Volume Enclosed by Example Subdivision Surfaces

by Jan Hakenberg - May 15th, 2014

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Abstract

Simple meshes such as the cube, tetrahedron, and tripod frequently appear in the literature to illustrate the concept of subdivision. The formula for the volume enclosed by subdivision surfaces has only recently been identified. We specify simple meshes and state the volume enclosed by the corresponding limit surfaces. We consider the subdivision schemes Doo-Sabin, Midedge, Catmull-Clark, and Loop.

Introduction

The volume enclosed by the subdivision surface defined by a closed, orientable mesh M is the sum over all facets f

$$\mathrm{vol}(S^{\infty}(M)) = \sum_{f \in M} \sum_{i,j,k}^{m(f)} \hat{Y}_{i,j,k}^{\tau(f)} \, \mathrm{px}_i \, \mathrm{py}_j \, \mathrm{pz}_k$$

The coefficients $\hat{Y}_{i,j,k}^{\tau(f)}$ consititute trilinear forms that are alternating. The coefficients are uniquely determined by the subdivision weights for topology type $\tau(f)$ of a facet f. The points $(\mathrm{px}_i, \mathrm{py}_i, \mathrm{pz}_i)$ for $i = 1, 2, \ldots, m(f)$ are the vertices of the mesh that determine the subdivision surface associated to facet f. For the derivation and proof of the formula, see [Hakenberg et al. 2014].

In this article, we state the volume enclosed by subdivision surfaces generated from various example meshes. Each of the four subdivision schemes is treated in a separate section

- Doo-Sabin
- Midedge
- Catmull-Clark
- Loop

Each example has the following structure:

• We state the vertex coordinates and topology of the mesh. The specification is omitted if the mesh was already introduced previously.

• The subdivision iteration is illustrated up to a certain level. After one or two rounds of subdivision, the mesh topology admits the application of the volume formula. In that case, the facets are colored based on their contribution to the volume.

• We state the volume enclosed by the corresponding subdivision surface. In some cases, we are unable to obtain the volume in symbolic form, and revert to numeric precision.

• Then, the approximation of the volume by the piecewise linear surface from the first refinements are tabulated and plotted. Non-planar quads are triangulated in an arbitrary fashion.

The results can help to validate implementations of the volume formula.

Doo-Sabin

The Doo-Sabin subdivision scheme is published as [Doo/Sabin 1978]. The algorithm applies to meshes with *n*-gons. One round of Doo-Sabin subdivision requires the contraction of each face in the mesh. The weights to contract an *n*-gon are $\omega_0, \omega_1, \ldots, \omega_{n-1}$. Specifically,

$$\omega_0 = \frac{1}{4n}(n+5), \text{ and } \omega_k = \frac{1}{4n}\left(3 + 2\,\text{Cos}\left[\frac{2\pi k}{n}\right]\right) \text{ for } k = 1, 2, \ldots, n-1.$$

The weights are applied in a rotational fashion. For instance,

- a triangle is contracted using the mask $\frac{2}{3}, \frac{1}{6}, \frac{1}{6}$ centered at all 3 vertices;
- a quad is contractred using $\frac{9}{16}, \frac{3}{16}, \frac{1}{16}, \frac{3}{16}$;
- for a pentagon, the mask is $\frac{1}{2}, \frac{1}{40}\left(5+\sqrt{5}\right), \frac{1}{40}\left(5-\sqrt{5}\right), \frac{1}{40}\left(5-\sqrt{5}\right), \frac{1}{40}\left(5+\sqrt{5}\right)$;
- a hexagon is contracted using $\frac{11}{24}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{12}, \frac{1}{6}$.

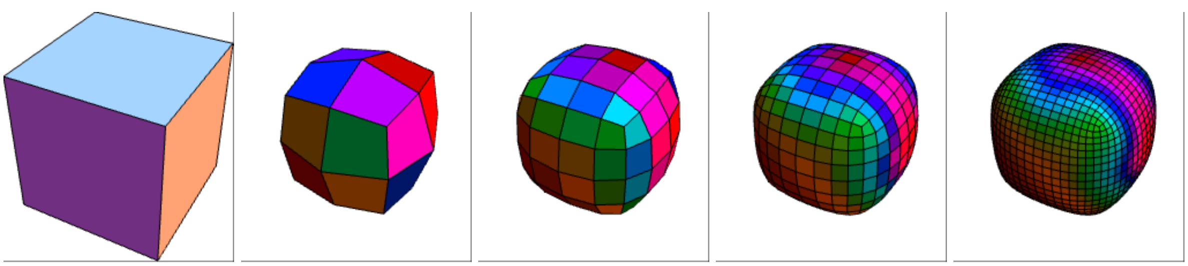
After one round of subdivision, all newly introduced faces are quads.

Cube

Vertices ↓ Faces ↓

0	0	0
1	0	0
0	1	0
1	1	0
0	0	1
1	0	1
0	1	1
1	1	1

1	2	6	5
3	4	2	1
4	8	6	2
5	6	8	7
7	8	4	3
7	3	1	5



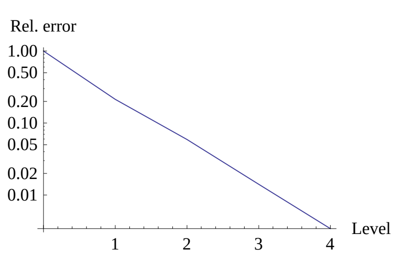
Important: In order to color the facets based on their contribution to the global volume, we display the *dual* mesh.

Required valences τ(f) ∈ {3}

Limit volume ↓ (≈0.629133064516129032258064516129)

$$\frac{6241}{9920}$$

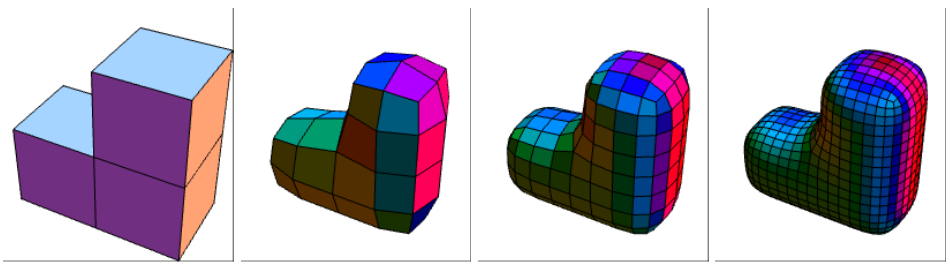
Level	Volume	Delta to ∞
0	1	0.370867
1	$\frac{17}{24}$	0.0792003
2	$\frac{125}{192}$	0.0219086
3	$\frac{7795}{12288}$	0.00522566
4	$\frac{20657}{32768}$	0.00126855



Corner Mesh

Vertices ↓ Faces ↓

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 4 & 2 & 1 \\ 5 & 6 & 8 & 7 \\ 7 & 8 & 4 & 3 \\ 7 & 3 & 1 & 5 \\ 2 & 11 & 13 & 6 \\ 6 & 13 & 15 & 9 \\ 4 & 12 & 11 & 2 \\ 12 & 14 & 13 & 11 \\ 14 & 16 & 15 & 13 \\ 9 & 15 & 16 & 10 \\ 6 & 9 & 10 & 8 \\ 4 & 8 & 14 & 12 \\ 8 & 10 & 16 & 14 \end{pmatrix}$$

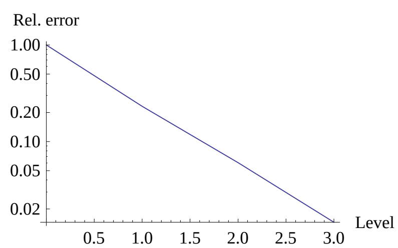


Required valences $\tau(f) \in \{3, 4, 5\}$

Limit volume ↓ (≈2.32171405683749043717321019725)

$$\frac{11\,271\,914\,948\,361 + 2\,535\,566\,756\,\sqrt{5}}{4\,857\,439\,104\,000}$$

Level	Volume	Delta to ∞
0	3	0.678286
1	$\frac{119}{48}$	0.157453
2	$\frac{37\left(3921+\sqrt{5}\right)}{61\,440}$	0.0409118
3	$\frac{36\,654\,495+8219\sqrt{5}}{15\,728\,640}$	0.00988443



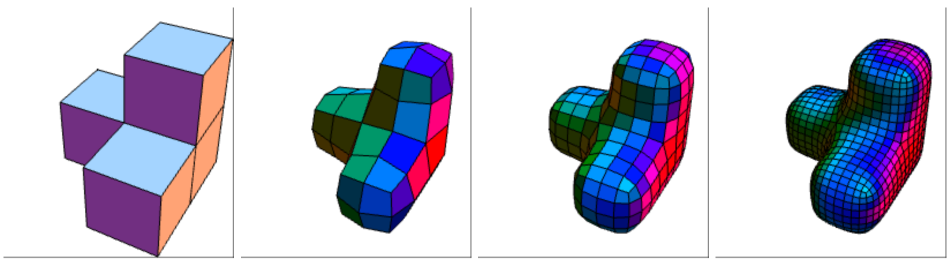
Tripod

Vertices ↓

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 4 & 2 & 1 \\ 5 & 6 & 8 & 7 \\ 7 & 8 & 4 & 3 \\ 7 & 3 & 1 & 5 \\ 6 & 13 & 15 & 9 \\ 4 & 12 & 11 & 2 \\ 12 & 14 & 13 & 11 \\ 14 & 16 & 15 & 13 \\ 9 & 15 & 16 & 10 \\ 6 & 9 & 10 & 8 \\ 4 & 8 & 14 & 12 \\ 8 & 10 & 16 & 14 \\ 17 & 18 & 20 & 19 \\ 19 & 20 & 13 & 6 \\ 11 & 13 & 20 & 18 \\ 11 & 18 & 17 & 2 \\ 17 & 19 & 6 & 2 \end{pmatrix}$$

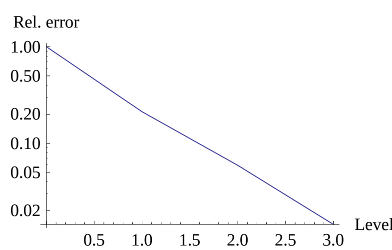


Required valences $\tau(f) \in \{3, 4, 5, 6\}$

Limit volume ↓ (≈3.20028781912720339769529916684)

$$\frac{10\,357\,799\,098\,161 + 2\,535\,566\,756\,\sqrt{5}}{3\,238\,292\,736\,000}$$

Level	Volume	Delta to ∞
0	4	0.799712
1	$\frac{647}{192}$	0.169504
2	$\frac{398\,821+99\sqrt{5}}{122\,880}$	0.0471273
3	$\frac{100\,979\,885+24\,369\sqrt{5}}{31\,457\,280}$	0.0115085



