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The Mathematical Structure and Analysis of an MHD Flow in Porous Media

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Abstract. This article is concerned with the formulation and analytical solution of equations for modeling a steady two-dimensional MHD flow of an electrically conducting viscous incompressible fluid in porous media in the presence of a transverse magnetic field. The governing equations, namely, Navier-Stokes equations and the Darcy-Lapwood-Brinkman model are employed for the flow through the porous media. The solutions obtained for the Riabouchinsky-type flows are then classified into different types.

Keywords: Porous medium, MHD flow, Navier-Stokes equations, Riabouchinsky flow.

1 Introduction

One of the oldest chapters of engineering sciences, the magnetohydrodynamic fluid flow in porous media due to its rapid expanse and widespread industrial and environmental applications, is still a subject of current research interest. For example, the rock that constitutes the earth's crust is essentially a porous medium that deforms over geological timescales. The flow through, and erosion of, this medium by magma leads to such phenomena on layered magma chambers and volcanic eruptions. The flow of groundwater through soil and / or rock has important applications in agriculture and pollution control. Other topics of interest include compaction of sedimentary basins and the phenomenon of frost heave, which occurs when groundwater freezes. As well as damaging roads and pavements, frost heave is responsible for geological formations [1], [5], [6]. One of the original motivations for studying porous medium flows was the extraction of oil from rocks. It was found that the suction of viscous fluid from a porous medium is often unstable, tending to leave behind a sizeable proportion of the oil in small isolated packets and increasing the extracted fraction is a constant challenge. Another important issue for the oil industry is upscaling from locally measured properties (e.g. permeability) [4]. Further application of porous media is most of the tissues in the body (e.g. bone, cartilage, muscle) are deformable porous medium. The proper functioning of such materials depends crucially on the flow of blood, nutrients through them. Porous medium models are used to understand various medical conditions (such as tumour growth) and treatment (such as injections) [3]. There are four independent fields of fluid dynamics namely, general fluid flow, flow through porous media, magnetohydrodynamics flow and analytical approaches to flow problems. In the field of general fluid flow, an enormous amount of work has been carried out over the past two centuries, and many aspects of fluid flow theory are well developed [10]. and in their study of inviscid aligned flow, by complex technique. Various aspects of magnetohydrodynamic flow have been studied, and many models have been developed to study the interactions between the magnetic field, the electric field, and how they interact with a flowing fluid. Analysis of this type of flow has been proved by [16]. Some rotational and irrotational viscous incompressible aligned plane flow were discussed in [2], [7].

As per fluid flow through porous media, interest in the field dates as far back as 1856 by Darcy and currently one finds various models governing types of fluid flow in many porous structures. The main type of single phase flow models have been developed and reviewed by [7], [12].

Although many models of flow through porous media are available, and various others are continually being developed, a model of particular importance to this work is the Darcy-Lapwood-Brinkman model [8], [13]. This accounts for both viscous shear and inertial effects of flow through porous media. A modified version of model in our current analysis, with the exception that a magnetic field is imposed on the flow field will be used.

In this work, we consider a single-phase fluid flow through porous media in the presence of a magnetic field. The aim is to analyze the nonlinear model equation in an attempt to find possible solutions corresponding to a particular form of the stream function. The choice of the stream function in this work is one that is linear with respect to one of the independent variables. This type of flow is referred to as the Riabouchinsky flow. In this case, the two dimensional Navier-Stokes equations, written as a fourth order partial differential equations in terms of the stream function, may be replaced by fourth order ordinary differential equations in two unknown functions of a single variable. Solutions to the coupled set are then obtained based on the knowledge of particular integrals of one of the equations. A different type of flow may then be obtained with the knowledge of one of the functions.

Riabouchinsky [14] assumed one of the functions to be zero and studied the resulting flow which represents a plane flow in which the flow is separated in the two symmetrical regions by a vertical or a horizontal plane. In addition to the study of Navier-Stokes flows and their applications [15], Riabouchinsky flows have also received considerable attention in the study of non-Newtonian flows [9] and in magnetohydrodynamics [11].

In this work we find analytical solutions to the two-dimensional viscous fluid flows through porous media in the presence of a magnetic field. The governing equations are based on the Darcy-Lapwood-Brinkman model of flow through porous media. The medium is assumed to be traversed and aligned by a magnetic field. Solutions are obtained for Riabouchinsky-type flows, with a modified solution algorithm that has been developed to handle the type of flow. Solutions obtained are then classified into different types.

