Initiating the effective unification of black hole horizon area and entropy quantization with quasi-normal modes

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Abstract

Black hole (BH) quantization may be the key to unlocking a unifying theory of quantum gravity (QG). Surmounting evidence in the field of BH research continues to support a horizon (surface) area with a *discrete* and *uniformly* spaced spectrum, but there is still no general agreement on the level spacing. In this specialized and important BH case study, our objective is to report and examine the pertinent groundbreaking work of the strictly thermal and non-strictly thermal spectrum level spacing of the BH horizon area quantization with included entropy calculations, which aims to tackle this *gigantic* problem. In particular, this work exemplifies a series of imperative corrections that eventually permits a BH's horizon area spectrum to be generalized from *strictly thermal* to *non-strictly* *thermal* with entropy results, thereby capturing multiple preceding developments by launching an effective unification between them. Moreover, the identified results are significant because quasi-normal modes (QNM) and "effective states" characterize the transitions between the established levels of the non-strictly thermal spectrum.

1 Introduction

BHs are mighty creatures that generate chaos in space-time physics. In general, the laws of classical and modern physics break down when attempts are made to rigorously characterize the behavior of BHs and their effects. In order to advance science, fundamental problems such as the BH information paradox and event horizon firewalls [1, 2, 3, 4] must be nullified and understood so the physical laws can be "upgraded" via the scientific method and tested in laboratory experiments [5].

There is a *vast* array of modern attacks that aim to tame and conquer the great BH beasts by establishing a unified field theory with a new set of physical laws. Among these approaches, numerous mainstream unification candidates (and variations of those candidates) exist, including, super-string theory [6], QG, loop quantum gravity (LQG) [7, 8, 9, 10, 11], Chern-Simons theory [12], Yukawa $SO(10)$ theory [13], E8 theory [14], and others. Frequently, components and ideas from different theories are combined, adjusted, and "hacked" together (i.e. copy-and-paste methods) to forge new hybrid theoretical frameworks with customized capabilities, such as semi-classical physics, which intertwines aspects of quantum mechanics and classical mechanics. Currently, none of these candidates are accepted to be *complete* by mainstream science. For example, some frameworks like super-string theory [6], Yukawa $SO(10)$ theory [13], and E8 theory [14] are incomplete because they require more spatial degrees of freedom to operate than 4D space-time can offer so they cannot be tested in the laboratory, while other theories are incomplete because they fail to fully describe paradoxical phenomena like BHs, which remain imposing, elusive, and continue to violate the modern laws of physics. Hence, the theories must be subjected to additional stringent scientific research, scrutiny, debate, and experimentation so they can continue to evolve and achieve improved representational capabilities.

In this review paper, we focus on the surface area and entropy quantization of BH event horizons, where we identify and examine some key points, issues, and corrections in a chronological narrative of *strictly thermal* and *non-strictly thermal* results. For this assignment, the pertinent, groundbreaking work of numerous researchers and teams is investigated. As mentioned above, there is a diverse landscape of candidate unification theories that may be applied this particular BH aspect. Thus, from among the said candidates, we've selected a *semi-classical* platform to launch a probe of BHs that exemplifies the underlying QG theory. For this work, we prefer this semi-classical, QG-based approach over existing unification candidates such as super-string theory [6], Yukawa $SO(10)$ theory [13], and E8 theory [14] because their gravitational treatment adds too many spatial degrees of freedom. Now LQG [7, 8, 9, 10, 11] does have 4D spacetime gravity "built-in" by default because it fundamentally operates on the principles of general relativity. Moreover, recent emerging LQG-based approaches do yield promising results with new quantization techniques and connections to semi-classical Bekenstein-Hawking entropy [7, 8, 9, 10, 11]. However, for the purpose of this case study, LQG still does not have the requisite gravitational tools for constructing and/or interpreting the desired BH area and entropy quantization framework. In fact, besides string theory in which gravity should be modied, in LQG theory general relativity is fundamental but the quantization process should be adapted. Although LQG is a powerful approach, it is not yet complete. For example we refer the reader to the holography, the origin of thermodynamics, spin-foams (quantum dynamics is not fully under control), many open question concerning the classical limit and extension of static uncharged black holes to charged, rotating case is unclear. Therefore, starting with Section 2, we review initial incursions that establish preliminary upper and lower bounds on the horizon area and entropy quantization for non-extremal BHs in terms of our semi-classical, QG perspective. Next, in Section 3, we proceed to work that further characterizes a strictly thermal spectrum, where the perturbation field states and transitions are encoded with damped harmonic oscillators and QNMs. Thereafter, in Section 4, we advance to additional corrections that lead to a non-strictly thermal spectrum, where effective states are deployed to encode BH characteristics and initiate the unification of the preceding strictly thermal quantization results. Finally, we conclude with the brief discussion and results recapitulation of Section 5.

2 Initiating horizon area quantization boundaries

In the early 1970s, Bekenstein [15, 16] observed that the (non-extremal) BH horizon area behaves as a classical adiabatic invariant and therefore conjectured that it should exemplify a *discrete* eigenvalue spectrum with quantum transitions [17, 18]. To date, a major objective in BH physics research is to determine the unique spacing between the BH horizon area levels because surmounting scientific evidence seems to indicate that the BH horizon area spectrum is in fact *quantized* and *uniformly* distributed [17, 18]. Thus, our investigation launches from the *particle* platform of wave-particle duality.

When a BH captures or releases a (point) particle with mass, then the BH's mass unavoidably increases or decreases, respectively, which directly influences its horizon area [17, 18]. For the *BH uncharged particle absorption process*, it was ascertained [16] from Ehrenfest's theorem that the particle's center of mass must follow a *classical* trajectory and therefore it was demonstrated that the *BH horizon area increase lower-bound* is [17, 18]

$$\Delta A = 8\pi\mu b, \tag{1}$$

where ΔA is the BH horizon area change, μ is the particle rest mass, and b is the particle finite proper radius. In a quantum theory, Heisenberg's *uncertainty* *principle*, which excludes a completely reversible process, applies to relativistic quantized particles [17, 18]—specifically, the radial position for the particle's center of mass is subject to an uncertainty of $b \geq \hbar/\mu$ because it cannot be localized with a degree of precision that supercedes its own Compton wavelength [17, 18]. Thus, for the *uncharged* particle absorption process, the uncertainty principle is the physical mechanism which defines the *uncharged BH horizon area increase lower-bound* as [17, 18]

$$\Delta A = 8\pi l_p^2, \tag{2}$$

where $l_p = \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length in gravitational units $G = c = 1$. However, for the *BH charged particle absorption process* the "uncertainty principle mechanism" must be supplemented by a secondary physical mechanism—a Schwinger discharge for the *BH vacuum polarization* process [17, 18]. Hence, for the *charged* particle absorption process, this "vacuum polarization mechanism" lets one bypass the reversible limit constraint and defines the *charged BH horizon area increase lower-bound as* [17, 18]

$$\Delta A = 4 \ln e l_p^2 = 4 l_p^2. \tag{3}$$

Here, the lower-bounds of eqs. (2–3) are fully consistent with the analysis of [19, 20].