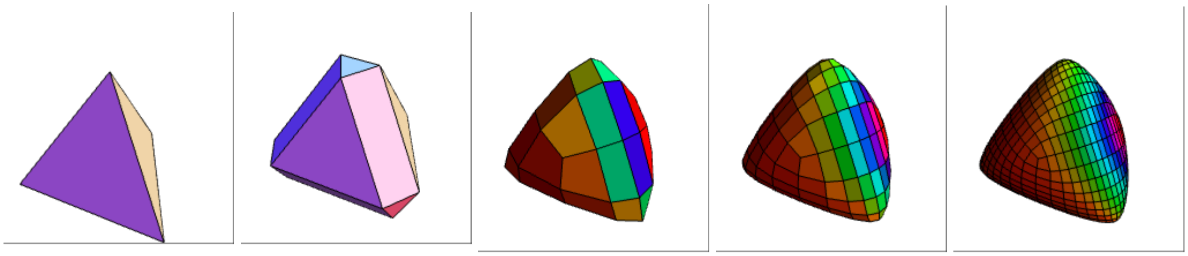
Tetrahedron

Vertices ↓

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Faces ↓

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

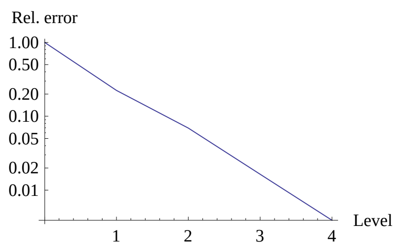


Required valences $\tau(f) \in \{3, 4\}$

Limit volume ↓ (≈0.0615838840593582221075615350861)

$$\frac{20\,317}{233\,280\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.0562672
1	$\frac{17}{162\sqrt{2}}$	0.0126187
2	$\frac{5}{54\sqrt{2}}$	0.00388897
3	$\frac{611}{6912\sqrt{2}}$	0.000922228
4	$\frac{927\,949}{10\,616\,832\sqrt{2}}$	0.000219772



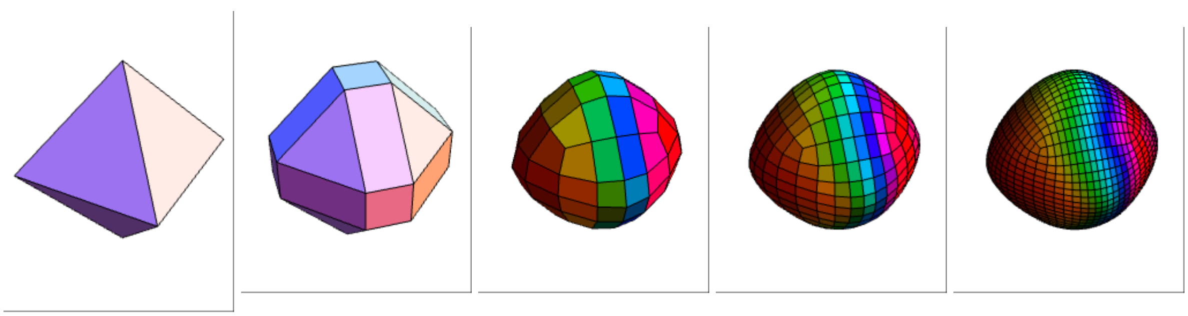
Octahedron

Vertices ↓

0	0	0
1	0	0
1	1	0
0	1	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$

Faces ↓

1	2	5
2	3	5
3	4	5
4	1	5
6	2	1
6	3	2
6	4	3
6	1	4

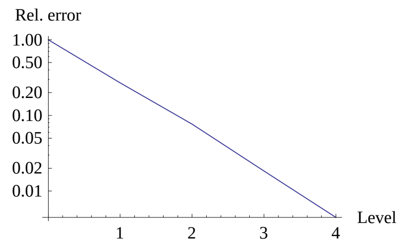


Required valences $\tau(f) \in \{3, 4\}$

Limit volume ↓ (≈0.327811296353841155440449188121)

$$\frac{93\,127}{200\,880\,\sqrt{2}}$$

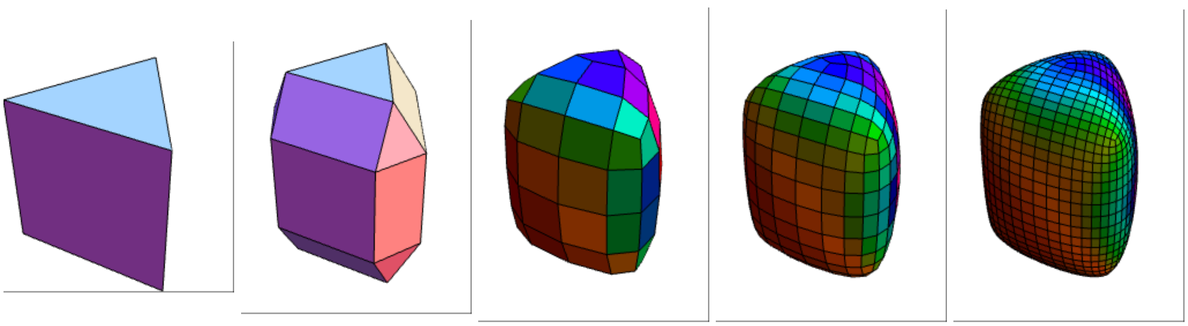
Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.143593
1	$\frac{7\sqrt{2}}{27}$	0.0388367
2	$\frac{23}{48\sqrt{2}}$	0.0110107
3	$\frac{4307}{9216\sqrt{2}}$	0.00264757
4	$\frac{34247}{73728\sqrt{2}}$	0.000643103



Regular Prism

Vertices ↓ Faces ↓

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \end{pmatrix} \quad \begin{pmatrix} \{1, 3, 2\} \\ \{1, 2, 5, 4\} \\ \{4, 5, 6\} \\ \{2, 3, 6, 5\} \\ \{3, 1, 4, 6\} \end{pmatrix}$$

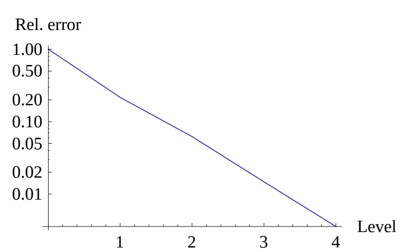


Required valences $\tau(f) \in \{3, 4\}$

Limit volume ↓ (≈0.24865765318779189912827478704l)

$$\frac{18229\sqrt{3}}{126976}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.184355
1	$\frac{1}{2\sqrt{3}}$	0.0400175
2	$\frac{173}{384\sqrt{3}}$	0.0114507
3	$\frac{21401}{49152\sqrt{3}}$	0.00272323
4	$\frac{1207471\sqrt{3}}{8388608}$	0.000656789



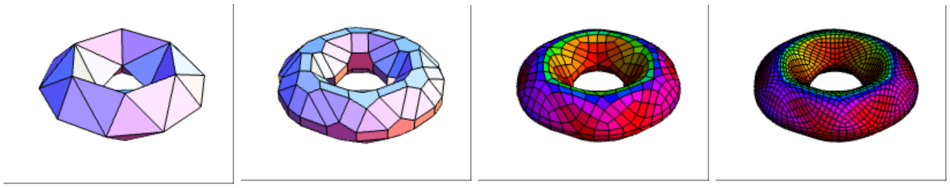
Torus

Vertices ↓

$\frac{3}{2}$	0	0
1.21353	0.881678	0
0.463525	1.42658	0
−0.463525	1.42658	0
−1.21353	0.881678	0
$-\frac{3}{2}$	0	0
−1.21353	−0.881678	0
−0.463525	−1.42658	0
0.463525	−1.42658	0
1.21353	−0.881678	0
1	0	$\frac{1}{2}$
$\frac{1}{2}$	0.866025	$\frac{1}{2}$
$-\frac{1}{2}$	0.866025	$\frac{1}{2}$
−1	0	$\frac{1}{2}$
$-\frac{1}{2}$	−0.866025	$\frac{1}{2}$
$\frac{1}{2}$	−0.866025	$\frac{1}{2}$
$\frac{1}{2}$	0	0
$-\frac{1}{4}$	0.433013	0
$-\frac{1}{4}$	−0.433013	0
1	0	$-\frac{1}{2}$
$\frac{1}{2}$	0.866025	$-\frac{1}{2}$
$-\frac{1}{2}$	0.866025	$-\frac{1}{2}$
−1	0	$-\frac{1}{2}$
$-\frac{1}{2}$	−0.866025	$-\frac{1}{2}$
$\frac{1}{2}$	−0.866025	$-\frac{1}{2}$

Faces ↓

3	4	13
12	2	3
18	13	14
23	18	19
21	20	17
22	4	3
24	19	25
8	24	9
7	6	23
6	14	5
14	13	5
11	1	2
12	3	13
13	4	5
18	12	13
11	2	12
10	16	9
19	18	14
17	11	12
18	17	12
2	20	21
23	5	22
2	1	20
3	2	21
22	3	21
22	5	4
24	23	19
21	17	18
18	22	21
23	6	5
18	23	22
24	7	23
7	24	8
24	25	9
25	10	9
25	19	17
25	20	10
10	20	1
25	17	20
16	10	11
19	16	17
16	15	9
19	14	15
17	16	11
19	15	16
7	14	6
15	14	7
15	8	9
15	7	8
10	1	11

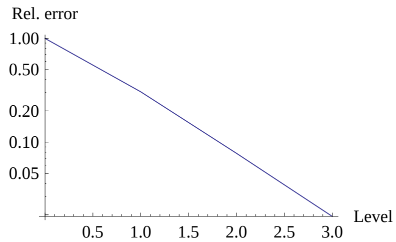


Required valences τ(f) ∈ {3, 4, 6, 7, 8}

Limit volume ↓

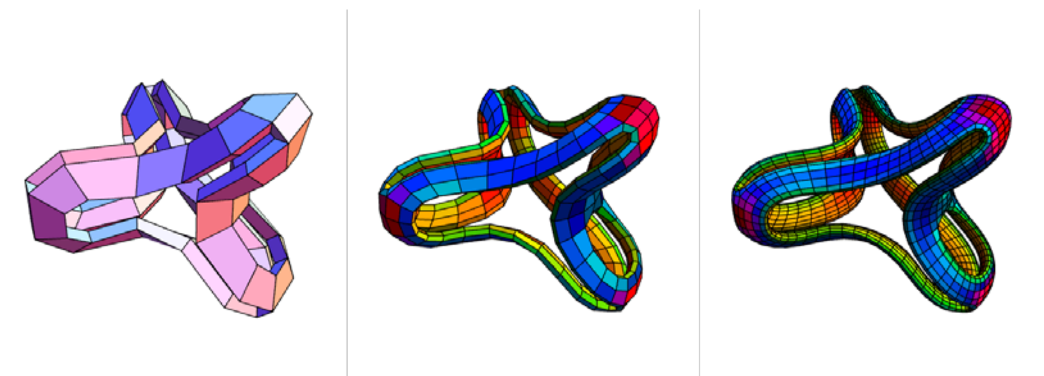
2.79721

Level	Volume	Delta to ∞
0	3.13298	0.335773
1	2.90002	0.102814
2	2.82346	0.0262512
3	2.8037	0.00649251