2. Flow Equations

2.1 The Darcy-Lapwood-Brinkman (DLB) Model

The steady flow of an incompressible viscous fluid through porous media is governed by the conservation of mass and conservation of linear momentum principles. In the absence of sources and sinks, conservation of mass principle takes the form

$$\nabla \cdot \vec{V} = 0 \,, \tag{1}$$

Where $\vec{V}$ is the macroscopic velocity vector. The conservation of linear momentum is given by the Navier-Stokes equations of the form

$$(\vec{V} \cdot \nabla)\vec{V} = -\frac{}{\rho} + \frac{}{\rho}\nabla^2\vec{V}, \tag{2}$$

Where P is the pressure , υ is the viscosity coefficient and ρ is the fluid density. When the macroscopic inertia and viscous shear are important, fluid flow through porous medium may be described by the DLB of the form

$$(\vec{V} \cdot \nabla)\vec{V} = -\frac{}{\rho} + \frac{}{\rho}\nabla^2\vec{V} - \frac{}{k\,\rho}\vec{V} \,, \tag{3}$$

Where k is the permeability. Equation (3) is postulated to govern the flow through porous medium of high permeability and the flow through a mushy zone undergoing rapid freezing. Through the viscous shear term, equation (3) is capable of handling the presence of a macroscopic boundary on which a non-slip condition is imposed. Comparison of the structure of equation (3) with that of the Navier-Stokes equation (2) which govern the flow of a viscous fluid in free-space shows the presence in equation (3) of the viscous damping term $(\frac{\upsilon}{k}\vec{V})$ that is postulated to the Darcy resistance to motion exerted by the medium on the transversing fluid. However, the presence of this term does not alter the nonlinear structure exhibited in the Navier-Stokes equations.

2.2 Flow through porous media in the presence of a magnetic field

The steady flow of a viscous, incompressible, electrically conducting fluid through porous media in the presence of a magnetic field is governed by the following system of partial differential equations

$$\nabla \cdot \vec{V} = 0 \, , \tag{4}$$

$$\rho(\vec{V}\cdot\nabla)\vec{V} = -\nabla P + \mu_1 \nabla^2 \vec{V} - \frac{\mu_2}{k_1}\vec{V} + k_2(\nabla \times \vec{H}) \times \vec{H} \, , \tag{5}$$

$$\nabla \times (\vec{V}\times\vec{H}) - \frac{1}{k_2\sigma}\nabla \times (\nabla \times \vec{H}) = 0 \tag{6}$$

$$\nabla \cdot \vec{H} = 0 \tag{7}$$

Where $\vec{V}$ is the velocity vector field, $\vec{H}$ is the magnetic vector field, P is the pressure, σ is the electrical conductivity, k_1 is the medium permeability to the fluid, k_2 is the magnetic permeability, μ_1 is the fluid viscosity, μ_2 is the porous medium viscosity and ρ is the fluid density. In virtue of the vector identity

$$(\vec{V}\cdot\nabla)\vec{V} = \nabla(\tfrac{1}{2}|\vec{V}|^2) - \vec{V}\times(\nabla\times\vec{V}) \tag{8}$$

then equation (5) takes the form

$$\rho\left\{\nabla\left(\tfrac{1}{2}|\vec{V}|^2 - \vec{V}\times(\nabla\times\vec{V})\right)\right\} = -\nabla P + \mu_1\nabla^2\vec{V} + k_2(\nabla\times\vec{H})\times\vec{H} \, . \tag{9}$$

The governing equations are thus (4), (6), (7) and (9). It is required to solve this system of equations for the unknowns $\vec{V}$, $\vec{H}$ and P .

2.3 The case of two-dimensional flow

Considering plane-transverse flow in two space dimensions x and y , with a velocity vector field $\vec{V} = (u, v, 0)$ and a magnetic field $\vec{H} = (0, 0, H)$ which acts in a constant direction $\frac{\partial}{\partial z} = 0$, then equation (7) is automatically satisfied and the governing equations (4), (6) and (9) take the following components form:

$$u_x + v_y = 0 \tag{10}$$

$$\rho\left\{\left[\tfrac{1}{2}q^2\right]_x - v\left(v_x - u_y\right)\right\} = -P_x + \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u - k_2\left[\tfrac{1}{2}H^2\right]_x \tag{11}$$

$$\rho\left\{\left[\tfrac{1}{2}q^2\right]_y - u\left(v_x - u_y\right)\right\} = -P_y + \mu_1\nabla^2 v - \frac{\mu_2}{k_1}v - k_2\left[\tfrac{1}{2}H^2\right]_y \tag{12}$$

