316\,956\,093\,739\,416\,011\,744\,388\,915\,443\,611\,874\,508\,574\,699\,995\,384\,354\,674\,328\,437\,567\,769\,927\,876\,804\,243\,089 - 708\,956\,980\,185\,761\,266\,362\,957\,266\,716\,307\,430\,695\,371\,709\,659\,315\,543\,811\,309\,939 + 7\,687\,992\,390\,492\,913\,983\,674\,502\,840\,778\,737\,284\,979\,584\,854\,108\,757\,756\,749\,706\,629\,207\,059\,152\,851\,782\,711\,893\,553 - 541\,211\,879\,339\,358\,018\,767\,623\,004\,620\,541\,737\,751\,805\,367\,212\,933\,178\,563\,400\,401\,\sqrt{5}\right) \Big/ \left(1\,986\,015\,378\,114\,841\,841\,441\,905\,863\,084\,071\,202\,560\,391\,238\,175\,498\,688\,161\,367\,101\,010\,066\,701\,413\,731\,516\,249\,182\,551 -\right.$$

$$349\,226\,247\,038\,465\,024\,850\,208\,019\,652\,873\,638\,303\,861\,806\,221\,409\,507\,082\,240\,000\,\sqrt{3}\Big)$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.18015
1	$\frac{32\,601+275\,\sqrt{5}}{65\,536\,\sqrt{3}}$	0.0397588
2	$\frac{119\,610\,987+1\,058\,135\,\sqrt{5}}{268\,435\,456\,\sqrt{3}}$	0.00948552
3	$\frac{476\,452\,855\,739+4\,274\,997\,303\,\sqrt{5}}{1\,099\,511\,627\,776\,\sqrt{3}}$	0.00234107
4	$\frac{3\,875\,936\,562\,233\,061+34\,905\,517\,220\,098\,\sqrt{5}}{9\,007\,199\,254\,740\,992\,\sqrt{3}}$	0.000583293



Wedged Prism

Vertices ↓

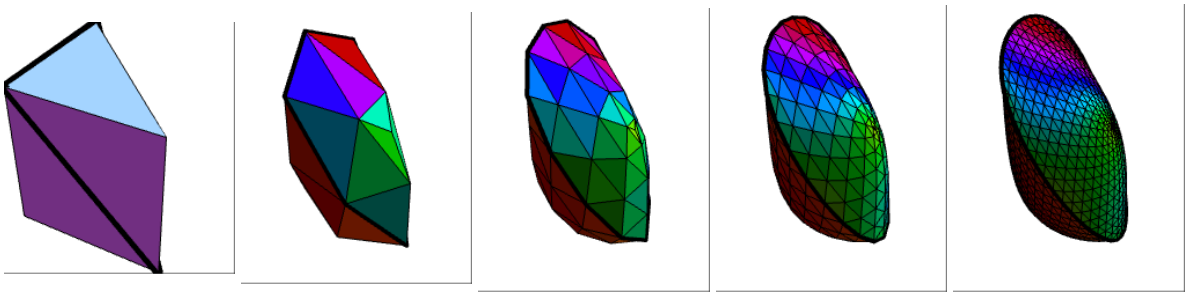
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 5 \\ 4 & 5 & 6 \\ 2 & 1 & 3 \\ 5 & 2 & 6 \\ 6 & 2 & 3 \\ 1 & 4 & 6 \\ 1 & 6 & 3 \end{pmatrix}$$

Cycles ↓

$$(4\ 2\ 6)$$

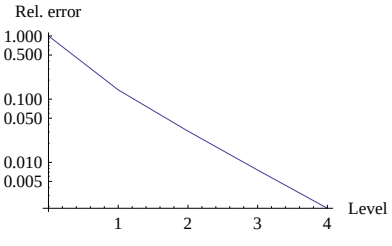


Required valences $\tau(f) \in \{3, 4, 5, 6\}$

Limit volume ↓ $(\approx 0.14790464770911802863529029 9837)$

$$22\,019\,733\,629\,566\,009\,632\,050\,792\,507\,567\,438\,044\,342\,955\,396\,113\,/\,148\,877\,901\,882\,244\,479\,728\,587\,648\,112\,669\,004\,444\,352\,512\,000\,000$$