Print11

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

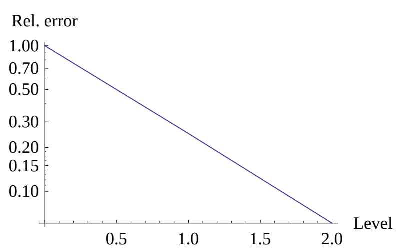


Required valences τ(f) ∈ {4, 5}

Limit volume ↓

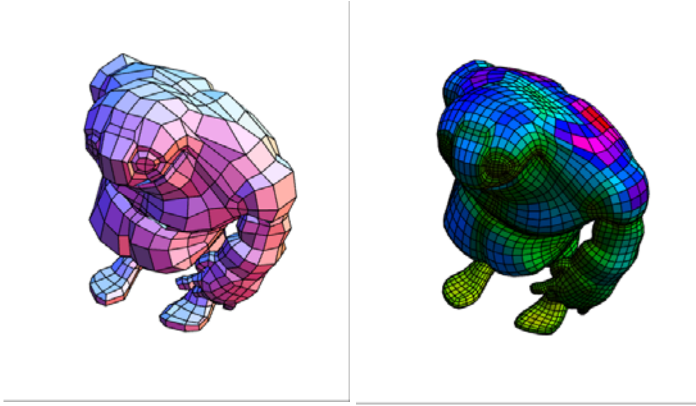
2.99511

Level	Volume	Delta to ∞
0	3.62315	0.628037
1	3.15143	0.156317
2	3.03301	0.037895



Bigguy00

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

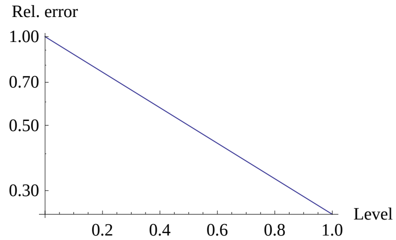


Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓

1.37984

Level	Volume	Delta to ∞
0	1.40311	0.0232651
1	1.38565	0.00580343



Midedge

The Midedge subdivision scheme is specified by [Peters/Reif 1997]. The algorithm applies to meshes with *n*-gons. One round of Midedge subdivision requires the contraction of each face in the mesh. In the *simplest-pure* configuration, the weights to contract an *n*-gon are $\omega_0 = \frac{1}{2}$, $\omega_1 = \frac{1}{4}$, $\omega_2 = 0$, ..., $\omega_{n-2} = 0$, $\omega_{n-1} = \frac{1}{4}$.

The weights are applied in a rotational fashion. For instance,

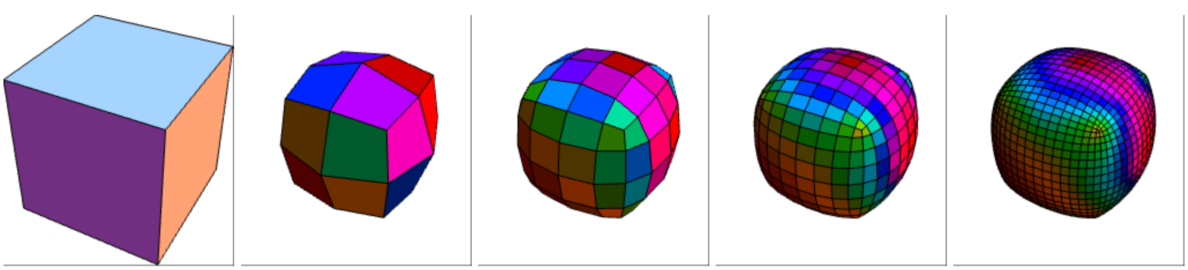
• a triangle is contracted using the mask $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ centered at all 3 vertices;

• a quad is contractred using $\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}$.

After one round of subdivision, all newly introduced faces are quads.

Cube

The vertex coordinates and topology of the mesh are specified in a section above.



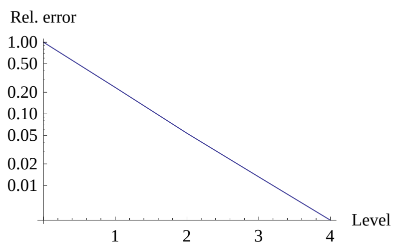
Important: In order to color the facets based on their contribution to the global volume, we display the *dual* mesh.

Required valences τ(f) ∈ {3}

Limit volume ↓ (≈0.619652305366591080876795162509)

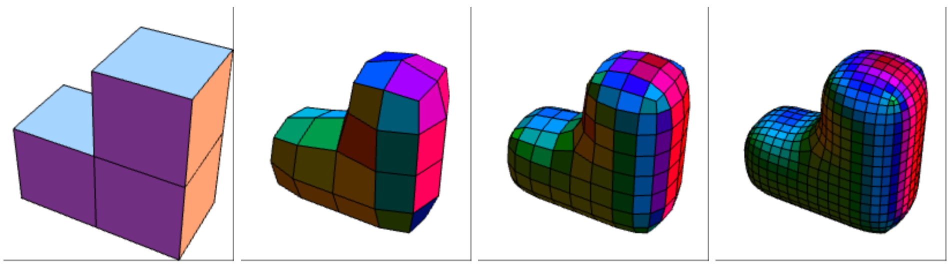
$$\frac{4099}{6615}$$

Level	Volume	Delta to ∞
0	1	0.380348
1	$\frac{17}{24}$	0.088681
2	$\frac{983}{1536}$	0.0203217
3	$\frac{30703}{49152}$	0.00500183
4	$\frac{3906353}{6291456}$	0.00124578



Corner Mesh

The vertex coordinates and topology of the mesh are specified in a section above.

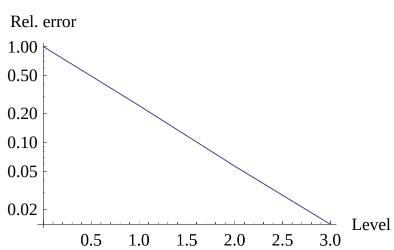


Required valences τ(f) ∈ {3, 4, 5}

Limit volume ↓ (≈2.31160902320739391947126157265)

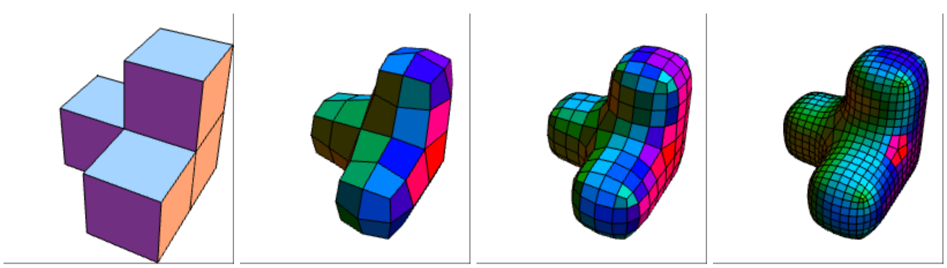
$$\frac{2448212576}{1059094575}$$

Level	Volume	Delta to ∞
0	3	0.688391
1	$\frac{119}{48}$	0.167558
2	$\frac{28883}{12288}$	0.0388955
3	$\frac{608493}{262144}$	0.00960756



Tripod

The vertex coordinates and topology of the mesh are specified in a section above.

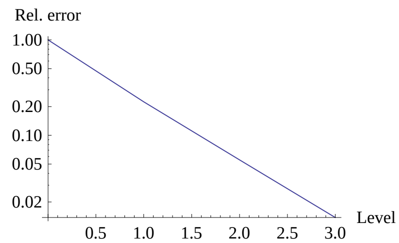


Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓ (≈3.18750045846944310898769356835)

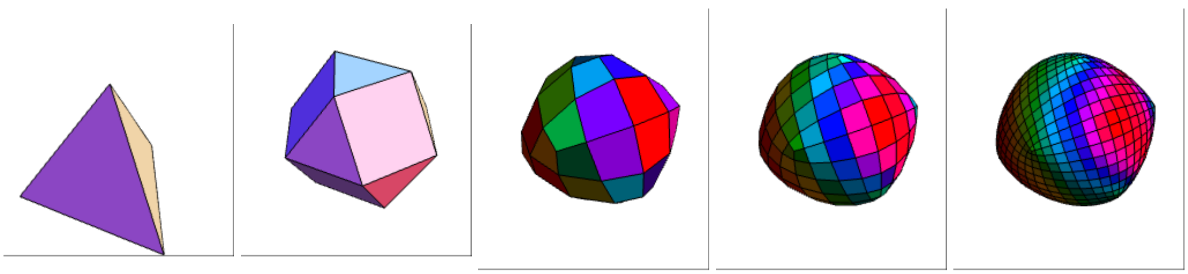
$$\frac{27\,006\,915\,547}{8\,472\,756\,600}$$

Level	Volume	Delta to ∞
0	4	0.8125
1	$\frac{647}{192}$	0.182291
2	$\frac{39\,719}{12\,288}$	0.04484
3	$\frac{2\,515\,547}{786\,432}$	0.011183



Tetrahedron

The vertex coordinates and topology of the mesh are specified in a section above.

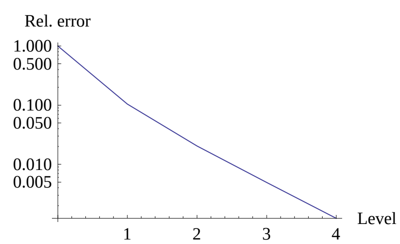


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.0273850216939121154810946810305)

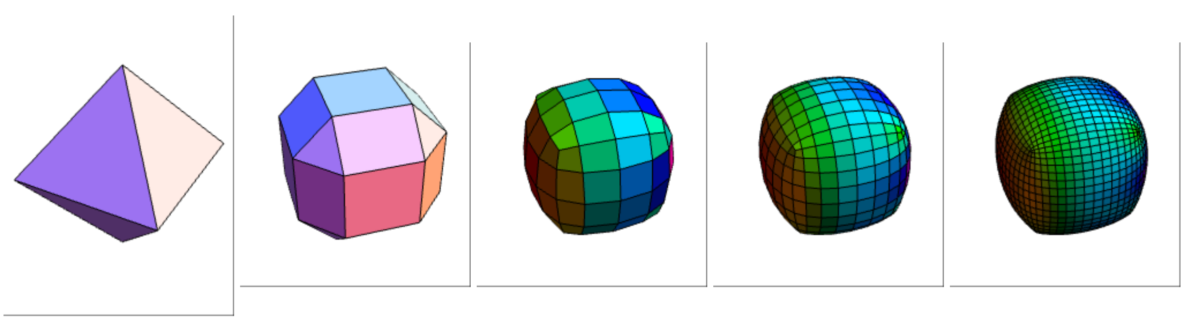
$$\frac{4099}{105\,840\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.0904661
1	$\frac{5}{96\sqrt{2}}$	0.00944346
2	$\frac{127}{3072\sqrt{2}}$	0.00184758
3	$\frac{3869}{98\,304\sqrt{2}}$	0.000444936
4	$\frac{489\,277}{12\,582\,912\sqrt{2}}$	0.00011029



Octahedron

The vertex coordinates and topology of the mesh are specified in a section above.