$$u\,H_x + v\,H_y - \frac{1}{k_2\sigma}\nabla^2 H = 0\,, \tag{13}$$

Where the subscript notation denotes partial differentiation, and $q^2 = u^2 + v^2$ is the square of the speed. The system of equations (10) – (13) represent four scalar equations in the unknowns $u(x,y), v(x,y), P(x,y)$ and $H(x,y)$. It should be noted that if the fluid is infinitely conducting, then $\frac{1}{k_2\sigma} \to 0$ and equation (13) reduces to

$$u\,H_x + v\,H_y = 0 \tag{14}$$

If we now define

$$\overline{P}(x,y) = P + \tfrac{1}{2}k_2 H^2$$

$$\overline{P}_x(x,y) = P_x + k_2[\tfrac{1}{2}H^2]_x$$

$$\overline{P}_y(x,y) = P_y + k_2[\tfrac{1}{2}H^2]_y\,,$$

Then the equations (11) and (12) take the following forms respectively:

$$\rho\left\{[\tfrac{1}{2}q^2]_x - v\,(v_x - u_y)\right\} = -\overline{P}_x + \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u \tag{15}$$

$$\rho\left\{[\tfrac{1}{2}q^2]_y - u\,(v_x - u_y)\right\} = -\overline{P}_y + \mu_1\nabla^2 v - \frac{\mu_2}{k_1}v \tag{16}$$

The governing equations for the case of finite electrical conductivity are therefore (10), (13), (15) and (16), and equations (10), (14), (15) and (16) are for the case of infinite conducting fluid.

2.4 Equations for vorticity function and stream function

Introducing the vorticity function $\omega(x,y)$ and the pressure function $h(x,y)$ defined respectively by

$$\omega(x,y) = v_x - u_y \tag{17}$$

$$h(x,y) = \overline{P} + \tfrac{1}{2}\rho q^2 \tag{18}$$

$$h_x = \overline{\mathrm{p}}_x + \rho\,[\tfrac{1}{2}q^2]_x$$

$$h_y = \bar{\mathrm{p}}_y + \rho[\tfrac{1}{2}q^2]_y \ .$$

Then equations (15) and (16) take the following forms respectively

$$h_x - \rho v\,\omega = \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u \ , \tag{19}$$

$$h_y + \rho u\,\omega = \mu_1\nabla^2 v - \frac{\mu_2}{k_1}v \ . \tag{20}$$

Hence the governing equations are thus (10), (14), (19) and (20) with ω and h are given in (17) and (18) respectively.

Let $\psi(x,y)$ be the stream function defined in terms of the components as

$$\psi_y = u \qquad \text{and} \qquad \psi_x = -v$$

Then we can see clearly that the conductivity equation (10) is identically satisfied since

$$u_y + v_y = \psi_{xy} - \psi_{yx} = 0 \tag{21}$$

and the vorticity equation (14) becomes

$$\omega = -\nabla^2\psi \tag{22}$$

Equations (19), (20) and (13) are then become in terms of the stream function as follows :

$$h_x - \rho\psi_x(\nabla^2\psi) = \mu_1\nabla^2\psi_y - \frac{\mu_2}{k_1}\psi_y \tag{23}$$

$$h_y - \rho\psi_y(\nabla^2\psi) = -\mu_1\nabla^2\psi_x - \frac{\mu_2}{k_1}\psi_x \tag{24}$$

$$\psi_y H_x - \psi_x H_y - \frac{1}{k_2\sigma}\nabla^2 H = 0 \ . \tag{25}$$

Equations (22)-(25) are now required to be solved for $\omega(x,y)$, $\psi(x,y)$, $H(x,y)$ and $h(x,y)$. The velocity components u and v will be obtained from (21), $\overline{P}$ from equation (18) and then follows P .