Thus, as soon as one introduces quantum implications into the absorption process it becomes evident that eqs. (2–3) are in fact *universal lower-bounds* because they are independent of the BH parameters [17, 18]; this fundamental lower-bound's *universality* strongly favors a uniformly-spaced quantum BH horizon area spectrum [17, 18]. Moreover, it is striking that although the results of eqs. (2–3) emerge from two distinct physical mechanisms, *they are clearly of the same magnitude order [17, 18] and differ by a factor of* 2π *due to the existence of charge*, which is further realized in [19, 20]. Hence, it was concluded that the BH horizon area quantization condition is of the form [17, 18]

$$A_n = \gamma n l_p^2 \quad ; \quad n = 1, 2, ... \tag{4}$$

where γ is a dimensionless constant.

In [17, 18], it was recognized that the exact values of eqs. (2–3) can be challenged because they operate on the assertion that the smallest possible particle radius is precisely equal to its Compton wavelength and because the particle size is inherently fuzzy. But it is clear that the γ in both eqs. (2–3) cases must be of the magnitude order $\gamma = O(4)$ [17, 18]. Moreover, that "the small uncertainty in the value of γ is the price we must pay for not giving our problem a full quantum treatment" [17, 18]. Therefore, the quantum analysis [17, 18] shifts from *discrete particles* to *continuous waves* due to the uncertainty of γ; this is legal because of nature's wave-particle *duality*—one must be able to infer the wave results from the particle results, and conversely. Consequently, the QNMs authorize one to explore BH perturbations from the perspective of such waves [17, 18, 21, 24]. Specifically, QNMs enable one to characterize a BH's free oscillations, where the behavior of the radiated perturbations is reminiscent to the last pure dying tones of a ringing bell because the QNM *frequencies* are representative of the BH itself [17, 18, 21, 24]. The perturbation field QNM states encode the scattering amplitude's pole singularities in the BH background [18]. More specifically, the quantized *states* of the perturbation fields *outside* the BH are encoded with complex numbers for QNMs, where *the BH perturbation fields transition between states in the "BH perturbation field state space" over "state time"*. The BH states of such complex-valued QNMs are equipped with the standard 2D coordinate-vector components: the *amplitude* ("radius" or "modulus") and *phase* ("azimuth" or "direction") for 2D polar vector–coordinate form, and *real* and *imaginary* for 2D Cartesian vector–coordinate form. In BH physics and thermodynamics, it is imperative to be able to encode such QNM states and transitions for determining the asymptotic behavior of BH ringing frequencies—this is a monstrous physical encoding problem that requires a proper, rigorous quantum treatment in order to further demystify and generalize the horizon area results of eqs. (1–4).

3 Strictly thermal horizon area and entropy quantization

To attack this massively *complex* encoding problem in Hawking's *strictly thermal* radiation spectrum, Maggiore [21] went on to demonstrate that the behavior of the BH perturbation field QNM states is identical to that of *damped harmonic oscillators* whose real frequencies are encoded as the 2D polar amplitude

$$|\omega| = \sqrt{\omega_{\mathbb{R}}^2 + \omega_{\mathbb{I}}^2}, \tag{5}$$

rather than just $\omega_{\mathbb{R}}$, such that

$$\omega_{\mathbb{R}} = \sqrt{|\omega|^2 - \left(\frac{K}{2}\right)^2} \quad \text{and} \quad \omega_{\mathbb{I}} = \frac{K}{2} \tag{6}$$

are the 2D Cartesian real and imaginary components, respectively, where K is the damping coefficient. In eqs. (5–6), the case $|\omega| = \omega_{\mathbb{R}}$ for $\omega_{\mathbb{I}} \ll |\omega|$ corresponds to *lowly-excited*, *very long-lived* perturbation states, whereas the "opposite" limit case $|\omega| = \omega_{\mathbb{I}}$ for $\omega_{\mathbb{R}} \ll \omega_{\mathbb{I}}$ corresponds to *highly-excited*, *very short-lived* perturbation states [21]—so $|\omega| \simeq \omega_{\mathbb{I}}$ rather than $|\omega| \simeq \omega_{\mathbb{R}}$. The results of eqs. (5–6) exemplify the three distinct QNM components—$|\omega|$, $\omega_{\mathbb{R}}$, and $\omega_{\mathbb{I}}$—that comply with Pythagorean's theorem of triangles for the precise determination of physical properties with the well-known elementary trigonometric (circular) functions. Thus, it is straightforward to identify the *fourth* QNM component as

$$\langle\omega\rangle = \arctan 2(\omega_{\mathbb{I}}, \omega_{\mathbb{R}}), \tag{7}$$

which encodes the 2D polar phase of the BH perturbation field's azimuthal eigenvalue, namely m.

With the aim to establish order in the chaos, *Bohr's correspondence principle* of 1923 was deployed, which claims that the "transition frequencies at large quantum numbers should equal classical oscillation frequencies" [17, 18]. Thus, the analysis [17, 18] focused on the ringing frequencies asymptotic behavior for the $n \to \infty$ limit, which are classified as highly-damped BH perturbation field QNM frequencies that operate under the assertion that such *quantum transitions between states are instantaneous*. The transitions do not require time because it was established that $\omega = \omega_{\mathbb{R}} - i\omega_{\mathbb{I}}$ [17, 18], such that $\tau \equiv \omega_{\mathbb{I}}^{-1}$ is the effective relaxation time which is required for the BH to return to a state of equilibrium, where τ is arbitrarily small as $n \to \infty$. On one hand, for each value of l, there exists an infinite number of QNMs for $n = 0, 1, 2, ...$ with decreasing relaxation times (so the value of $\omega_{\mathbb{I}}$ increases) [17, 18]. On the other hand, $\omega_{\mathbb{R}}$ approaches a *constant* value as n is increased [17, 18]. Hence, eqs. (5) and (7) are re-written for large n as the amplitude