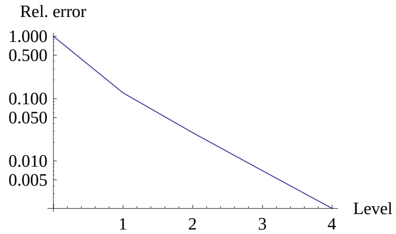


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.219080173551296923848757448244)

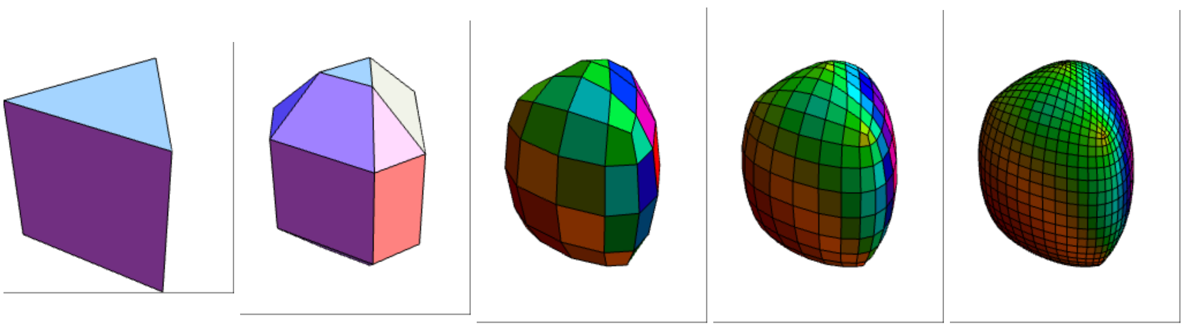
$$\frac{4099}{13\,230\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.252324
1	$\frac{17}{48\sqrt{2}}$	0.0313535
2	$\frac{983}{3072\sqrt{2}}$	0.00718479
3	$\frac{30\,703}{98\,304\sqrt{2}}$	0.00176841
4	$\frac{3\,906\,353}{12\,582\,912\sqrt{2}}$	0.000440451



Regular Prism

The vertex coordinates and topology of the mesh are specified in a section above.

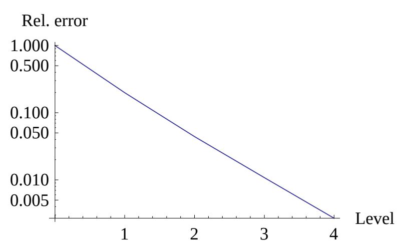


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.213362263064362018550162079268)

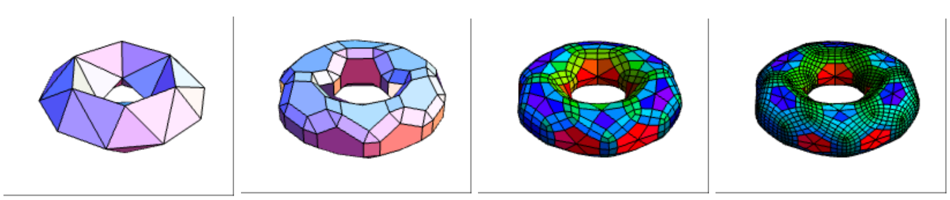
$$\frac{104\,303}{282\,240\,\sqrt{3}}$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.21965
1	$\frac{19\sqrt{3}}{128}$	0.043739
2	$\frac{1055\sqrt{3}}{8192}$	0.00969848
3	$\frac{65\,303\sqrt{3}}{524\,288}$	0.00237434
4	$\frac{4\,144\,837\sqrt{3}}{33\,554\,432}$	0.000590644



Torus

The vertex coordinates and topology of the mesh are specified in a section above.

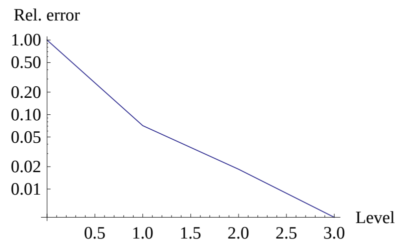


Required valences τ(f) ∈ {3, 4, 6, 7, 8}

Limit volume ↓

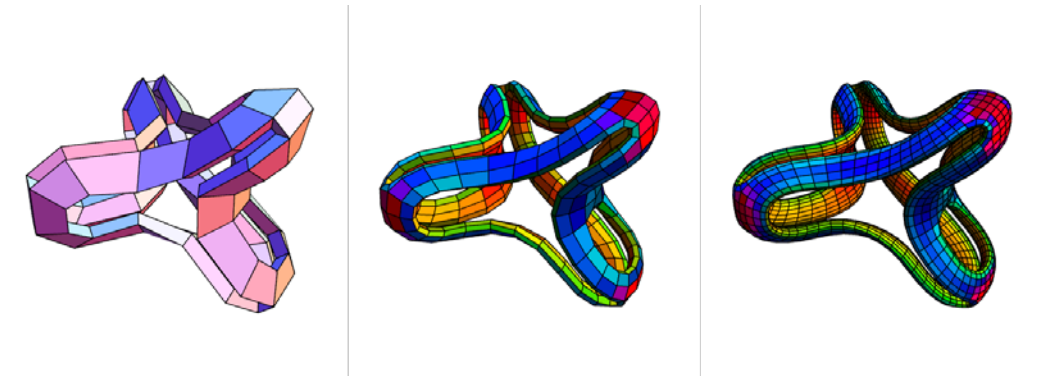
2.50137

Level	Volume	Delta to ∞
0	3.13298	0.631605
1	2.54626	0.0448856
2	2.51305	0.0116758
3	2.50401	0.00263748



Print11

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

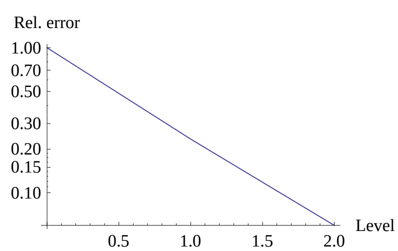


Required valences τ(f) ∈ {4, 5}

Limit volume ↓

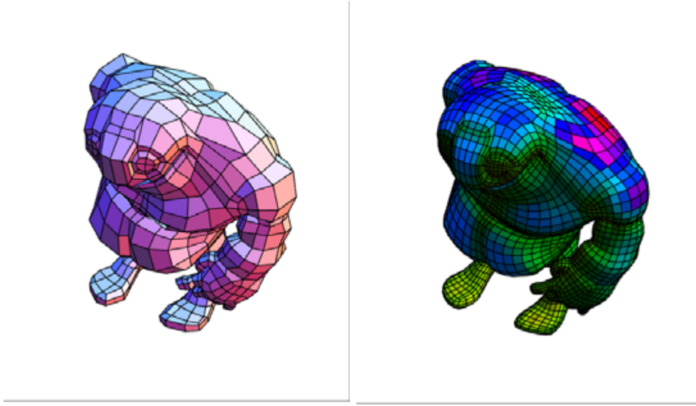
2.98337

Level	Volume	Delta to ∞
0	3.62315	0.639776
1	3.13344	0.150069
2	3.02152	0.0381422



Bigguy00

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

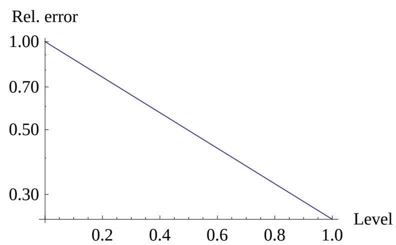


Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓

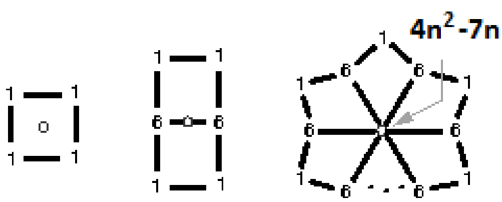
1.37973

Level	Volume	Delta to ∞
0	1.40311	0.0233745
1	1.3855	0.00576977



Catmull-Clark

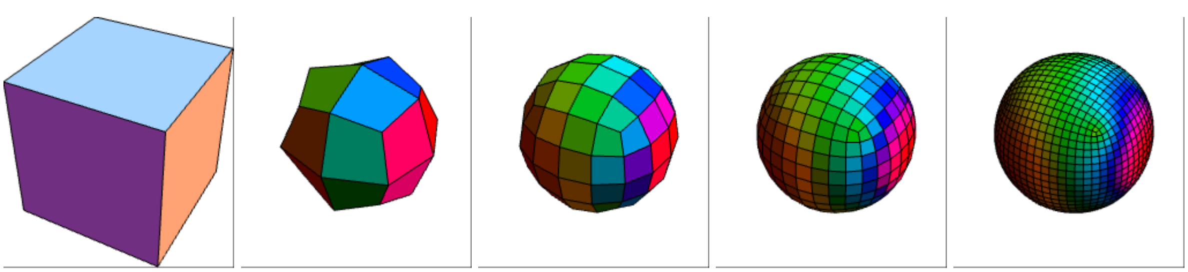
The Catmull-Clark subdivision scheme is published as [Catmull/Clark 1978]. The algorithm applies to meshes with quads. Weights are specified for the insertion of a face midpoint, and edge midpoint, as well as the repositioning of a vertex.



The weights are subject to normalization so that their sum adds up to 1.

Cube

The vertex coordinates and topology of the mesh are specified in a section above.