Level	Volume	Delta to ∞
0	$\frac{1}{2}$	0.352095
1	$\frac{9691}{49\,152}$	0.0492593
2	$\frac{31\,979\,407}{201\,326\,592}$	0.0109388
3	$\frac{248\,316\,449\,995}{1\,649\,267\,441\,664}$	0.00265702
4	$\frac{4\,014\,440\,847\,617\,141}{27\,021\,597\,764\,222\,976}$	0.000659508



Twisted ZigZag

Vertices ↓

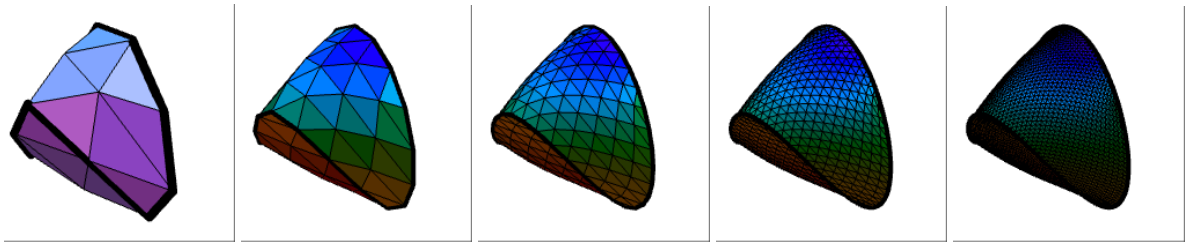
$$\begin{pmatrix} 0 & \frac{1}{8} & \frac{1}{4} \\ \frac{7}{8} & 0 & \frac{1}{4} \\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16} \\ \frac{1}{8} & 0 & \frac{3}{4} \\ \frac{7}{8} & \frac{1}{8} & \frac{7}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{7}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{4} & \frac{3}{8} \\ \frac{1}{2} & \frac{1}{8} & \frac{7}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{7}{8} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 8 & 1 & 7 \\ 7 & 2 & 14 \\ 14 & 4 & 8 \\ 7 & 14 & 8 \\ 16 & 4 & 14 \\ 14 & 2 & 13 \\ 13 & 5 & 16 \\ 14 & 13 & 16 \\ 17 & 4 & 16 \\ 16 & 5 & 18 \\ 18 & 6 & 17 \\ 16 & 18 & 17 \\ 11 & 2 & 7 \\ 7 & 1 & 10 \\ 10 & 3 & 11 \\ 7 & 10 & 11 \\ 18 & 5 & 13 \\ 13 & 2 & 12 \\ 12 & 6 & 18 \\ 13 & 12 & 18 \\ 15 & 6 & 12 \\ 12 & 2 & 11 \\ 11 & 3 & 15 \\ 12 & 11 & 15 \\ 9 & 1 & 8 \\ 8 & 4 & 17 \\ 17 & 6 & 9 \\ 8 & 17 & 9 \\ 10 & 1 & 9 \\ 9 & 6 & 15 \\ 15 & 3 & 10 \\ 9 & 15 & 10 \end{pmatrix}$$

Cycles ↓

$$(6\ 9\ 1\ 8\ 4\ 14\ 2\ 13\ 5\ 18)$$

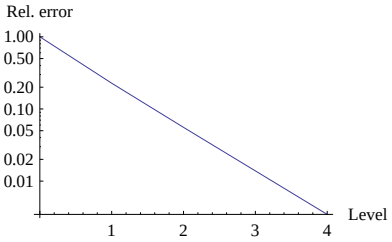


Required valences $\tau(f) \in \{3, 4, 5, 6\}$

Limit volume ↓ $(\approx 0.207642328556229883797226287710)$

$$\frac{18\,419\,838\,425\,496\,441\,675\,729\,228\,523\,308\,561\,088\,835\,786\,228\,213\,485\,043\,170\,450\,060\,897}{88\,709\,458\,006\,816\,366\,251\,118\,298\,141\,221\,567\,374\,489\,199\,038\,073\,786\,811\,938\,652\,160\,000}$$

Level	Volume	Delta to ∞
0	$\frac{785}{3072}$	0.0478915
1	$\frac{114\,583}{524\,288}$	0.0109074
2	$\frac{2\,709\,794\,323}{12\,884\,901\,888}$	0.00266539
3	$\frac{2\,748\,404\,038\,585}{13\,194\,139\,533\,312}$	0.000662581
4	$\frac{3\,743\,531\,424\,811\,475}{18\,014\,398\,509\,481\,984}$	0.000165411