$$|\omega_n| = \sqrt{\omega_{n_{\mathbb{R}}}^2 + \omega_{n_{\mathbb{I}}}^2}, \tag{8}$$

and the azimuthal phase is

$$\langle \omega_n \rangle = \arctan 2(\omega_{n_{\mathbb{I}}}, \omega_{n_{\mathbb{R}}}), \tag{9}$$

respectively. Eqs. (8–9) exhibit a BH energy level structure that is physically very reasonable, because both the amplitude component $|\omega_n|$ and the imaginary component $\omega_{n_{\mathbb{I}}}$ increase monotonically with the overtone number n [21]. Thus, the context of equivalent harmonic oscillators, $n = 1$ is the *least* damped state for the *lowest* value of $|\omega|$, while $|\omega_n|$ is the *larger* state with a *shorter* lifetime [21]. The asymptotic behavior of the highly-damped states is difficult to determine because of the effect of exponential divergence of the QNM eigenfunctions at $r_* \to \infty$ [17, 18]. However, it is known for the simplest case of a Schwarzschild BH (SBH) as [17, 18]

$$M\omega_n = 0.0437123 - \frac{i}{4}(n + \frac{1}{2}) + O[(n+1)^{-1/2}], \tag{10}$$

a characteristic of the BH itself (in the $n \gg 1$ limit), which is only dependent upon M and is *independent* of l and σ.

Moreover, it was shown in [17, 18] that the numerical limit $Re(M\omega_n) \to 0.0437123$ (as $n \to \infty$) agrees with the quantity $\ln 3/(8\pi)$ and is thereby supported by thermodynamic and statistical physics. So when equipped with $\Delta A = 4 \ln 3 l_p^2$ from $A(M) = 16\pi M^2$ and $dM = E = \hbar\omega$, one can identify

$$\gamma_{Hod}(3) = 4 \ln 3 \tag{11}$$

for the quantum SBH horizon area spectrum of eq. (4), which is upgraded to [17, 18]

$$A_n = \gamma_{Hod}(3) l_p^2 \cdot n. \tag{12}$$

So the wave analysis is consistent with the particle analysis $O(4)$ [17, 18]—a result that supports wave-particle duality with an exactitude of *mechanics*, rather than statistics. From the statistical standpoint, eq. (12) is paramount because it complies with the semi-classical version of Christodoulou's reversible process, which is mechanistic in nature, and is independent of the thermodynamic relation between the BH horizon area A_n and entropy $S_{BH}(n)$ [17, 18]. The accepted relation between A_n and $S_{BH}(n)$ is pertinent if $\forall n$ the constraint

$$\gamma_{Hod}(k) = 4\ln k \quad ; \quad k = 2, 3, ... \quad , \tag{13}$$

is satisfied, such that $g(n) \equiv e^{S_{BH}(n)}$ [17, 18]. Hence, the first independent derivation of k was established [17, 18], which still requires additional contemplation because there is still no general agreement on the spectrum level spacing. But eq. (13) is *still* the *only* expression that is consistent with both the area-entropy thermodynamic relation, statistical physics, and Bohr's correspondence principle [17, 18].

The lower-bound universality of eqs. (2–3) and the entropy universality suggest that the area spectrum of eq. (12) is valid not only for SBHs, but more sophisticated physical structures such as Kerr BHs (KBH) and Kerr-Newman BHs (KNBH) [17, 18]. Moreover, an assumption was proposed regarding the asymptotic behavior of highly-damped QNMs of generic KNBHs [17, 18]. Upon considering the first law of BH thermodynamics [17, 18]

$$dM = \Theta(M, a, Q)dA(M) + \Omega dJ \tag{14}$$

for

$$\Theta(M, a, Q) = \frac{r_+(M, a, Q) - r_-(M, a, Q)}{4A(M)} \tag{15}$$

and

$$\Omega(M, a) = \frac{4\pi a}{A(M)}, \tag{16}$$

where the KNBH inner and outer horizons are

$$r_+(M, a, Q) = M + \sqrt{M^2 - a^2 - Q^2} r_-(M, a, Q) = M - \sqrt{M^2 - a^2 - Q^2} \tag{17}$$

such that $a = J/M$ is the KNBH angular momentum per unit mass, one can find [17, 18]

$$\omega_{n_{\mathbb{R}}} \to \Theta(M, a, Q)\gamma_{Hod}(3) + \Omega(M, a)m, \tag{18}$$

where $n \to \infty$, such that m is the perturbation field's azimuthal eigenvalue that corresponds to its phase.

Along this approach, for large n, the strictly thermal asymptotic behavior [21] was employed

$$8\pi M\omega_n = \frac{\omega_n}{8\pi M} = \ln 3 + 2\pi i(n + \frac{1}{2}) + O[(n+1)^{-1/2}], \tag{19}$$

for the Hawking temperature

$$T_H = \frac{\hbar}{8\pi M} \tag{20}$$

to re-write eq. (8) as

$$\hbar|\omega_n| = \sqrt{m_0^2 + p_n^2}, \tag{21}$$

for the underlying QNM Pythagorean components

$$m_0 = \omega_{n_{\mathbb{R}}} = T_H \ln 3 \quad \text{and} \quad p_n = \omega_{n_{\mathbb{I}}} = 2\pi T_H \left(n + \frac{1}{2}\right) \tag{22}$$

so eq. (9) becomes

$$\langle\omega_n\rangle = \arctan 2(p_n, m_0), \tag{23}$$

for the trigonometric functions. In the very large n approximation, the leading term in the imaginary part of the complex frequencies in eq. (19) becomes dominant and spin independent, while, strictly speaking, eq. (19) works only for scalar (spin 0) and gravitational (spin 2) perturbations, see [21] for details. In the p_n of eq. (22), recall that the 2π mathematically relates a circular radius to a circular circumference and is the difference between the uncharged and charged area quantization lower bounds of eqs. (2–3) that complies with [19, 20]—so one could hypothesize that this intriguing 2π critical value may suggest a fundamental relationship to a circularly-symmetric or spherically-symmetric physical topology. The formulation of p_n [21] is fascinating because it harmonizes a quantized particle with antiperiodic boundary conditions on a circle of circumference length

$$L = \frac{\hbar}{T_H(M)} = 8\pi M. \tag{24}$$

At this point, preparations were made to re-examine some aspects of quantum BH physics by assuming the relevant frequencies are $|\omega_n|$, rather than $\omega_{n_{\mathbb{R}}}$ [21].