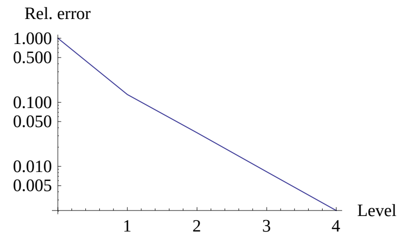


Required valences $\tau(f) \in \{3\}$

Limit volume ↓ (≈0.327552379087373087848083217632)

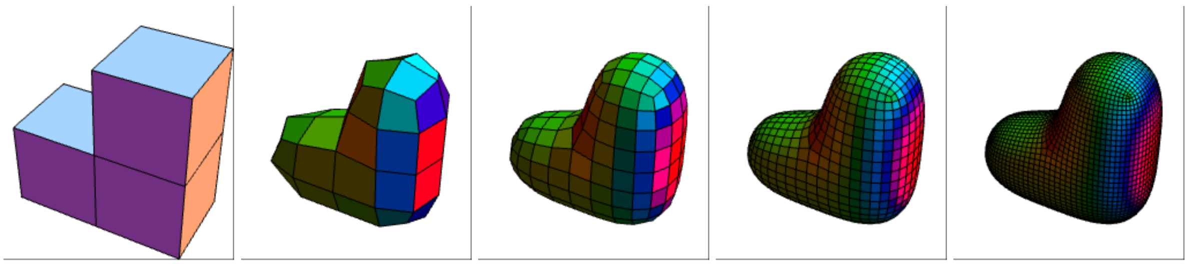
422055974386524841158589150775845920116884578238015637159706113035161643534077381378475128253612936122944021053/1288514452444087895669556451725569755403478463105824354494245921153221232752339311828293424979186926464089292800

Level	Volume	Delta to ∞
0	1	0.672448
1	$\frac{5}{12}$	0.0891143
2	$\frac{8363555}{23887872}$	0.0225648
3	$\frac{782158429027}{2348273369088}$	0.00552572
4	$\frac{8436764484639443}{25649407252758528}$	0.00137392



Corner

The vertex coordinates and topology of the mesh are specified in a section above.

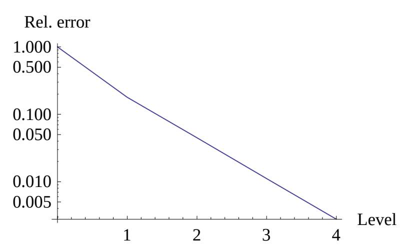


Required valences $\tau(f) \in \{3, 4, 5\}$

Limit volume ↓ (≈1.72871650297817260099302006563)

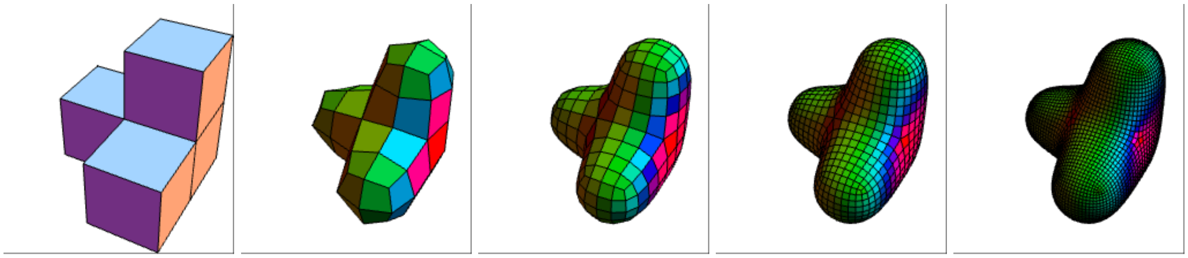
238183562818285379345104287170766063946255047105197448398979857879244506582805172399581114793975399698786909665273619053622092301710506169993088941960907385804766639666785367554430343147026590653088715413159009832508909157850148996816370517714027221224366047058854221723203124335077064802820221056245121533023701817782747006399704713335408037487924219779293143306138979680744192499487/137780580221193602526165904020601497954640209435648591333259207907660850845852197061466377968334522284468800995353916425496437384178259571462144000005314504326139212878523777189088041166002039202096464746642530489349431889280430360325927796364789904158852086765869213006735818866703286592991855415865108086126721814850651456015926150686134124030809921418026857198265997231277260800000

Level	Volume	Delta to ∞
0	3	1.27128
1	$\frac{587}{300}$	0.22795
2	$\frac{12798302651}{7166361600}$	0.0571692
3	$\frac{245562297842002151}{140896402145280000}$	0.0141406
4	$\frac{444309855599330136299}{256494072527585280000}$	0.00352569



Tripod

The vertex coordinates and topology of the mesh are specified in a section above.

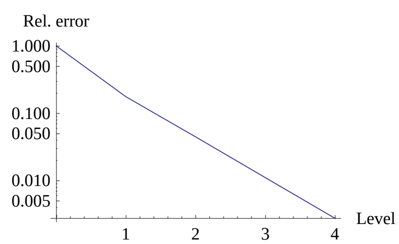


Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓ (≈2.5040054761592054376437149098 8)

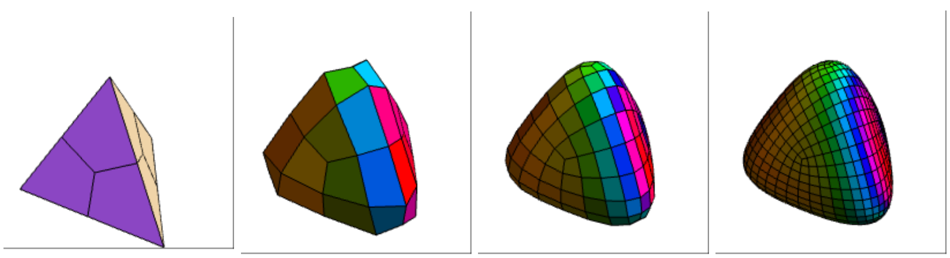
40180082442670051252900902177393516510211208965892170836569226629214212708997856639459550700829 1
02574031201693056634561140899647555135055500656660483140756960778898649244955464121982187033671 7
31647624767955162967234692920559299362507134859937248588698224893139701577318904153684167935446 0
76618334003376802825747208498276611956468960741228369454585883538515036663126880925423313255516 9
0567671425629575023937687497124360041820710 21/160463237102423130899091753923079591181080214100747
42425376177351348347992382685285901785697765860338901271935282090099437828184717866085833813124 0
18343088953677884563654098038825035627833355249407449932715943989016756237298883810746718774035 5
82028805731566895736089226365074195187059803111007004229005368248251629007854376338913422782400 8
775572142274115412160255773388520595334004563330096707755296887477615110463588476731392000

Level	Volume	Delta to ∞
0	4	1.49599
1	$\frac{1661}{600}$	0.264328
2	$\frac{12\,282\,222\,401}{4\,777\,574\,400}$	0.0668017
3	$\frac{236\,756\,772\,823\,958\,651}{93\,930\,934\,763\,520\,000}$	0.0165355
4	$\frac{1\,286\,640\,528\,317\,971\,702\,297}{512\,988\,145\,055\,170\,560\,000}$	0.00412369



Tetrahedron

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, the faces are quadrangulated as illustrated.

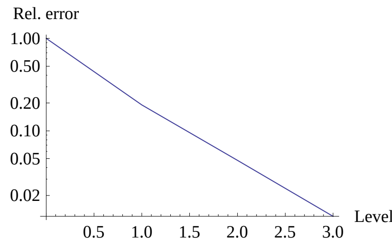


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.072510596904440933978107534005 1)

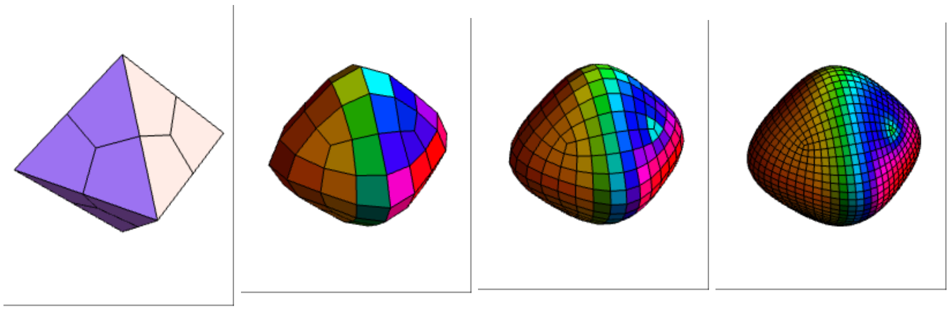
9 664 462 230 541 570 393 482 570 158 742 446 128 534 211 334 349 389 086 514 185 107 011 789 915 680 638 - 956 061 823 981 903 509 122 228 571 136 857 / (94 245 628 521 624 714 654 687 557 611 927 387 823 797 281 872 883 152 785 864 844 518 635 610 167 028 - 246 808 012 319 084 191 958 049 944 816 844 800 $\sqrt{2}$)

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.0453405
1	$\frac{57131}{497664\sqrt{2}}$	0.00866409
2	$\frac{1993519}{18874368\sqrt{2}}$	0.00217433
3	$\frac{248415737467693}{2404631929946112\sqrt{2}}$	0.000538609



Octahedron

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, the faces are quadrangulated as illustrated.

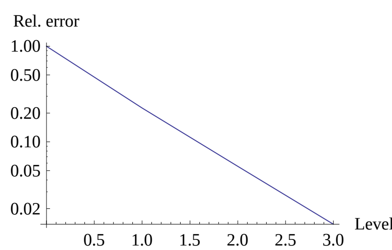


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.364061243906681220004932169994)

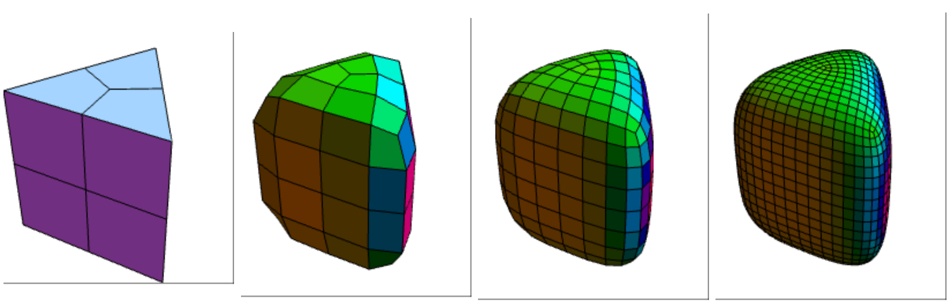
$$190923582374957052419353517704947556555324809676851440847239331209 \Big/ \left(370825958668551833121618823275801848452611938096556139677768314880\sqrt{2}\right)$$

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.107343
1	$\frac{15185}{27648\sqrt{2}}$	0.0243002
2	$\frac{52674337}{100663296\sqrt{2}}$	0.00594831
3	$\frac{3836667347369}{7421703487488\sqrt{2}}$	0.0014793



Regular Prism

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, the faces are quadrangulated as illustrated.

