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The property of Moore Penrose Generalized Inverse And The Correlativity Between Two Random Variables[1]

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Abstract. It is an important problem to study the correlativity of two random variables. The correlation coefficient can be used to measure the correlativity of two random variables and the generalized correlation coefficient can be used to measure the correlativity of two random vectors . In this paper, The property of Moore Penrose generalized inverse has been proved and we make use of it to mainly discuss the property of generalized correlation coefficient of two random vectors and obtain some meaningful results.

Keywords: Moore Penrose Generalized Inverse, Generalized Correlation Coefficient, Latent Root, Canonical Correlation Coefficient.

1 Introduction

Let x and y respectively be p-dimension and q-dimension random vectors, V is covariance matrix of x and y, That is

$$V = V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} V(x) & Cov(x,y) \\ Cov(y,x) & V(y) \end{pmatrix} \triangleq \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix} \tag{1}$$

Because non-zero eigenvalues of $V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}$ is the same as $\left(V_{yy}^{\frac{1}{2}}\right)^{+} V_{yx}V_{xx}^{+}V_{xy}\left(V_{yy}^{\frac{1}{2}}\right)^{+}$, the non-zero eigenvalues of $V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}$ is not negative . Here A^{+} expresses Moore Penrose generalized inverse matrix of matrix A. When the rank of $V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}$ is r, it can be proved that the eigenvalues of $V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}$, $\lambda_1^*, \lambda_2^*, \cdots$ is not larger than number 1. Let $\lambda_i = \sqrt{\lambda_i^*}, i = 1, 2, \cdots$, then $\lambda_i\left(i = 1, 2, \cdots\right.$ is called canonical correlation coefficient of x and y. The correlativity of random vectors x and y can be measured by function of canonical correlation coefficient and five kinds of generalized correlation coefficient are defined by using

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canonical correlation coefficient in article [1]. Hoetelling[2] has defined another two kind generalized correlation coefficient. References [3-6] discuss the properties and applications of generalized correlation coefficient. In this paper ,we use $rz(x, y)$ to stand for generalized correlation coefficient of x and y.

2 The Main Results

Theorem 1 Let A be $n \times n$ square matrix and C be $n \times n$ nonnegative symmetric matrix. If $AC = CA$, then

$$AC^{+} = C^{+}A \tag{2}$$

Proof. As matrix $C \geq 0$, there is an orthogonal square P to make

$$C = P\begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} P', \quad \Lambda r = diag(\lambda_1, \lambda_2, \cdots$$

So, $AC = CA$ can be written

$$AP\begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} P' = P\begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} P'A. \tag{3}$$

Multiplying both sides of (3) on the left by P' and on the right by P, we have

$$P'AP\begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} P'AP \tag{4}$$

Let $P'AP = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Lambda r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{5}$$

That is,

$$\begin{pmatrix} A_{11}\Lambda r & 0 \\ A_{21}\Lambda r & 0 \end{pmatrix} = \begin{pmatrix} \Lambda r A_{11} & \Lambda r A_{12} \\ 0 & 0 \end{pmatrix} \tag{6}$$

So we can get

$$A_{11}\Lambda r = \Lambda r A_{11}$$

$$A_{21}\Lambda r = \Lambda r A_{12} = 0$$

and the more,

$$A_{11}\Lambda r^{-1} = \Lambda r^{-1} A_{11}, \quad A_{21}\Lambda r^{-1} = \Lambda r^{-1} A_{12} = 0. \tag{7}$$

Yet,

$$P'AP\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'P = P'AP\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}A_{11}\Lambda r^{-1} & 0\\ A_{21}\Lambda r^{-1} & 0\end{pmatrix}=\begin{pmatrix}A_{11}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}$$

$$P'P\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'AP=\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'AP$$

$$=\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

$$=\begin{pmatrix}\Lambda r^{-1}A_{11} & \Lambda r^{-1}A_{12}\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}\Lambda r^{-1}A_{11} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}A_{11}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}$$

$$=P'AP\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'P$$

By the above, we know that

$$AP\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'=P\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'A \tag{8}$$

According to the property of Moor Penrose generalized matrix, there is

$$C^{+}=\left[P\begin{pmatrix}\Lambda r & 0\\ 0 & 0\end{pmatrix}P'\right]^{+}=P\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P' \tag{9}$$

Therefore

$$AC^{+}=AP\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'=P\begin{pmatrix}\Lambda r^{-1} & 0\\ 0 & 0\end{pmatrix}P'A=C^{+}A\,.$$

Theorem1 is true.□

Theorem 2 Let A and B respectively be $p\times p$ and $q\times q$ square matrix which exist the inverse, and

$$AV_{xx}=V_{xx}A\,,\ \ BV_{yy}=V_{yy}B\,.$$

Then

$$rz(Ax,By)=rz(x,y) \tag{10}$$

Proof. Because the covariance of AX and BY is

$$V\begin{pmatrix} Ax \\ By \end{pmatrix} = \begin{pmatrix} V(Ax) & Cov(Ax, By) \\ Cov(By, Ax) & V(By) \end{pmatrix} = \begin{pmatrix} AV_{xx}A' & AV_{xy}B' \\ BV_{yx}A' & BV_{yy}B' \end{pmatrix} \tag{11}$$

The canonical correlation coefficient of Ax and By $(\rho_i > 0)$ satisfies the following equation

$$\left|(BV_{yy}B')^{+} BV_{yx}A'(AV_{xx}A')^{+} AV_{xy}B' - \rho_i^2 I\right| = 0 \tag{12}$$