Next, in [21] some important quantized spacing results for the discrete BH area spectrum were recalled. First, the conjecture of [16] was noted [21], which proposed that the level spacing is in quantized units of l_p^2 and thereby resulted in the SBH area quantum $\Delta A = 8\pi l_p^2$ of eq. (2) so we label $\gamma_{Bek} = 8\pi$ as Bekenstein's dimensionless constant. Second, he [21] recognized that the results of [17, 18] revealed a similar quantization, but utilized the SBH QNM properties to discover the different numerical coefficient, namely $\Delta A = \gamma_{Hod}(3) l_p^2$ of eq. (12).

Although the hypothesis [17, 18] is exciting (primarily due to some possible connections with LQG), it still exhibits some complications [21]. Additional analysis on the term $\gamma_{Hod}(3)$ with its origin in $\omega_{n_{\mathbb{R}}}$ for eq. (19) is in fact *not universal* because it does not comply with charged and/or rotating BHs [21]. For example, in the case of a KBH or KNBH with $a = J/M$, one finds that the large n limit and the limit $a \to 0$ do not commute because if one first considers $a \to 0$, then $\omega_{n_{\mathbb{R}}}$ does not reduce to $\ln 3/(8\pi M)$ and instead vanishes as $a^{1/3}$, which means that the area quantum becomes arbitrarily small if one gives the BH an infinitesimal rotation [21]. Similarly, in the case of a Reissner-Nordström BHs (RNBH) or KNBH, one finds that $\omega_{n_{\mathbb{R}}}$ changes discontinuously if the limits $Q \to 0$ and $n \to \infty$ are interchanged [21]. Thereafter, a couple of additional exploits were pointed out in Hod's conjecture [17, 18], so it was initially concluded that it "does not reflect any intrinsic property of the BH, and the would-be area quantum vanishes in various instances" and that its "area quantization holds only for a transition from (or to) a BH in its fundamental state, while transitions among excited levels do not obey it" [21]. But, after additional scrutiny and venture [21], it was determined that all of the above complications are deleted when, in the conjecture of [17, 18], one replaces $\omega_{n_{\mathbb{R}}}$ with $|\omega_n|$! For large n and the transition $n \to n-1$, eq. (19) and $|\omega_n| \simeq \omega_{n_{\mathbb{I}}}$ yield the absorbed energy $\Delta M = \hbar[|\omega_n| - |\omega_{n-1}|] = \hbar/(4M)$, such that [21]

$$\Delta A = 32\pi M \Delta M = 8\pi l_p^2, \tag{25}$$

which complies with the old results of [16] because $\gamma_{Bek} = 8\pi$. Thus, given the equal spacing for $|\omega_n|$ at large n, all other transitions require a larger energy; i.e. $n \to n-2$ consumes about twice the energy [21]. Even if one dares to extrapolate at low n, where semi-classical reasoning may be destroyed, we still realize a non-vanishing area quantum of eq. (25)'s magnitude order [21]. Therefore, the final results of [21] concluded that the spacing of eq. (25) indicates a *consistent* SBH horizon area quantization, which implies that l_p is the minimum magnitude order length for the existential and generalized uncertainty principle.

Consequently, in terms of BH entropy and micro-states, the work of [21] determined that for large n, the horizon area quantum is $\Delta A = \gamma_{Bek} l_p^2$, such that $\gamma_{Bek} = 8\pi$ of [16] replaces $\gamma_{Hod}(3) = 4\ln 3$ of [17, 18]. Thus, the total horizon area must be of the form [21]

$$A = N\Delta A = N\gamma l_p^2, \tag{26}$$

where the area quanta number $N = A/\Delta A$ is an integer but is *not* the same as the integer n (which is used to label the BH perturbation field QNM states). Hence, the BH entropy is defined as [16, 21]

$$S_{BH} = \frac{A}{\delta}, \tag{27}$$

where

$$\delta = 4l_p^2 \tag{28}$$

agrees with the approach of [19, 20] and additionally the LQG approaches of [9, 10, 11] to the same order of magnitude. Therefore, at level $N(M)$, it was expected that the number of possible BH micro-states (or "BH micro-state space cardinality") is [21]

$$g(N) \propto e^{A/\delta} = e^{N\Delta A/\delta} = e^{\gamma N/4}. \tag{29}$$

Subsequently, upon fixing the constant for $N = 1$ in eq. (27), there is only one micro-state in the state space, namely $g(N) = 1$, which gives [21]

$$g(N) = e^{(\gamma/4)(N-1)}. \tag{30}$$

This operates under the required *assumption* that $g(N)$ is an integer, which restricts γ to in the form $\gamma_{Hod}(k)$ of eq. (13), such that k is an integer [21]; the value $\gamma_{Hod}(3)$ is in the form of $\gamma_{Hod}(k)$ but the value γ_{Bek} is not—γ_{Bek} is only in the form of $\gamma_{Hod}(k)$ if $k = e^{2\pi}$ because

$$\gamma_{Bek} = 8\pi = 4(2\pi) = 4(\ln k) \tag{31}$$

holds for the periodicity $\ln k = 2\pi$ but clearly violates the "k must be an integer" or "k-constraint" assertion—we also note that $\gamma_{Hod}(e) = 4 \ln e$ takes a similar form to $\gamma_{Hod}(k)$ but also violates the k-constraint.

These attempts to restrict γ raise a number of objections [21]. First, even in the trusted semi-classical framework, N is *gigantic*, therefore $g(N)$ is the exponential of a colossal number [21]. Even if the number of micro-states must be an integer, there is no hope that a semi-classical (or even a classical and statistical) calculation can identify this quantity with a precision of order one, which is requisite to distinguishing between an integer and non-integer result [21]. Moreover, the above $g(N)$ expression assumes that the horizon area quantum ΔA is legal from large N down to $N = 1$, where this semi-classical approximation is unwarranted [21]. So although we see that eqs. (21–22) determine equally spaced levels in the limit of highly-excited states, the level spacing for lowly-excited states are *not* equally spaced [21].