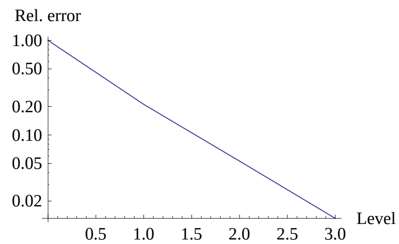


Required valences τ(f) ∈ {3, 4}

Limit volume ↓ (≈0.315380789328794278740715134259)

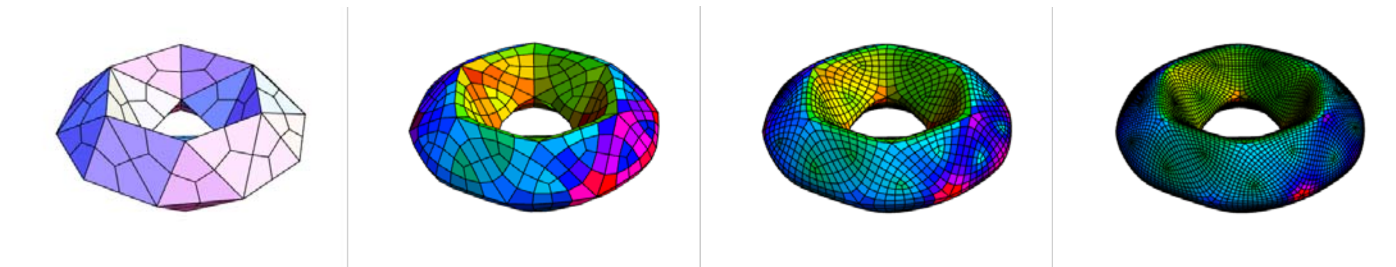
281 359 744 297 373 025 806 887 950 476 597 141 067 926 989 781 961 609 819 901 659 104 108 487 666 858 - 454 785 383 414 542 105 343 543 118 211 808 427 911 /
(515 069 812 764 849 736 831 322 074 119 111 309 158 500 996 783 801 408 336 378 387 302 587 801 342 - 469 929 592 233 252 002 346 864 068 513 228 652 544 000 $\sqrt{3}$)

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.117632
1	$\frac{43435}{73728\sqrt{3}}$	0.0247506
2	$\frac{2018647757}{3623878656\sqrt{3}}$	0.00622679
3	$\frac{782217777968233}{1424967069597696\sqrt{3}}$	0.00154839



Torus

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, the faces are quadrangulated as illustrated.

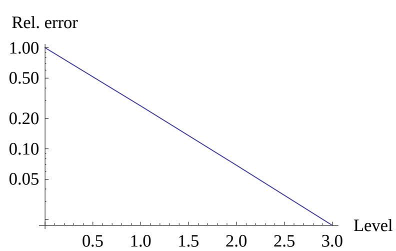


Required valences τ(f) ∈ {3, 4, 6, 7, 8}

Limit volume ↓

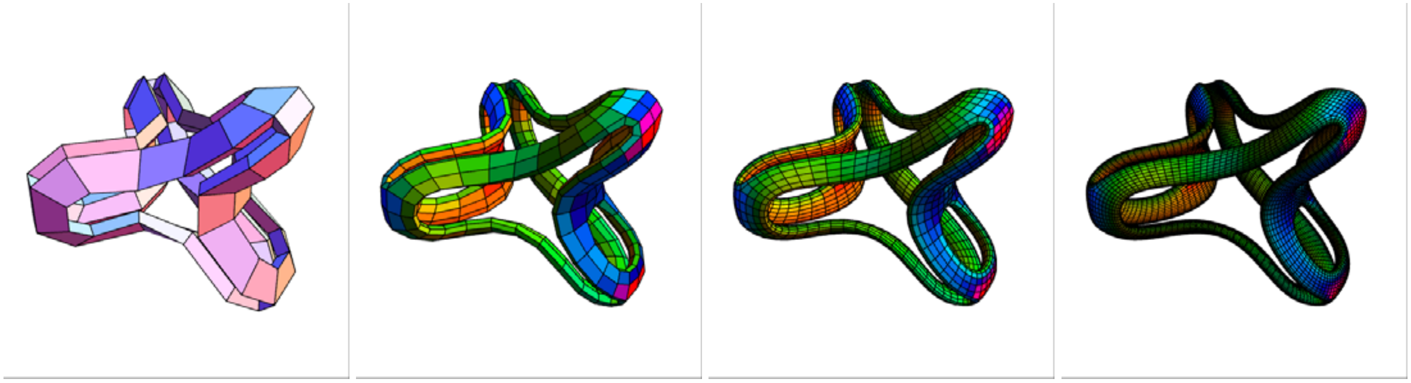
2.92617

Level	Volume	Delta to ∞
0	3.13298	0.20681
1	2.98097	0.0548035
2	2.94034	0.0141741
3	2.92978	0.00361127



Print11

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

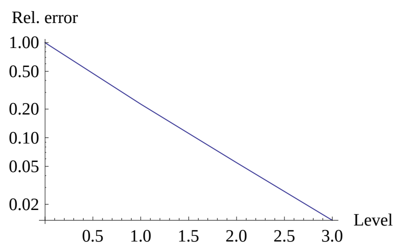


Required valences τ(f) ∈ {4, 5}

Limit volume ↓

2.45329

Level	Volume	Delta to ∞
0	3.62315	1.16986
1	2.71701	0.263724
2	2.51737	0.0640766
3	2.4692	0.0159126



Bigguy00

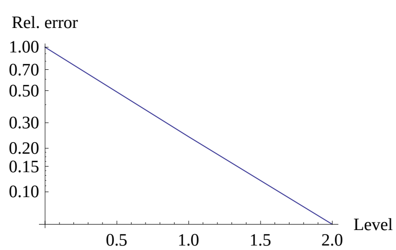
The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

Required valences τ(f) ∈ {3, 4, 5, 6}

Limit volume ↓

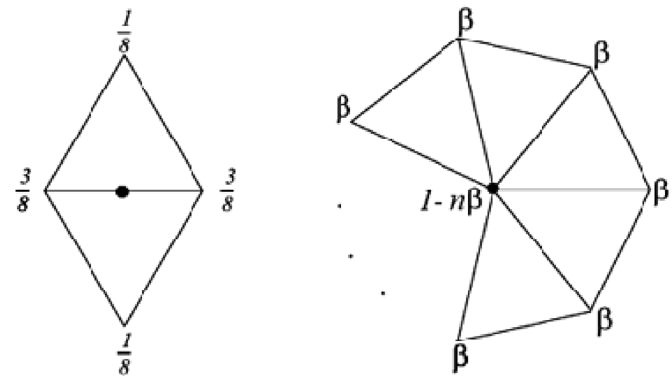
1.35762

Level	Volume	Delta to ∞
0	1.40311	0.0454884
1	1.36855	0.0109267
2	1.36033	0.00271077



Loop

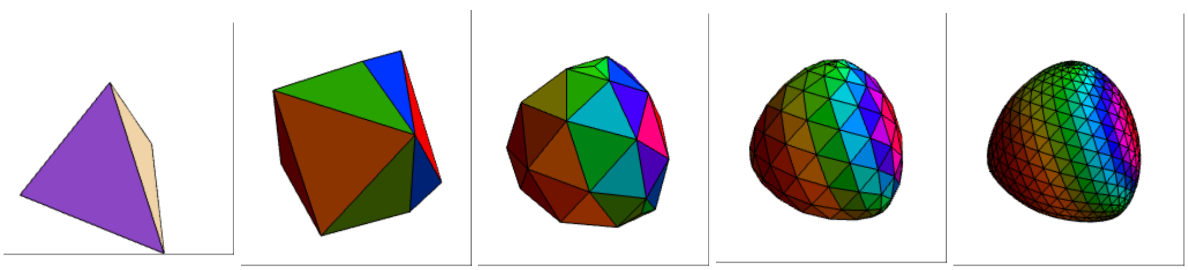
The Loop subdivision scheme is published as [Loop 1987]. The algorithm applies to meshes with triangles. The weights for the insertion of an edge midpoint, as well as the repositioning of a vertex that already existed in the input mesh are



where $\beta = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\,\mathrm{Cos}\left[\frac{2\pi}{n}\right]\right)^2\right)$.

Tetrahedron

The vertex coordinates and topology of the mesh are specified in a section above.

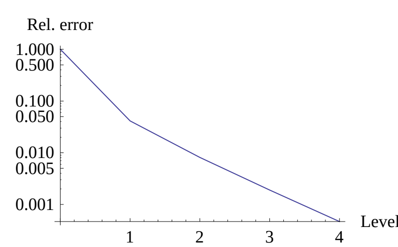


Required valences τ(f) ∈ {3, 6}

Limit volume ↓ (≈0.00455169559584284472638894206836)

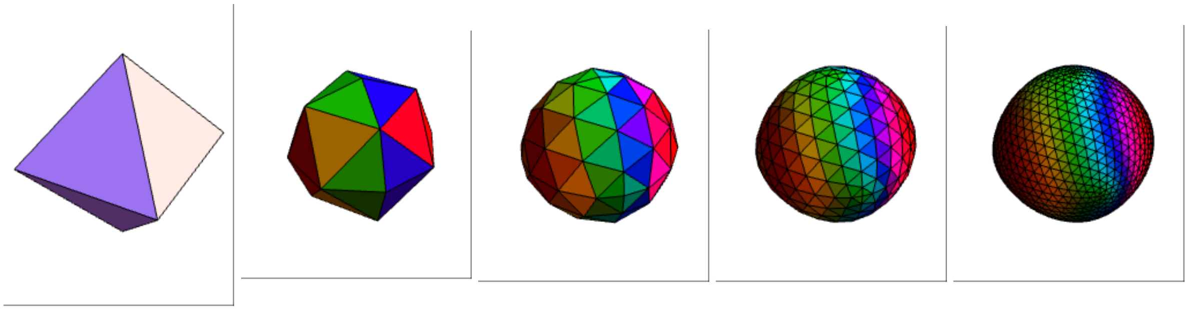
$$\frac{44\,192\,429\,513\,855\,101}{6\,865\,302\,375\,425\,894\,400\,\sqrt{2}}$$

Level	Volume	Delta to ∞
0	$\frac{1}{6\sqrt{2}}$	0.113299
1	$\frac{5}{384\sqrt{2}}$	0.00465542
2	$\frac{507}{65\,536\sqrt{2}}$	0.000918628
3	$\frac{5\,428\,333}{805\,306\,368\sqrt{2}}$	0.000214703
4	$\frac{21\,479\,003\,773}{3\,298\,534\,883\,328\sqrt{2}}$	0.0000527575



Octahedron

The vertex coordinates and topology of the mesh are specified in a section above.