Cube ZigZag

Vertices ↓

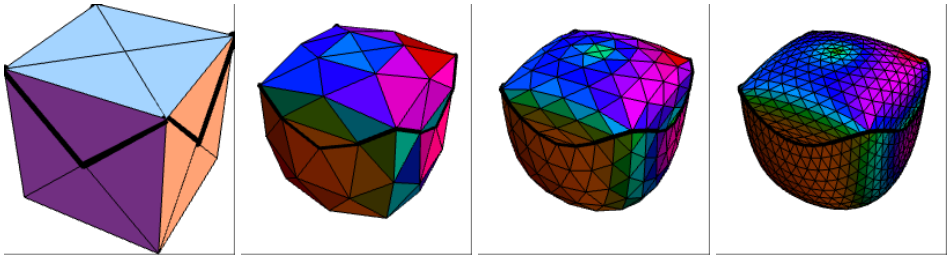
$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 9 & 5 & 1 \\ 9 & 1 & 2 \\ 9 & 2 & 6 \\ 9 & 6 & 5 \\ 10 & 1 & 3 \\ 10 & 3 & 4 \\ 10 & 4 & 2 \\ 10 & 2 & 1 \\ 11 & 2 & 4 \\ 11 & 4 & 8 \\ 11 & 8 & 6 \\ 11 & 6 & 2 \\ 12 & 7 & 5 \\ 12 & 5 & 6 \\ 12 & 6 & 8 \\ 12 & 8 & 7 \\ 13 & 3 & 7 \\ 13 & 7 & 8 \\ 13 & 8 & 4 \\ 13 & 4 & 3 \\ 14 & 5 & 7 \\ 14 & 7 & 3 \\ 14 & 3 & 1 \\ 14 & 1 & 5 \end{pmatrix}$$

Cycles ↓

$$(5\ 9\ 6\ 11\ 8\ 13\ 7\ 14)$$

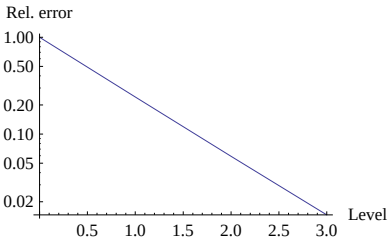


Required valences $\tau(f) \in \{4, 6\}$

Limit volume ↓ $(\approx 0.719587549369929675887134096181)$

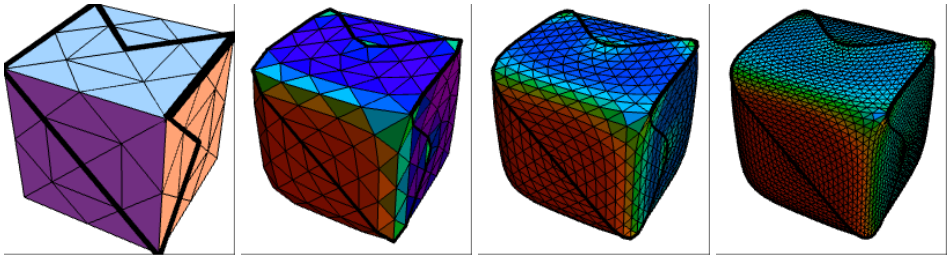
$$\frac{18\,302\,899\,293\,318\,629\,518\,466\,729\,731\,126\,356\,334\,826\,796\,187\,797\,121\,113\,000\,675\,471\,677\,089\,129}{25\,435\,264\,005\,532\,939\,743\,508\,876\,618\,068\,557\,876\,581\,936\,313\,069\,304\,458\,059\,484\,543\,938\,560\,000}$$

Level	Volume	Delta to ∞
0	1	0.280412
1	$\frac{2419}{3072}$	0.0678473
2	$\frac{192\,975}{262\,144}$	0.0165537
3	$\frac{1\,165\,595\,405}{1\,610\,612\,736}$	0.00410933



Cube Melted

The mesh and the cycle can be reconstructed from the figure. The corner points are from the unit cube.