Thus, when the value γ_{Bek} [16] was employed in $S_{BH}(M) = \gamma_{Bek} N(M)/4$, the result [21]

$$S_{BH} = 2\pi N + O(1) \tag{32}$$

was discovered, such that $g(N) \propto e^{2\pi N(M)}$, for the leading order in the large N limit. Basically, eq. (32) gives a discrete spectrum which indicates that the entropy is an adiabatic invariant in accordance to Bohr's correspondance principle [21]. All of this replicates the BH behavior and perturbation field states in terms of highly-damped harmonic oscillators whose real frequencies are the amplitude-modulus $|\omega_n|$ (instead of $\omega_{n_{\mathbb{R}}}$) for the area quantization $\Delta A = \gamma_{Bek} l_p^2$ (instead of $\Delta A = \gamma_{Hod}(3) l_p^2$). At this point, we also note that $\Delta A = \gamma_{Bek} l_p^2$ was also obtained in the alternative approach of [22] *without* the use of QNMs—another remarkable result that supports this development.

4 Non-strictly thermal horizon area and entropy quantization with effective states

The striking work of Parikh and Wilczek [23] demonstrated that Hawking's radiation spectrum cannot be strictly thermal, where such a *non-strictly thermal* character implies that the BH spectrum is also non-strictly continuous [24, 25]: this generates a natural correspondence between Hawking radiation and the BH perturbation field QNM states, which supports the idea that BHs result in highly-excited states in an underlying unitary quantum gravity theory [24, 25]. Moreover, the strictly thermal spectrum *deviation* results of [5] strongly suggested that single particle quantum mechanical approaches may be essential for finding potential solutions to the BH information puzzle.

Thus, after a careful and extensive examination of the non-strictly thermal and non-strictly continuous BH energy spectrum and the spherically-symmetric particle tunneling results of [23] in [24, 25] with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), the conventional Hawking temperature $T_H(M)$ of eq. (20) was replaced by defining the SBH's *effective temperature* in eq. (3) of [24] as

$$\begin{aligned} T_{E_{SBH}}(M,-\omega) &= \tfrac{2M}{2M+(-\omega)}T_H = \tfrac{1}{4\pi(2M+(-\omega))} \\ &= \tfrac{1}{8\pi M_E(M,-\omega)} = \tfrac{1}{2\pi R_{E_{SBH}}(M,-\omega)} = \tfrac{1}{\beta_{E_{SBH}}(M,-\omega)} \end{aligned} \tag{33}$$

for the *emission* of an uncharged particle with energy-frequency ω so the SBH *contracts*, where M is the SBH's initial mass before the emission, $M - \omega$ is the SBH's final mass after the emission, M_E is the SBH's *effective mass* defined in eq. (5) of [24] as

$$M_E(M,-\omega) = M + \frac{-\omega}{2} = M - \frac{\omega}{2}, \tag{34}$$

$R_{E_{SBH}}$ is the SBH's *effective horizon* defined in eq. (5) of [24] as

$$R_{E_{SBH}}(M,-\omega) = 2M_E(M,-\omega), \tag{35}$$

and β_E is the SBH's *effective Botzmann factor* defined in eq. (12) of [25]. The new *effective quantities* $T_{E_{SBH}}$, M_E, $R_{E_{SBH}}$, and $\beta_{E_{SBH}}$ are *average* quantities which characterize the *effective state* of a *discrete* process rather than a *continuous* process [24, 25]. Thus, for example, eqs. (33–35) indicate that the circular antiperiodic boundary conditions of eq. (24) can be replaced with with the *effective horizon circumference*

$$\begin{aligned} L_{E_{SBH}}(M,-\omega) &= \tfrac{1}{T_{E_{SBH}}(M,-\omega)} = 8\pi M_E(M,-\omega) = \beta_{E_{SBH}}(M,-\omega), \\ &= 4\pi R_{E_{SBH}}(M,-\omega) = \tfrac{2\pi}{\kappa_{E_{SBH}}(M,-\omega)} \end{aligned} \tag{36}$$

which is simply the geometric equivalence of Boltzmann's effective physical quantity $\beta_{E_{SBH}}$, such that the fundamentally related $\kappa_{E_{SBH}}$ is the SBH's *effective surface gravity*. Subsequently, the results of eqs. (33–35) were instrumental in the establishment of two additional effective quantities [25]: the SBH's *effective line element* from eq. (14) of [25]

$$ds^2_{E_{SBH}} = -\left(1 - \frac{R_{E_{SBH}}(M,-\omega)}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{R_{E_{SBH}}(M,-\omega)}{r}} + r^2(\sin^2\varphi d\phi^2 + d\varphi^2), \tag{37}$$

which encompasses the *dynamical* geometry of the SBH during the emission or absorption of the particle. Through a rigorous examination of Hawking's arguments [36, 29], the Euclidean form of eq. (18) in [25] was successfully presented as

$$ds^2_{E_{SBH}} = x^2\left[\frac{d\tau}{4M\left(1 - \frac{\omega}{2M}\right)}\right]^2 + \left(\frac{r}{R_E(M,-\omega)}\right)^2 dx^2 + r^2(\sin^2\varphi d\phi^2 + d\varphi^2), \tag{38}$$

which is regular at $x = 0$ and $r = R_E(M, -\omega)$ and permits to rigorously obtain eq. (37). In [36, 29] it was shown that τ serves as an angular variable with the periodicity of $\beta_{E_{SBH}} = L_{E_{SBH}}$ in eq. (36) with the underlying antiperiodic boundary conditions.

Henceforth, the procedure of [28] authorized the acquisition of the *corrected* physical states for bosons and fermions from eq. (15) of [25] as

$$\begin{aligned}|\Psi\rangle_{boson} &= (1-\exp(\tfrac{-\omega n}{T_E(M,-\omega)})^{\frac{1}{2}}\Sigma_n \exp(-\omega 4\pi n M_E(M,-\omega)|n_{out}^{Left}\rangle \otimes |n_{out}^{Right}\rangle \\ |\Psi\rangle_{fermion} &= (1+\exp(\tfrac{-\omega n}{T_E(M,-\omega)})^{-\frac{1}{2}}\Sigma_n \exp(-\omega 4\pi n M_E(M,-\omega)|n_{out}^{Left}\rangle \otimes |n_{out}^{Right}\rangle\end{aligned} \tag{39}$$

which respectively correspond to the encoding of the emission probability distributions from eq. (16) of [25], which are

$$\begin{aligned}\langle n\rangle_{boson} &= \frac{1}{\exp(\frac{-\omega n}{T_E(M,-\omega)})-1} \\ \langle n\rangle_{fermion} &= \frac{1}{\exp(\frac{-\omega n}{T_E(M,-\omega)})+1}.\end{aligned} \tag{40}$$

At this point, we note that in order to compute the SBH effective parameters for the *absorption* of an uncharged particle with energy-frequency ω, the $-\omega$ argument of eqs. (33–38) may be quickly replaced with $+\omega$—if we wish to reference both emission and absorption simultaneously in such formulas, it is straightforward to specify $\pm\omega$.