By Theorem 1 and $AV_{xx} = V_{xx}A, BV_{yy} = V_{yy}B$, we can get that

$$AV_{xx}^{+} = V_{xx}^{+}A, BV_{yy}^{+} = V_{yy}^{+}B,$$

So there is

$$(AV_{xx}A')^{+} = (A^{-1})' V_{xx}^{+}A^{-1}, (BV_{yy}B')^{+} = (B^{-1})' V_{yy}^{+}B^{-1}. \tag{13}$$

In fact

$$(A^{-1})' V_{xx}^{+}A^{-1}AV_{xx}A'(A^{-1})' V_{xx}^{+}A^{-1} = (A^{-1})' V_{xx}^{+}A^{-1}$$

$$AV_{xx}A'(A^{-1})' V_{xx}^{+}A^{-1}AV_{xx}A' = AV_{xx}A'$$

$$\begin{aligned}\left(AV_{xx}A'(A^{-1})' V_{xx}^{+}A^{-1}\right)' &= (AV_{xx}V_{xx}^{+}A^{-1})' \\ &= (V_{xx}V_{xx}^{+})' = V_{xx}V_{xx}^{+} \\ &= AV_{xx}A'(A^{-1})' V_{xx}^{+}A^{-1}\end{aligned}$$

$$\begin{aligned}&\left((A^{-1})^{-1} V_{xx}^{+}A^{-1}AV_{xx}A'\right)' \\ &= (V_{xx}^{+}V_{xx})' = V_{xx}^{+}V_{xx} \\ &= (A^{-1})' V_{xx}^{+}A^{-1}AV_{xx}A'\end{aligned}$$

.

Here we make use of $AV_{xx} = V_{xx}A, AV_{xx}^{+} = V_{xx}^{+}A$ and $V_{xx}V_{xx}^{+} = V_{xx}^{+}V_{xx}$.

According to the definition of Moore Penrose matrix we know that $(AV_{xx}A')^{+} = (A^{-1})' V_{xx}^{+}A^{-1}$ is correct and we can use the same method to testify $(BV_{yy}B')^{+} = (B^{-1})' V_{yy}^{+}B^{-1}$.

Substituting (13) into (12), we have

$$\left|(B^{-1})' V_{yy}^{+}B^{-1}BV_{yx}A'(A^{-1})' V_{xx}^{+}A^{-1}AV_{xy}B' - \rho_i^2 I\right| = 0.$$

That is

$$\left|V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}-\rho_i^2 I\right|=0 \tag{14}$$

This means that the canonical correlation coefficient of Ax and By Equals the canonical correlation coefficient of x and y, so their generalized correlation coefficient equals too.□

The above result can be generalized. Let A and B respectively be $n\times p$ and $m\times q$ matrix with full column rank . They have the singular value decomposition $A=P_1\Lambda_1Q_1$ and $B=P_2\Lambda_2Q_2$, where P_1,Q_1,P_2 and Q_2 are orthogonal matrix, Λ_1 and Λ_2 are diagonal matrix.

Theorem 3 For the above matrix A and B, when

$$\Lambda_1Q_1V_{xx}=V_{xx}\Lambda_1Q_1,\ \Lambda_2Q_2V_{yy}=V_{yy}\Lambda_2Q_2,$$

we have the following result

$$\text{rz(Ax,By)=rz(x,y)} \tag{15}$$

Proof. let $\rho_i^2>0$ be the canonical correlation coefficient of Ax and By, then ρ_i satisfies the following equation

$$\left|\left(BV_{yy}B'\right)^{+}BV_{yx}A'\left(AV_{xx}A'\right)^{+}AV_{xy}B'-\rho_i^2 I\right|=0.$$

Substituting $A=P_1\Lambda_1Q_1$ and $B=P_2\Lambda_2Q_2$ into the above equation, we can get

$$\left|\left(\Lambda_2Q_2V_{yy}Q_2'\Lambda_2\right)^{+}\Lambda_2Q_2V_{yx}Q_1'\Lambda_1\left(\Lambda_1Q_1V_{xx}Q_1'\Lambda_1\right)^{+}\Lambda_1Q_1V_{xy}Q_2'\Lambda_2-\rho_i^2 I\right|=0 \tag{16}$$

Note $\Lambda_1Q_1V_{xx}=V_{xx}\Lambda_1Q_1$, $\Lambda_2Q_2V_{yy}=V_{yy}\Lambda_2Q_2$.

According to Theorem 1, there is a result

$$\Lambda_1Q_1V_{xx}^{+}=V_{xx}^{+}\Lambda_1Q_1\ ,\ \Lambda_2Q_2V_{yy}^{+}=V_{yy}^{+}\Lambda_2Q_2$$

By the proving procedure of Theorem 2, we have

$$\left(\Lambda_1Q_1V_{xx}Q_1'\Lambda_1\right)^{+}=\left(\left(\Lambda_1Q_1\right)^{-1}\right)'V_{xx}^{+}\left(\Lambda_1Q_1\right)^{-1}$$

$$\left(\Lambda_2Q_2V_{yy}Q_2'\Lambda_2\right)^{+}=\left(\left(\Lambda_2Q_2\right)^{-1}\right)'V_{yy}^{+}\left(\Lambda_2Q_2\right)^{-1}$$

To substitute them into (16), we can get

$$\left|V_{yy}^{+}V_{yx}V_{xx}^{+}V_{xy}-\rho_i^2 I\right|=0 \tag{17}$$

This completes the proof of theorem 3.□

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