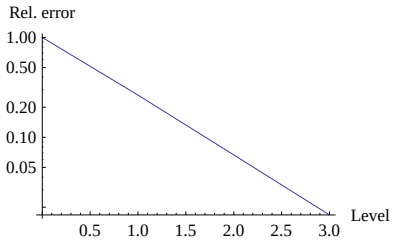


Required valences $\tau(f) \in \{4, 6\}$

Limit volume $\downarrow$ $(\approx 0.939714557998252758936258231791)$

650461664254399165450556628636172838039214082052271206509844602286058650422592309257491157713803980799733535615081171 43 /
692190685690759075630426802920717201144775905753608520614304604337671285985099954821272952346339329747207631137111040 00

Level	Volume	Delta to ∞
0	1	0.0602854
1	$\frac{46\,973}{49\,152}$	0.0159536
2	$\frac{15\,833\,375}{16\,777\,216}$	0.00402813
3	$\frac{16\,161\,510\,779}{17\,179\,869\,184}$	0.00100918



Torus Loops

Vertices ↓

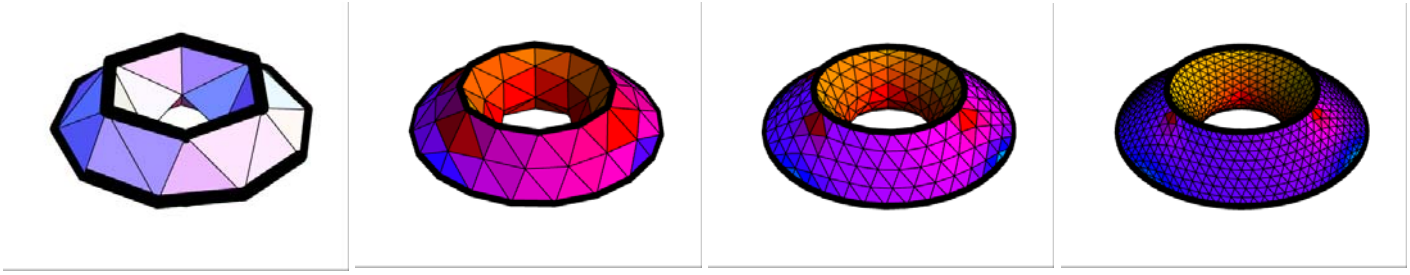
$$\begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 1.21353 & 0.881678 & 0 \\ 0.463525 & 1.42658 & 0 \\ -0.463525 & 1.42658 & 0 \\ -1.21353 & 0.881678 & 0 \\ -\frac{3}{2} & 0 & 0 \\ -1.21353 & -0.881678 & 0 \\ -0.463525 & -1.42658 & 0 \\ 0.463525 & -1.42658 & 0 \\ 1.21353 & -0.881678 & 0 \\ 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0.866025 & \frac{1}{2} \\ -\frac{1}{2} & 0.866025 & \frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -0.866025 & \frac{1}{2} \\ \frac{1}{2} & -0.866025 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & 0.433013 & 0 \\ -\frac{1}{4} & -0.433013 & 0 \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0.866025 & -\frac{1}{2} \\ -\frac{1}{2} & 0.866025 & -\frac{1}{2} \\ -1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -0.866025 & -\frac{1}{2} \\ \frac{1}{2} & -0.866025 & -\frac{1}{2} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 3 & 4 & 13 \\ 12 & 2 & 3 \\ 18 & 13 & 14 \\ 23 & 18 & 19 \\ 21 & 20 & 17 \\ 22 & 4 & 3 \\ 24 & 19 & 25 \\ 8 & 24 & 9 \\ 7 & 6 & 23 \\ 6 & 14 & 5 \\ 14 & 13 & 5 \\ 11 & 1 & 2 \\ 12 & 3 & 13 \\ 13 & 4 & 5 \\ 18 & 12 & 13 \\ 11 & 2 & 12 \\ 10 & 16 & 9 \\ 19 & 18 & 14 \\ 17 & 11 & 12 \\ 18 & 17 & 12 \\ 2 & 20 & 21 \\ 23 & 5 & 22 \\ 2 & 1 & 20 \\ 3 & 2 & 21 \\ 22 & 3 & 21 \\ 22 & 5 & 4 \\ 24 & 23 & 19 \\ 21 & 17 & 18 \\ 18 & 22 & 21 \\ 23 & 6 & 5 \\ 18 & 23 & 22 \\ 24 & 7 & 23 \\ 7 & 24 & 8 \\ 24 & 25 & 9 \\ 25 & 10 & 9 \\ 25 & 19 & 17 \\ 25 & 20 & 10 \\ 10 & 20 & 1 \\ 25 & 17 & 20 \\ 16 & 10 & 11 \\ 19 & 16 & 17 \\ 16 & 15 & 9 \\ 19 & 14 & 15 \\ 17 & 16 & 11 \\ 19 & 15 & 16 \\ 7 & 14 & 6 \\ 15 & 14 & 7 \\ 15 & 8 & 9 \\ 15 & 7 & 8 \\ 10 & 1 & 11 \end{pmatrix}$$