Next, the work of [24] deployed eq. (33) to re-write eq. (22) in the corrected form

$$m_n = T_E(M, -|\omega_n|)\ln 3 \quad \text{and} \quad p_n = T_E(M, -|\omega_n|)2\pi i\left(n+\frac{1}{2}\right), \tag{41}$$

which takes into account the non-strictly thermal behavior of the SBH, where

$$\omega_n = m_n + p_n + \mathcal{O}(n^{-\frac{1}{2}}). \tag{42}$$

We stress that, although eqs. (41) and (42) have been derived in [24] only intuitively, they have been rigorously derived in the appendix of [35]. In that paper it has been also shown that in the very large n approximation, the leading term in the imaginary part of the complex frequencies in eq. (42) becomes dominant and spin independent, while, strictly speaking, eq. (42) works only for scalar and gravitational perturbations, see [35] for details. Then, considering the leading term in the imaginary part of the complex frequencies, eq. (24) of [24] gives

$$|\omega_n| = M - \sqrt{M^2 - \frac{1}{2}(n+\frac{1}{2})} \tag{43}$$

for emission. In eq. (43) it was observed that the emission $n \to n-1$ gives the energy variation of eq. (29) in [24] as

$$\Delta M_n = |\omega_{n-1}| - |\omega_n| = -f(M, n) \tag{44}$$

for the spacing of eq. (25) as

$$\Delta A_{SBH}(M, \Delta M_n) = 32\pi M \Delta M_n = -32\pi M \times f(M,n) \approx -\gamma_{Bek} \tag{45}$$

in the very large n limit, which is the same order of magnitude as the original area quantization result [16]—the $f(M,n)$ of eqs. (44–45) was constructed in eq. (30) of [24]. We recall that the SBH's horizon area A_{SBH} is related to its mass M via the relation $A_{SBH} = 16\pi M^2$ [16]. From this, one gets that if A_{SBH} is quantized as $|\Delta A| = \gamma_{Bek}$ [16, 21] and $|\Delta A| = \gamma_{Hod}(3)$ [17, 18], then the SBH's *total horizon area* must be [24]

$$A_{SBH}(M,n) = N_{SBH}(M,n)|\Delta A_{SBH}(M,n)| = 16\pi M^2 = 4\pi R_{SBH}^2, \tag{46}$$

for the SBH's *event horizon* $R_{SBH} = 2M$, such that eq. (33) of [24] is

$$N_{SBH}(M,n) = \frac{A_{SBH}(M,n)}{|\Delta A_{SBH}(M,n)|} = \frac{16\pi M^2}{32\pi M \Delta M_n} = \frac{M}{2f(M,n)}, \tag{47}$$

where the well-known SBH's *Bekenstein-Hawking entropy* [15, 16, 27] was re-written as [24]

$$\begin{array}{ccl} S_{SBH}(M,n) & = & \frac{A_{SBH}(M,n)}{4} \\ & = & 8\pi N_{SBH}(M,n) M |\Delta M_n| \\ & = & 8\pi N_{SBH}(M,n) M \times f(M,n), \end{array} \tag{48}$$

which indicates the crucial result that S_{SBH} is a function of the quantum overtone number n [24].

On the other hand, it is a common and general belief that there is no reason to expect that Bekenstein-Hawking entropy will be the whole answer for a correct unitary theory of quantum gravity [37]. For a better understanding of black hole's entropy one needs to go beyond Bekenstein-Hawking entropy and identify the sub-leading corrections [37]. The quantum tunnelling approach can be used to obtain the sub-leading corrections to the second order approximation [38]. One gets that the black hole's entropy contains three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and the inverse area term [38]

$$S_{total} = S_{BH} - \ln S_{BH} + \frac{3}{2A}. \tag{49}$$

In fact, if one wants to satisfy the unitary quantum gravity theory the logarithmic and inverse area terms are requested [38]. Apart from a coefficient, this correction to the black hole's entropy is consistent with the one of loop quantum gravity [38], where the coefficient of the logarithmic term has been rigorously fixed at $\frac{1}{2}$ [38, 39]. The expression (48) for Bekenstein-Hawking entropy permits to re-write eq. (49) as [24]

$$S_{total(SBH)} = 8\pi NM \cdot f(M,n) - \ln\left[8\pi NM \cdot f(M,n)\right] + \frac{3}{64\pi NM \cdot f(M,n)}. \tag{50}$$

In the top line of eq. (48), observe that denominator 4, which divides the numerator A_{SBH} to compute the resulting S_{SBH}, is reminiscent of the δ from [16, 19] in eqs. (27–28). Additionally, note that the results of eqs. (45) and (48) indicate the SBH's *Bekenstein-Hawking entropy change* is

$$\Delta S_{SBH}(M,n) = \frac{\Delta A_{SBH}(M,n)}{4}, \tag{51}$$

where clearly a change of *negative entropy* ($\Delta S_{SBH} < 0$) recurs for absorption transitions because energy is conserved in 4D space-time.