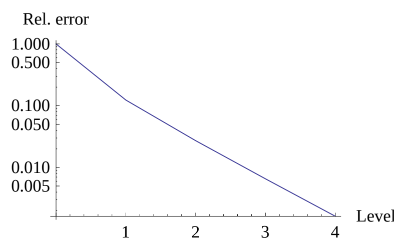


Required valences τ(f) ∈ {4, 6}

Limit volume ↓ (≈0.107428933423629243357597485668)

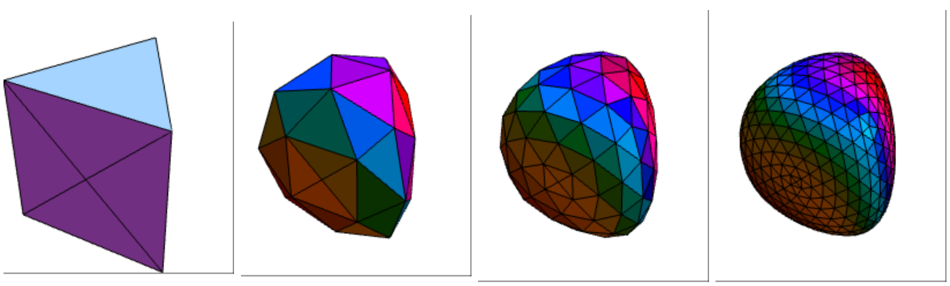
3 969 077 707 781 314 018 093 365 433 145 909 318 003 197 /
 (26 124 821 989 633 711 204 270 304 840 270 381 056 000 000 $\sqrt{2}$)

Level	Volume	Delta to ∞
0	$\frac{\sqrt{2}}{3}$	0.363976
1	$\frac{441}{2048\sqrt{2}}$	0.0448338
2	$\frac{11130723}{67108864\sqrt{2}}$	0.00985229
3	$\frac{170757897345}{1099511627776\sqrt{2}}$	0.00238716
4	$\frac{2751964955442441}{18014398509481984\sqrt{2}}$	0.000592052



Regular Prism

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, each quad is split into 4 triangles as illustrated.

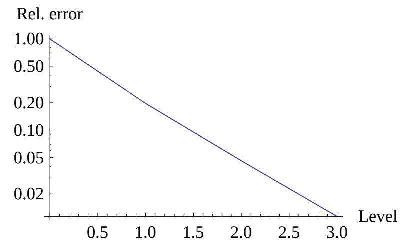


Required valences τ(f) ∈ {4, 5, 6}

Limit volume ↓ (≈0.210085040689707138380133913482)

(92 102 378 824 048 334 066 237 990 528 255 386 720 127 599 601 585 932 852 823 577 402 005 508 309 404 -
 724 577 925 531 613 854 236 163 744 960 612 871 +
 2 846 437 556 815 824 900 986 417 954 643 689 012 981 643 092 318 794 469 188 568 507 344 643 933 552 -
 055 595 108 409 008 708 450 180 943 118 742 577 $\sqrt{5}$) /
 (270 605 027 873 252 246 875 440 331 354 961 480 772 146 789 160 880 293 976 398 013 695 715 892 196 -
 903 931 187 098 174 407 867 290 066 752 634 880 000 $\sqrt{3}$)

Level	Volume	Delta to ∞
0	$\frac{\sqrt{3}}{4}$	0.222928
1	$\frac{\sqrt{3}\left(8993+275\sqrt{5}\right)}{65536}$	0.0438426
2	$\frac{\sqrt{3}\left(63904837+1967526\sqrt{5}\right)}{536870912}$	0.0102782
3	$\frac{7\sqrt{3}\left(288578472373+8908373583\sqrt{5}\right)}{17592186044416}$	0.00252878



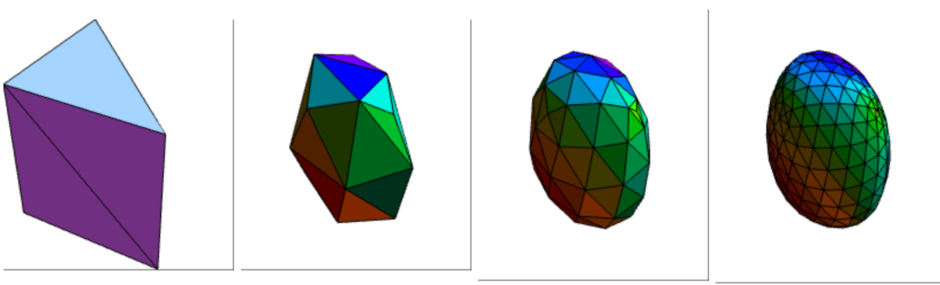
Twisted Prism

Vertices ↓

0	0	0
1	0	0
0	1	0
0	0	1
1	0	1
0	1	1

Faces ↓

1	2	4
4	2	5
4	5	6
2	1	3
5	2	6
6	2	3
1	4	6
1	6	3

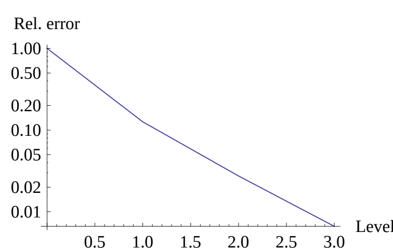


Required valences $\tau(f) \in \{3, 4, 5, 6\}$

Limit volume ↓ (≈0.11110232336294279338467164119 1)

$$\left(682736412347955315305274223162118927358124424103041608101831684273988492206372 - 602809357261229880603934427944490300197801443668773 + 27003068399200622182768159799285436165537150274434469160132941982977503898 - 035149976116453584053484891539569054516651044417649715\sqrt{5}\right) \Big/ 6688582978252140996295748683509263007202421208222548687454129894791195341373 - 438829861734650296568016765169529224531516850176000000$$

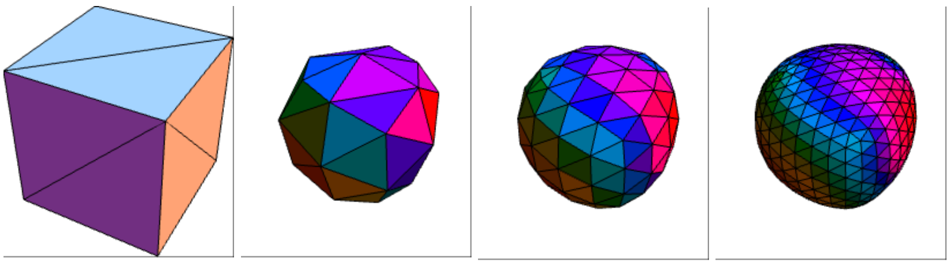
Level	Volume	Delta to ∞
0	$\frac{1}{2}$	0.388898
1	$\frac{3614+145\sqrt{5}}{24576}$	0.0491447
2	$\frac{1441549681+57050011\sqrt{5}}{12884901888}$	0.0106772
3	$\frac{22049210779315+871963985243\sqrt{5}}{211106232532992}$	0.00257969



Twisted Cube

Vertices ↓ Faces ↓

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 6 \\ 1 & 6 & 5 \\ 3 & 4 & 2 \\ 3 & 2 & 1 \\ 4 & 8 & 6 \\ 4 & 6 & 2 \\ 5 & 6 & 8 \\ 5 & 8 & 7 \\ 7 & 8 & 4 \\ 7 & 4 & 3 \\ 7 & 3 & 1 \\ 7 & 1 & 5 \end{pmatrix}$$

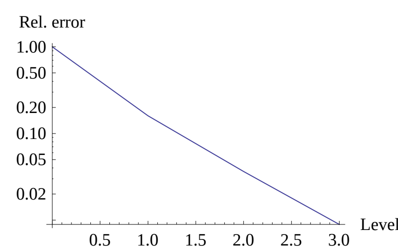


Required valences $\tau(f) \in \{4, 5, 6\}$

Limit volume ↓ (≈0.370924163799450705911051341355)

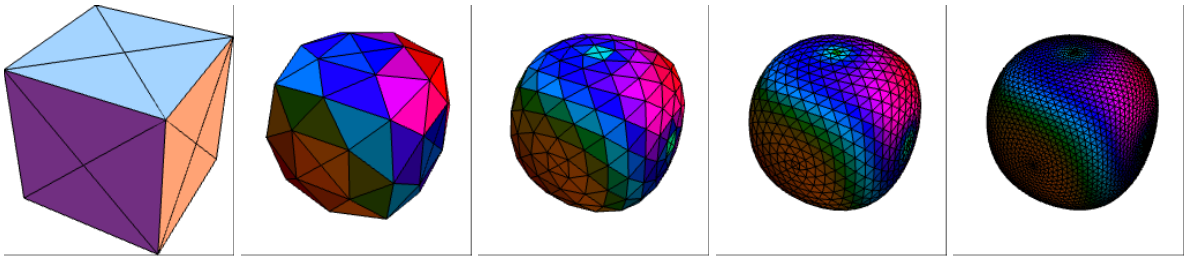
$$\left(145\,286\,606\,628\,148\,444\,780\,316\,147\,513\,318\,841\,772\,282\,876\,314\,767\,589\,452\,951\,336\,821\,210\,809\,075\,308 - 962\,790\,987\,486\,379\,649\,669\,656\,992\,249\,278\,780\,265\,454\,948\,703 + 3\,774\,886\,935\,311\,940\,206\,370\,415\,852\,122\,597\,834\,601\,671\,085\,542\,621\,522\,975\,679\,163\,918\,791\,066\,732 - 736\,575\,102\,828\,788\,488\,022\,564\,054\,056\,913\,715\,052\,398\,247\,175\sqrt{5}\right) \Big/ 414\,444\,583\,087\,335\,240\,186\,864\,797\,805\,018\,775\,231\,478\,511\,745\,448\,737\,243\,109\,502\,734\,087\,616\,162\,432 - 181\,295\,133\,921\,655\,851\,455\,103\,591\,965\,211\,368\,030\,208\,000\,000$$

Level	Volume	Delta to ∞
0	1	0.629076
1	$\frac{5\left(1094+29\sqrt{5}\right)}{12\,288}$	0.100611
2	$\frac{149\,841\,731+3\,906\,417\sqrt{5}}{402\,653\,184}$	0.0229054
3	$\frac{2\,347\,476\,078\,113+61\,032\,047\,889\sqrt{5}}{6\,597\,069\,766\,656}$	0.00559874



Regular Cube

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, each quad is split into 4 triangles as illustrated.