Cycles ↓

$$\begin{pmatrix} \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ \{11, 12, 13, 14, 15, 16\} \end{pmatrix}$$

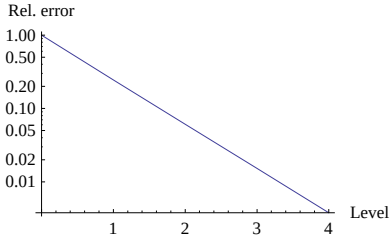


Required valences $\tau(f) \in \{4, 6, 7, 8\}$

Limit volume ↓

2.66612...

Level	Volume	Delta to ∞
0	3.13298	0.466856
1	2.78067	0.114552
2	2.69455	0.0284279
3	2.67321	0.00709181
4	2.66789	0.00177193

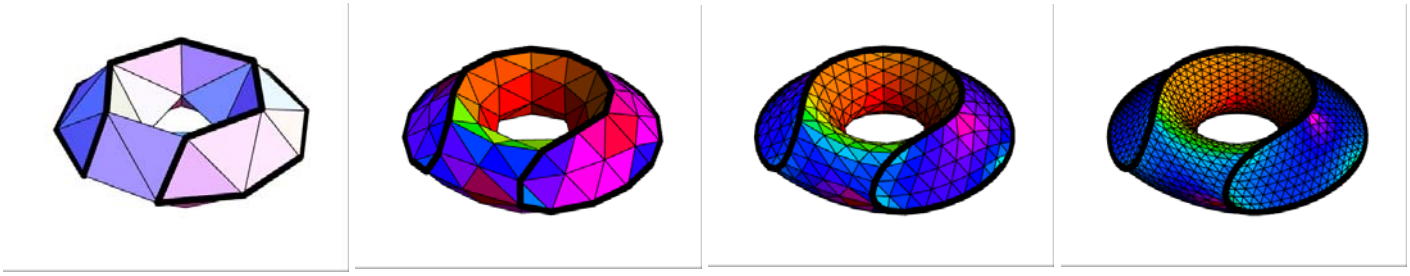


Torus Circuit

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

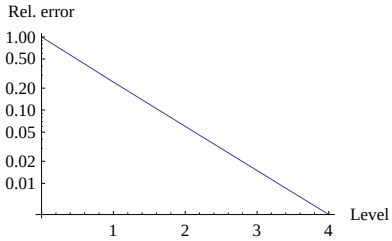
{1 2 3 4 5 6 7 8 15 14 13 12 11 16 9 10}



Limit volume ↓

2.61935...

Level	Volume	Delta to ∞
0	3.13298	0.513625
1	2.74383	0.124472
2	2.65016	0.0308099
3	2.62704	0.00768142
4	2.62127	0.00191898

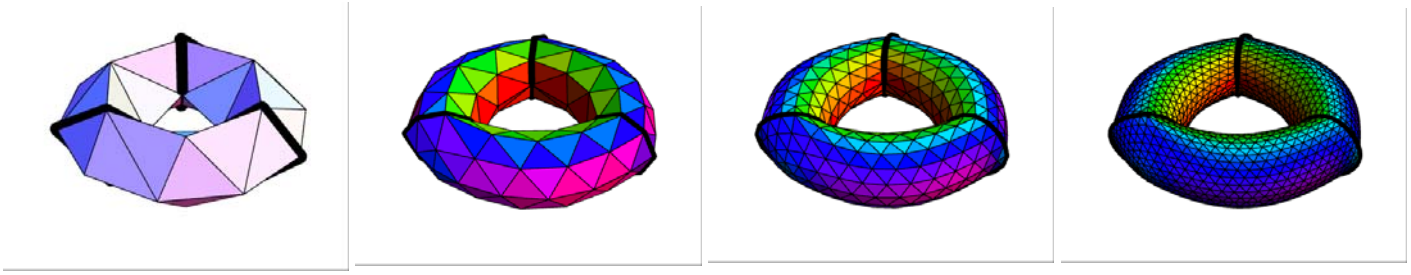


Torus Rings

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

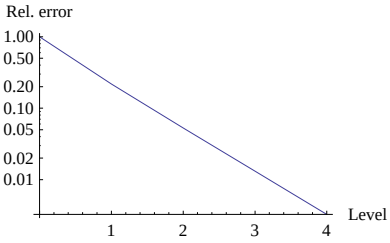
4	22	18	13
17	11	1	20
19	15	7	24



Limit volume ↓

2.06762...