Therefore, in order to incorporate the emerging SBH effective state framework, eqs. (46–48) become

$$\begin{aligned} A_{E_{SBH}}(M,\Delta M_n) &= N_{E_{SBH}}(M,\Delta M_n)|\Delta A_{E_{SBH}}(M,\Delta M_n)| \\ &= 16\pi M_E^2(M,\Delta M_n) = 4\pi R^2_{E_{SBH}}(M,\Delta M_n), \end{aligned} \tag{52}$$

$$\begin{aligned} N_{E_{SBH}}(M,\Delta M_n) &= \tfrac{A_{E_{SBH}}(M,\Delta M_n)}{|\Delta A_{E_{SBH}}(M,\Delta M_n)|} \\ &= \tfrac{16\pi M_E^2(M,\Delta M_n)}{32\pi M_E(M,\Delta M_n) n\times f(M,n)} \\ &= \tfrac{M_E(M,\Delta M_n)}{2f(M,n)}, \end{aligned} \tag{53}$$

and

$$\begin{aligned} S_{E_{SBH}}(M,\Delta M_n) &= \tfrac{A_{E_{SBH}}(M,\Delta M_n)}{4} = \pi R^2_{E_{SBH}}(M,\Delta M_n) \\ &= 8\pi N_{E_{SBH}}(M,\Delta M_n) M_E(M,\Delta M_n)|\Delta M| \\ &= 8\pi N_{E_{SBH}}(M,\Delta M_n) M_E(M,\Delta M_n)\times f(M,n) \\ &= \tfrac{f(M,n)}{T_E(M,\Delta M_n)}. \end{aligned} \tag{54}$$

One also obtains the total effective entropy as

$$S_{(total)E_{SBH}}(f(M,\Delta M_n)) = \frac{f(M,n)}{T_E(M,\Delta M_n)} - \ln\left[\frac{f(M,n)}{T_E(M,\Delta M_n)})\right] + \frac{3T_E(M,\Delta M_n)}{8f(M,n)}. \tag{55}$$

Hence, the effective state quantities of eqs. (52–55) recognize the seemingly pertinent, disjoint aspects of the candidate horizon area theories of Bekenstein [15, 16], Hod [17, 18], and Maggiore [21] by replacing Hawking's strictly thermal T_H [27, 29] with the non-strictly thermal T_E [24] to establish a preliminary generalization and unification.

Thereafter, subsequent work initiated an effective state framework generalization from SBHs [24] to KBHs [26], which was largely inspired by the discoveries of [30, 31, 32, 33]. It is known that the quantifiable difference between a SBH and a KBH is the angular momentum components [26]. Hence, for this the the KBH's *effective angular momentum* as $J_E(M,\Delta M_n) = M_E(M,\Delta M_n)\alpha_E(M,\Delta M_n)$

[26], where the KBH's *effective specific angular momentum* from eq. (3.13) in [26] is expressed as

$$\alpha_E(M,\Delta M_n)=\frac{J_E(M,\Delta M_n)}{M_E(M,\Delta M_n)} \tag{56}$$

for the additional KBH's *effective angular momentum components*

$$\Delta_E(M,\Delta M_n)=r^2-2M_E(M,\Delta M_n)r+\alpha_E^2(M,\Delta M_n) \tag{57}$$

and

$$\Sigma_E(M,\Delta M_n)=r^2+\alpha_E^2(M,\Delta M_n)\cos^2\varphi \tag{58}$$

from eqs. (3.14–3.15) in [26] that authorized the identification of the KBH's *effective outer and inner horizons*

$$\begin{aligned} R_{+E_{KBH}}(M,\Delta M_n) &= M_E(M,\Delta M_n)+\sqrt{M_E^2(M,\Delta M_n)-\alpha_E^2(M,\Delta M_n)} \\ R_{-E_{KBH}}M,\Delta M_n) &= M_E(M,\Delta M_n)-\sqrt{M_E^2(M,\Delta M_n)-\alpha_E^2(M,\Delta M_n)}, \end{aligned} \tag{59}$$

and the corresponding KBH's *effective line element*

$$\begin{aligned} ds^2_{E_{KBH}} &= -\left(1-\tfrac{2M_E(M,\Delta M_n)r}{\Sigma_E(M,\Delta M_n)}\right)dt^2-\tfrac{4M_E(M,\Delta M_n)\alpha_E(M,\Delta M_n)r\sin^2\varphi}{\Sigma_E(M,\Delta M_n)}dtd\phi \\ &\quad +\tfrac{\Sigma_E(M,\Delta M_n)}{\triangle_E(M,\Delta M_n)}dr^2+\Sigma_E(M,\Delta M_n)d\varphi^2 \\ &\quad +\left(r^2+\alpha_E^2(M,\Delta M_n)+2M_E(M,\Delta M_n)\alpha_E^2(M,\Delta M_n)r\sin^2\varphi\right)\sin^2\varphi d\phi^2 \end{aligned} \tag{60}$$

respectively, which takes into due account the KBH's dynamical geometry as it emits or absorbs particles [26]. From there, eqs. (56–59) permitted the definition of the KBH's *effective (outer) horizon area* of eq. (3.19) in [26] as

$$\begin{aligned} A_{+E_{KBH}}(M,\Delta M_n) &= 4\pi\left(R^2_{+E_{KBH}}(M,\Delta M_n)+\alpha_E^2(M,\Delta M_n)\right) \\ &= 8\pi\left(M_E^2(M,\Delta M_n)+\sqrt{M_E^4(M,\Delta M_n)-J_E^2(M,\Delta M_n)}\right), \end{aligned} \tag{61}$$

the KBH's *effective temperature* of eq. (3.20) in [26] as

$$\begin{aligned} T_{+E_{KBH}}(M,\Delta M_n) &= \tfrac{R_{+E_{KBH}}(M,\Delta M_n)-R_{-E_{KBH}}(M,\Delta M_n)}{A_{+E}(M,\Delta M_n)} \\ &= \tfrac{\sqrt{M_E^4(M,\Delta M_n)-J_E^2(M,\Delta M_n)}}{4\pi M_E(M,\Delta M_n)\left(M_E^2(M,\Delta M_n)+\sqrt{M_E^4(M,\Delta M_n)-J_E^2(M,\Delta M_n)}\right)}, \end{aligned} \tag{62}$$

and the KBH's *effective area quanta* of eq. (3.22) in [26] as

$$\Delta A_{+E_{KBH}}(M,\Delta M_n)=16\pi M_E(M,\Delta M_n)\left[1+\left(1-\frac{J_E^2(M,\Delta M_n)}{M_E^4(M,\Delta M_n)}\right)^{-\frac{1}{2}}\right]\Delta M_n \tag{63}$$

for the KBH's *effective area quanta number* of eq. (3.23) in [26] as

$$N_{+E_{KBH}}(M,\Delta M_n) = \frac{A_{+E_{KBH}}(M,\Delta M_n)}{|\Delta A_{+E_{KBH}}(M,\Delta M_n)|} = \frac{M_E(M,\Delta M_n)}{2f(M,n)}, \tag{64}$$

which enabled the KBH's *effective Bekenstein-Hawking entropy* of eq. (3.24) in [26] to be identified as

$$\begin{aligned} S_{+E_{KBH}}(M,\Delta M_n) &= \tfrac{A_{+E_{KBH}}(M,\Delta M_n)}{4} \\ &= 8\pi N_{+E_{KBH}}(M,\Delta M_n) M_E(M,\Delta M_n) \times f(M,n). \end{aligned} \tag{65}$$

Thus, for $J_E \ll M_E^2$, it was confirmed that eqs. (61–65) reduce to the SBH case of eqs. (46–54) [24].