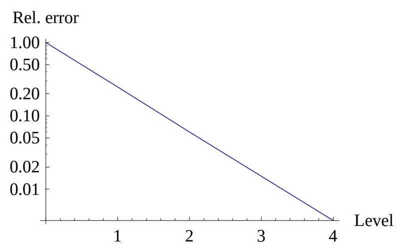


Required valences $\tau(f) \in \{4, 6\}$

Limit volume ↓ (≈0.67572730196555770900320032406)

$$\frac{3\,357\,526\,114\,225}{4\,968\,759\,001\,536}$$

Level	Volume	Delta to ∞
0	1	0.324273
1	$\frac{387}{512}$	0.0801321
2	$\frac{91\,131}{131\,072}$	0.0195471
3	$\frac{182\,691\,551}{268\,435\,456}$	0.00485176
4	$\frac{186\,075\,296\,807}{274\,877\,906\,944}$	0.00121068



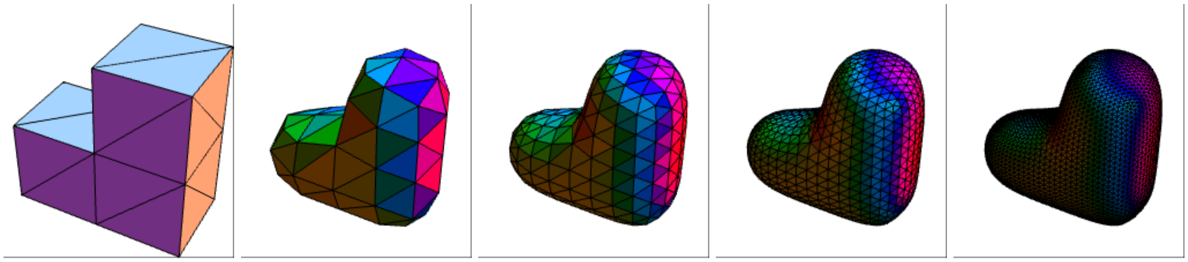
Twisted Corner

Vertices ↓

0	0	0
1	0	0
0	1	0
1	1	0
0	0	1
1	0	1
0	1	1
1	1	1
1	0	2
1	1	2
2	0	0
2	1	0
2	0	1
2	1	1
2	0	2
2	1	2

Faces ↓

1	2	6
1	6	5
3	4	2
3	2	1
5	6	8
5	8	7
7	8	4
7	4	3
7	3	1
7	1	5
2	11	13
2	13	6
6	13	15
6	15	9
4	12	11
4	11	2
12	14	13
12	13	11
14	16	15
14	15	13
9	15	16
9	16	10
6	9	10
6	10	8
4	8	14
4	14	12
8	10	16
8	16	14

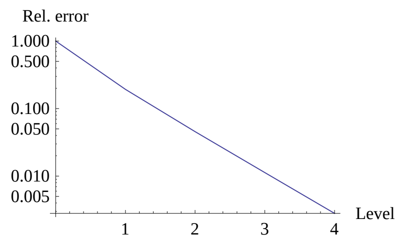


Required valences $\tau(f) \in \{4, 5, 6, 7, 8\}$

Limit volume ↓

1.81521

Level	Volume	Delta to ∞
0	3.	1.18479
1	2.0431	0.227886
2	1.86926	0.0540428
3	1.82856	0.0133427
4	1.81854	0.00332547

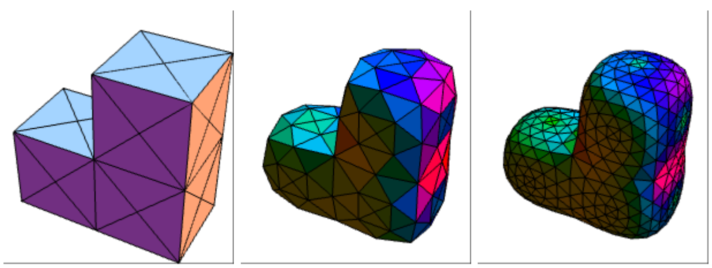


The volumes are stated only in numeric precision. For instance at subdivision level 1 in symbolic form, the volume is

$$\left(2\,737\,215 + 6174\sqrt{2} + 16\,240\sqrt{5} + 39\,264\,\mathrm{Sin}\left[\frac{3\pi}{14}\right] + 13\,088\,\mathrm{Sin}\left[\frac{3\pi}{14}\right]^2\right) \Big/ 1\,376\,256$$

Regular Corner

The vertex coordinates and topology of the mesh are specified in a section above. For subdivision, each quad is split into 4 triangles as illustrated.

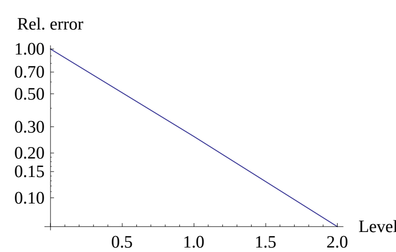


Required valences $\tau(f) \in \{4, 6, 8, 10\}$

Limit volume ↓

2.41296

Level	Volume	Delta to ∞
0	3	0.587037
1	$\frac{124\,291+960\sqrt{2}+175\sqrt{5}}{49\,152}$	0.151326
2	$\frac{2\,430\,281\,209+19\,853\,256\sqrt{2}+3\,756\,634\sqrt{5}+4956\sqrt{10}}{1\,006\,632\,960}$	0.037556



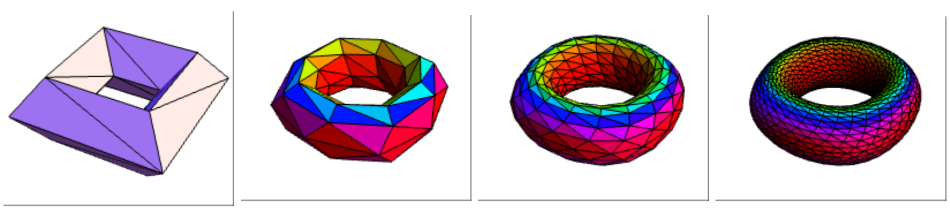
Regular Torus

Vertices ↓

$$\begin{pmatrix}
\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\
\sqrt{2} & \sqrt{2} & 1 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\sqrt{2} & \sqrt{2} & -1 \\
-\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\
-\sqrt{2} & \sqrt{2} & 1 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\sqrt{2} & \sqrt{2} & -1 \\
-\frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & 0 \\
-\sqrt{2} & -\sqrt{2} & 1 \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
-\sqrt{2} & -\sqrt{2} & -1 \\
\frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & 0 \\
\sqrt{2} & -\sqrt{2} & 1 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\sqrt{2} & -\sqrt{2} & -1
\end{pmatrix}$$

Faces ↓

$$\begin{pmatrix}
1 & 5 & 2 \\
5 & 6 & 2 \\
2 & 6 & 3 \\
6 & 7 & 3 \\
3 & 7 & 4 \\
7 & 8 & 4 \\
4 & 8 & 5 \\
8 & 9 & 5 \\
5 & 9 & 6 \\
9 & 10 & 6 \\
6 & 10 & 7 \\
10 & 11 & 7 \\
7 & 11 & 8 \\
11 & 12 & 8 \\
8 & 12 & 9 \\
12 & 13 & 9 \\
9 & 13 & 10 \\
13 & 14 & 10 \\
10 & 14 & 11 \\
14 & 15 & 11 \\
11 & 15 & 12 \\
15 & 16 & 12 \\
12 & 16 & 13 \\
16 & 1 & 13 \\
13 & 1 & 14 \\
1 & 2 & 14 \\
14 & 2 & 15 \\
2 & 3 & 15 \\
15 & 3 & 16 \\
3 & 4 & 16 \\
16 & 4 & 1 \\
4 & 5 & 1
\end{pmatrix}$$

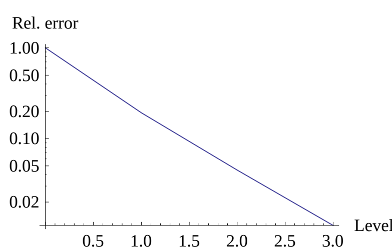


Required valences $\tau(f) \in \{6\}$

Limit volume ↓ (≈8.15591951258617925284591951259)

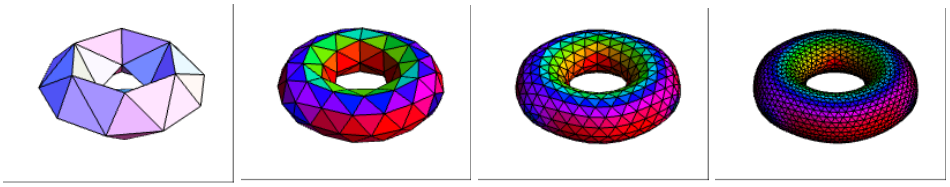
$$\frac{5\,086\,847}{623\,700}$$

Level	Volume	Delta to ∞
0	16	7.84408
1	$\frac{29}{3}$	1.51075
2	$\frac{34\,853}{4096}$	0.353114
3	$\frac{414\,870\,301}{50\,331\,648}$	0.0868128



Torus

The vertex coordinates and topology of the mesh are specified in a section above.

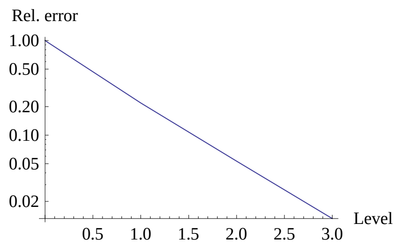


Required valences τ(f) ∈ {4, 6, 7, 8}

Limit volume ↓

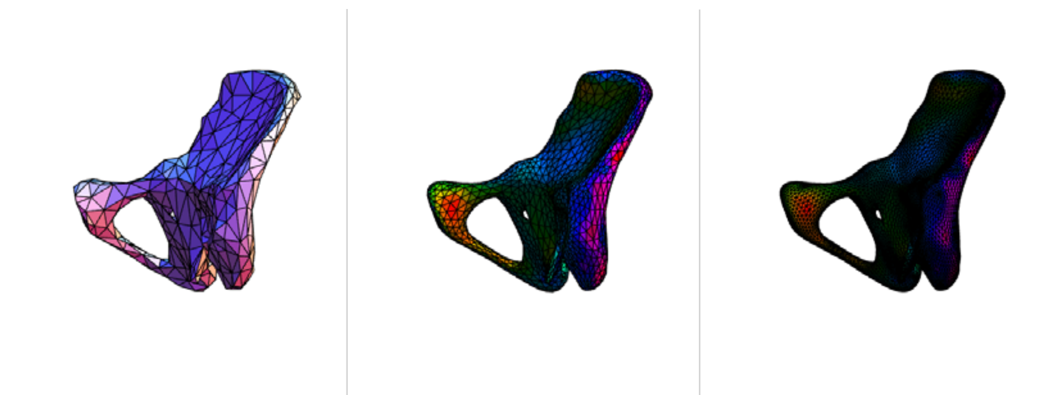
2.16461

Level	Volume	Delta to ∞
0	3.13298	0.968366
1	2.37699	0.212374
2	2.2161	0.0514863
3	2.17739	0.0127724



Hip_s

The specification of the mesh is omitted. The example is included for the purpose of illustration, and to study the approximation of the volume.

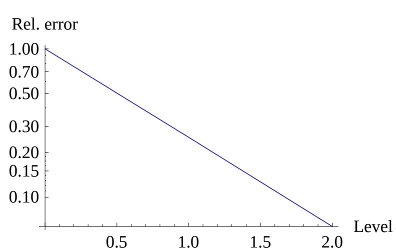


Required valences τ(f) ∈ {3, 4, 5, 6, 7, 8, 9, 12}

Limit volume ↓

0.169937

Level	Volume	Delta to ∞
0	0.17317	0.00323312
1	0.170755	0.000817913
2	0.170142	0.000205162



References

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[Doo/Sabin 1978] Doo. D., Sabin M.: *Behaviour of recursive division surfaces near extraordinary points*, Computer-Aided Design 10(6), 1978

[Hakenberg et al. 2014] Hakenberg J., Reif U., Schaefer S., Warren J.: *Volume Enclosed by Subdivision Surfaces*, viXra:1405.0012, viXra.org, 2014

[Loop 1987] Loop C.: *Smooth subdivision surfaces based on triangles*, Master's thesis, University of Utah, 1987

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