If the fluid is infinitely electrically conducting, then the diffusion equation (25) is reduced to

$$\psi_y H_x - \psi_x H_y = 0 \ .$$

A compatibility equation can be derived from equations (23) and (24) using the integrability condition $h_{xy} = h_{yx}$. Thus, differentiating equation (23) with respect to y and equation (24) with respect to x and using the integrability condition we obtain the compability equation

$$\psi_x(\nabla^2\psi)_y - \psi_y(\nabla^2\psi)_x = \frac{\upsilon_2}{k_1}\nabla^2\psi - \upsilon_1\nabla^2\psi \tag{27}$$

or

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)} = \frac{\upsilon_2}{k_1}\nabla^2\psi - \upsilon_1\nabla^4\psi \ , \tag{28}$$

Where

$$\upsilon_1 = \frac{\mu_1}{\rho}$$

and

$$\upsilon_2 = \frac{\mu_2}{\rho} \ ; \quad \nabla^2 = \partial_{xx} + \partial_{yy} \ .$$

$$\nabla^4 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy} \ .$$

Hence once equation (26) is solved for $\psi(x,y)$, the vorticity function can be calculated from equation (17) and $H(x,y)$ from equation (25). The pressure function $h(x,y)$ can be obtained from equations (23) and (24), while $\overline{P}(x,y)$ is obtained from equation (18) and hence $P(x,y)$ follows. The velocity components can then be obtained from equation (21).

3. An Overview of the Solution Methodology

3.1 The case of Navier-Stokes Equation

The two-dimensional steady flow of a viscous incompressible fluid in free space is governed by the continuity equation (1) and the Navier-Stokes equation (2). In this form, it is required to solve (1) and (2) for $\overline{V}$ and P. These equations may be written in vorticity-stream function form using definitions (17) and (21) to obtain stream function equation

$$\nabla^2\psi = -\omega \tag{29}$$

Vorticity equation

$$\psi_y\omega_x - \psi_x\omega_y = \upsilon_1\nabla^2\omega - \frac{\upsilon_2}{k_2}\omega \ . \tag{30}$$

Integrability equation for the Navier-Stokes equation can be obtained by substituting equation (29) into equation (30) to yield

$$\upsilon_1\nabla^4\psi + \psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x - \frac{\upsilon_2}{k_2}\nabla^2\psi = 0 \ . \tag{33}$$

Let

$$h(x,y)=\overline{P}+\tfrac{1}{2}\rho q^{2} \tag{34}$$

be the generalized pressure function for the Navier-Stokes equation. Then the momentum equation (2) may be written in component form as

$$h_x-\rho\,\psi_x(\nabla^2\psi)=\mu_1\nabla^2\psi_y-\frac{\mu_2}{k_1}\psi_y \tag{35}$$

$$h_y-\rho\,\psi_y(\nabla^2\psi)=-\mu_1\nabla^2\psi_x+\frac{\mu_2}{k_1}\psi_x\ . \tag{36}$$

Equation (33) can then be solved for $\psi(x,y)$. Using equation (29) we can obtain $\omega(x,y)$, $u(x,y)$ and $v(x,y)$ can then be obtained from (21). The pressure function $h(x,y)$ may then be obtained from equations (35) and (36) and the pressure function $P(x,y)$ is then follows from (34). In order to provide a solution for equation (33) we assume that the stream function $\psi(x,y)$ is linear with respect to the independent variable y and has the form

$$\psi(x,y)=y\,f(x)+g(x). \tag{37}$$

where $f(x)$ and $g(x)$ are four lines differentiable arbitrary functions of x .

3.2 Reduction of the governing partial differential equations to ordinary differential equations

Substituting equation (37) into equation (33) gives

$$\upsilon_1\left(yf^{(\text{IV})}+g^{(\text{IV})}\right)+\left(yf'+g'f''\right)-f\left(yf''+g''\right)-\frac{\upsilon_2}{k_1}\left(yf''+g''\right)=0 \tag{38}$$

Equating the coefficients of similar powers of y, leads to the following coupled set of fourth order ordinary differential equations :

$$\upsilon_1 f^{(\text{IV})}-\frac{\upsilon_2}{k_1}f''+[f'f''+f\,f'']=0 \tag{39}$$

$$\upsilon_1 g^{(\text{IV})}-\frac{\upsilon_2}{k_1}g''+[g'f''+f\ \ g'']=0 \tag{40}$$

3.3 Particular Solution

In the absence of general solutions for the coupled equations (39) and (40), it is customary to determine $f(x)$ and $g(x)$ in accordance with the following algorithm:

- A particular solution is found for $f(x)$ satisfying equation (39).
- The function $f(x)$ is then substituted into equation (40) and a general solution is then found for $g(x)$.
- With the knowledge of $f(x)$ and $g(x)$ the stream function $\psi(x,y)$ can then be obtained from equation (37).

The velocity components and the vorticity function can be calculated from (21) and (29), the pressure function can be determined from equation (35) and (36) and the pressure follows from equation (34).