Level	Volume	Delta to ∞
0	3.13298	1.06536
1	2.30027	0.232645
2	2.12388	0.0562549
3	2.08157	0.013947
4	2.0711	0.00347952

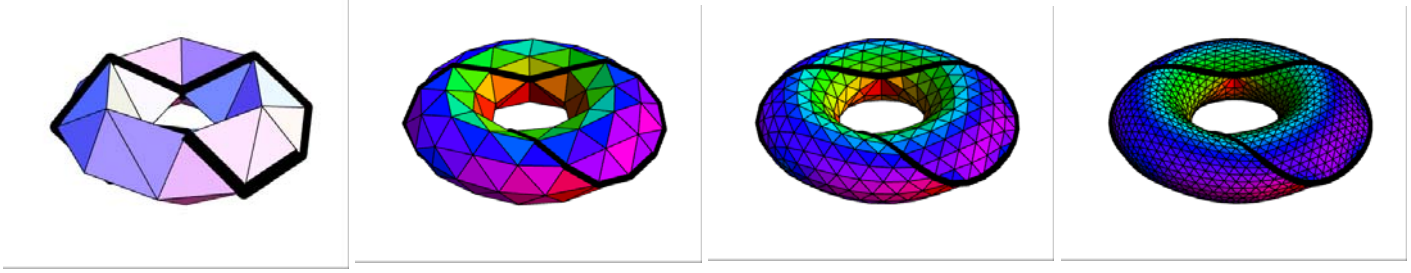


Torus Wrap

The vertex coordinates and topology of the mesh are specified in a section above.

Cycles ↓

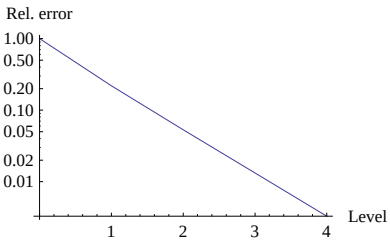
{12 18 14 7 24 19 16 10 1 2}



Limit volume ↓

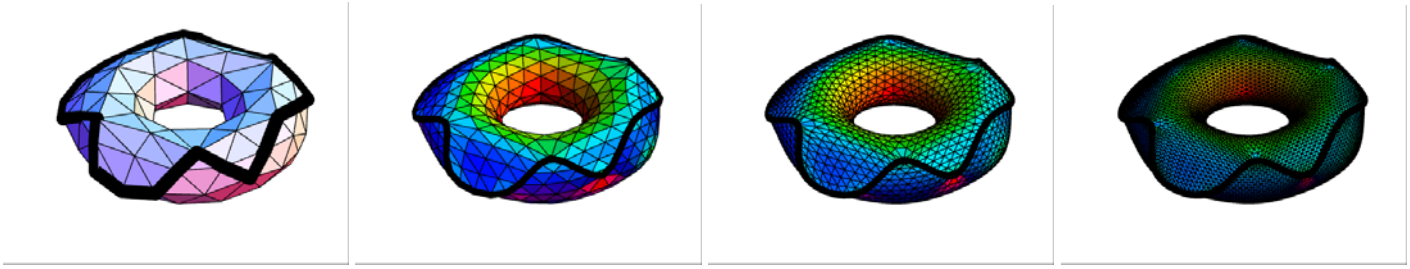
2.20094...

Level	Volume	Delta to ∞
0	3.13298	0.932042
1	2.40544	0.204504
2	2.25048	0.0495407
3	2.21322	0.0122819
4	2.204	0.00306329



Torus ZigZag

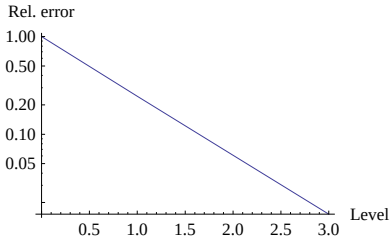
The vertex coordinates and topology of the mesh are specified in a section above. The cycle is {11,2,12,3,4,13,5,6,7,15,8,9,16,10}. One initial round of Loop subdivision is applied.



```
Limit volume ↓
```

2.36759...

Level	Volume	Delta to ∞
0	2.54337	0.175788
1	2.41063	0.0430459
2	2.37829	0.0107018
3	2.37026	0.00267137



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