Consequently, following the QNM KBH effective state framework [26], the constructions were generalized to a *non-extremal* RNBH version [34]. For this implementation, a *new* definition of ΔM_n for RNBH QNMs was formulated to construct the new RNBH effective quantities [34]. Starting from eq. (40) in [34] the RNBH's *effective charge* was defined for small Q as

$$Q_E(Q,q) = \frac{Q + (Q \pm q)}{2}, \tag{66}$$

where Q is the RNBH's *initial* charge *before* the transition and $Q \pm q$ is the RNBH's *final* charge *after* the transition. The BH's M_E of eq. (34) and the RNBH's Q_E of eq. (66) can be used to identify the RNBH's *effective line element* as

$$\begin{aligned} ds^2_{E_{RNBH}} &= \left(1 - \tfrac{2M_E(M,\Delta M)}{r} + \tfrac{Q_E^2(Q,q)}{r^2}\right) dt^2 - \tfrac{dr^2}{1 - \frac{2M_E(M,\Delta M)}{r} + \frac{Q_E^2(Q,q)}{r^2}} \\ &\quad - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{67}$$

Next, for a quantum transition between the levels n and $n-1$, the RNBH QNM definition of ΔM_n in [34] and the Q_E of eq. (66) were deployed to define the RNBH's *effective outer and inner horizons* from eq. (60) in [34] as

$$\begin{aligned} R_{+E_{RNBH}}(M,\Delta M_n,Q,q) &= M_E(M,\Delta M_n) + \sqrt{M_E^2(M,\Delta M_n) - Q_E^2(Q,q)} \\ R_{-E_{RNBH}}(M,\Delta M_n,Q,q) &= M_E(M,\Delta M_n) - \sqrt{M_E^2(M,\Delta M_n) - Q_E^2(Q,q)}, \end{aligned} \tag{68}$$

the RNBH's *effective (outer) horizon area* from eq. (61) in [34] as

$$\begin{aligned} A_{+E_{RNBH}}(M,\Delta M_n,Q,q) &= 4\pi R^2_{+E_{RNBH}}(M,\Delta M_n,Q,q) \\ &= 4\pi \left(M_E(M,\Delta M_n) + \sqrt{M_E^2(M,\Delta M_n) - Q_E^2(Q,q)}\right)^2, \end{aligned} \tag{69}$$

the RNBH's *effective horizon area change* from eq. (67) in [34] as

$$\Delta A_{+E_{RNBH}}(M, \Delta M_n, Q, q) = \frac{2\Delta M_n q + \pi Q^3}{(M^2 - Q^2)^{3/2}}, \tag{70}$$

the RNBH's *effective Bekenstein-Hawking entropy* from eq. (62) in [34] as

$$S_{+E_{RNBH}}(M, \Delta M_n, Q, q) = \frac{A_{+E_{RNBH}}(M, \Delta M_n, Q, q)}{4}, \tag{71}$$

the RNBH's *effective Bekenstein-Hawking entropy change* from eq. (66) in [34] as

$$\Delta S_{+E_{RNBH}}(M, \Delta M_n, Q, q) = \frac{\Delta A_{+E_{RNBH}}(M, \Delta M_n, Q, q)}{4}, \tag{72}$$

and the RNBH's *effective quantum area number* from eq. (68) in [34] as

$$N_{+E_{RNBH}}(M, \Delta M_n, Q, q) = \frac{A_{+E_{RNBH}}(M, \Delta M_n, Q, q)}{|\Delta A_{+E_{RNBH}}(M, \Delta M_n, Q, q)|}. \tag{73}$$

Thus, for $Q_E^2 \ll M_E^2$, it was confirmed that eqs. (69), (71), and (73) reduce to the corresponding effective quantities of the SBH case for eqs. (52–54) [34].

5 Conclusion

In this work, we reported and examined the pertinent groundbreaking work of the strictly thermal and non-strictly thermal spectrum level spacing of the BH horizon area and entropy quantization from a semi-classical approach. For this, we chronologically reviewed a series of imperative corrections that eventually permits a BH's horizon area and entropy spectrum to be generalized from strictly thermal to non-strictly thermal with QNMs and effective states [24, 25, 26, 34]. The reported strictly thermal results are significant because they ultimately build up to the new, emerging BH effective state framework that exemplifies the underlying QG theory [24, 25, 26, 34]. Moreover, these outcomes launch an effective unification that begins to merge and generalize an array of strictly thermal quantization approaches to a *single*, consolidated non-strictly thermal approach [24, 25, 26, 34]. In general, all of this research is important to physics and science because the characteristic physical laws of BHs must be understood in order to resolve, for example, the puzzles imposed by the BH information paradox and firewalls [1, 2, 3, 4, 5] in nature.

First, we discussed attack approaches that initiated universal upper and lower bounds on the area quanta for non-extremal BHs that emit or absorb particles, which may or may not be charged. We reviewed the mechanisms and predicted quanta for both uncharged and charged particles, along with the relevant aspects of wave-particle duality. Therefore, we conveyed the importance of linking the discrete particles to continuous waves with perturbation field QNMs that encode the BH's asymptotic behavior of spectral states and transitions.

Subsequently, we identified a series of damped harmonic oscillator QNM configurations and strictly thermal corrections that were systematically deployed to encode a BH's behavior and quantization of area and entropy. Next, we shifted to the strictly thermal spectrum deviation corrections [23] that permitted the generalized non-strictly thermal spectrum with cutting-edge effective states [24, 25] for encoding the area and entropy quantization of SBHs, KBHs [26], and non-extremal RNBHs [34].

In our opinion, the BH quantization work that we chronologically reviewed in this paper highlights a series of striking scientific results that are beneficial for tackling the gigantic problems imposed by BHs in the domain of space-time physics. In the future, such findings should be subjected to additional rigorous analysis, debate, experimentation, and hard work via the scientific method. In particular, we suggest that future work should focus on applying the non-strictly thermal spectrum and effective state framework [24, 25, 26, 34] to additional classes of BHs and alternative unification approaches.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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