3.4 The different types of possible flows and their solutions

A solution to the system of equations (39) and (40) is considered for the case $k_1 \to \infty$.

In this case, equations (39) and (40) give the coupled set of fourth order ordinary differential equations

$$\upsilon_1 f^{(iv)} + [f'f'' - f f''] = 0 \tag{41}$$

$$\upsilon_1 g^{(iv)} + [g'f'' - f g''] = 0 \tag{42}$$

Equation (41) admits the three particular solutions

$$f_1(x) = \frac{-6\upsilon_1}{x} \tag{43}$$

$$f_2(x) = \upsilon_1 \alpha (1 + \beta e^{\alpha x}) \tag{44}$$

and

$$f_3(x) = c_1 x + c_2 \;. \tag{45}$$

where α, β, c_1 and c_2 are arbitrary constants. The solution $f_3(x)$ is independent of the viscocity. Upon substituting for $f_1(x)$ from equation (43) into equation (42) gives

$$g_1(x) = \alpha_0 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3} \;, \tag{46}$$

where α_0, α_1, α_2 and α_3 are arbitrary constants. The corresponding stream function $\psi(x,y)$ and the velocity components take the following forms respectively:

$$\psi(x,y) = -6\upsilon_1 \frac{y}{x} + \alpha_0 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \alpha_3 x^{-3} \tag{47}$$

$$u(x,y) = \frac{\partial \psi}{\partial y} = \frac{\upsilon_1}{x} \tag{48}$$

$$v(x,y) = -\frac{\partial \psi}{\partial x} = -6\upsilon_1 y x^{-2} + \alpha_1 x^{-2} + +\alpha_3 x^{-3} - 3\alpha_3 x^{2} \tag{49}$$

Similarly, substituting from $f_2(x)$ equation (3.14) into equation (42) yields

$$g_2(x) = c_4 + c_3 e^x + c_2 \int e^x dx \int \exp[\beta e^x - x] dx$$

$$+ c_1 \int e^x dx \int \exp[\beta e^x - x] dx \int \exp[-\beta x^x] dx \ , \tag{50}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. The corresponding stream function with the velocity components are

$$\psi(x,y) = \upsilon_1 \alpha (1 + \beta e^{\alpha x}) y + c_4 + c_3 e^x + c_2 \int e^x dx \int \exp[\beta e^x - x] dx$$
$$+ c_1 \int e^x dx \int \exp(\beta e^x - x) dx \int \exp(-\beta x)^x dx \tag{51}$$

$$u(x,y) = \frac{\partial \psi}{\partial y} = \upsilon_1 \alpha (1 + \beta e^{\alpha x}) \tag{52}$$

$$v(x,y) = -\frac{\partial \psi}{\partial x} = -\upsilon_1 \alpha \ . \tag{53}$$

Likewise, substituting $f_3(x)$ from equation (45) into equation (42) gives

$$g_3(x) = \iiint c_3 \exp[\tfrac{1}{\upsilon_1} \{\tfrac{1}{2} c_1 x^2 + c_2 x\}] dx dx dx \tag{54}$$

where c_1, c_2 and c_3 are arbitrary constants. The corresponding stream function and the velocity components take the following forms respectively :

$$\psi(x,y) = (c_1 x + c_2) y + \iiint c_3 \exp[\tfrac{1}{\upsilon_1} \{\tfrac{1}{2} c_1 x^2 + c_2 x\}] dx dx dx \tag{55}$$

$$u(x,y) = \frac{\partial \psi}{\partial y} = c_1 x + c_2 \ , \tag{56}$$

$$v(x,y) = -\frac{\partial \psi}{\partial y} = -\left(c_1 y + \iint c_3 \exp[\tfrac{1}{\upsilon_1} \{\tfrac{1}{2} c_1 x^2 + c_2 x\}] dx dx \right) \tag{57}$$

Conclusion

In the current work we have discussed the analytical solution of the magnetohydrodynamic flow through porous media in the presence of a magnetic field. We have implemented the Riabouchinsky method. The solutions obtained using the above procedure involve a number of arbitrary constants, and restrictive assumptions are needed to determine them. It is also clear that determining $g(x)$ is not an easy task, and some integrals are rather involved. In the absence of a systematic procedure for determining the arbitrary constants, Riabouchinsky [14] assumed that $g(x)=0$, and obtained the special form of the stream function $\psi(x,y)=y\,f(x)$. This amounts to assigning the value of zero to each of the arbitrary constants $g_1(x)$ and $g_2(x)$ and $g_3(x)$ above.

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