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ON THE DUAL SPACELIKE – SPACELIKE INVOLUTE – EVOLUTE CURVE COUPLE ON DUAL LORENTZIAN SPACE ID_1^3

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Abstract. In this paper, firstly we have defined the involute curves of the dual spacelike curve M_1 with a dual timelike binormal in dual Lorentzian space ID_1^3 We have seen that the dual involute curve M_2 must be a dual spacelike curve with a dual spacelike or timelike binormal vector. Secondly,the relationship between the Frenet frames of couple the spacelike – spacelike involute – evolute dual curve has been found and finally some new characterizations related to the couple of the dual curve has been given.

Keywords: Dual Lorentzian space, dual involute – evolute curve couple, dual Frenet frames
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1 Introduction

The consept of the involute of a given curve is a well-known in 3-dimensional Euclidean space IR^3 in [7,8,12,13]. Some basic notions of Lorentzian space are given [3,10,14]. M_1 is a timelike curve then the involute curve M_2 is a spacelike curve with a spacelike or timelike binormal. On the other hand, it has been investigated the involute and evolute curves of the spacelike curve M_1 with a spacelike binormal in Minkowski 3-space and it has been seen that the involute curve M_2 is timelike. The involute curves of the spacelike curve M_1 with a timelike binormal is defined in Minkowski 3-space IR_1^3, [2,4,5]. Lorentzian angle defined in [11]. W.K. Clifford, introduced dual numbers as the set

$$ID = \left\{ \hat{\lambda} = \lambda + \varepsilon\lambda^* \,\middle|\, \lambda, \lambda^* \in IR, \ \ \varepsilon^2 = 0 \ \text{ for } \ \varepsilon \neq 0 \right\}, [6].$$

Addition, product, division and absolute value operations are defined on ID like below, respectively:

$$(\lambda+\varepsilon\lambda^*)+(\beta+\varepsilon\beta^*)=(\lambda+\beta)+\varepsilon(\lambda^*+\beta^*)$$
$$(\lambda+\varepsilon\lambda^*)(\beta+\varepsilon\beta^*)=\lambda\beta+\varepsilon(\lambda\beta^*+\lambda^*\beta),$$

$$\frac{\lambda+\varepsilon\lambda^*}{\beta+\varepsilon\beta^*}=\frac{\lambda}{\beta}+\varepsilon\left(\frac{\lambda^*\beta-\lambda\beta^*}{\beta^2}\right),$$
$$\left|\lambda+\varepsilon\lambda^*\right|=|\lambda|.$$

$ID^3=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a} \;\middle|\; \vec{a},\vec{a} \in IR^3\right\}$. The elements of ID^3 are called dual vectors .

On this set addition and scalar product operations are respectively

$$\vec{A}\oplus\vec{B}=\vec{a}+\vec{b}+\varepsilon\left(\vec{a}+\vec{b}\right),\ \lambda\square \qquad +\varepsilon\left(\lambda\vec{a}+\lambda\vec{a}\right)$$

The set $\left(ID^3,\oplus\right)$ is a module over the ring $(ID,+,\cdot)$. $(ID-Modul)$.

The Lorentzian inner product of dual vectors $\vec{A},\ \vec{B}\in ID^3$ is defined by

$$\left\langle\vec{A},\vec{B}\right\rangle=\left\langle\vec{a},\vec{b}\right\rangle+\varepsilon\left(\left\langle\vec{a},\vec{b}\right\rangle+\left\langle\vec{a},\vec{b}\right\rangle\right)$$

with the Lorentzian inner product $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in IR^3$

$$\left\langle\vec{a},\vec{b}\right\rangle=-a_1b_1+a_2b_2+a_3b_3.$$

Therefore, ID^3 with the Lorentzian inner product $\left\langle\vec{A},\vec{B}\right\rangle$ is called 3-dimensional dual Lorentzian space and denoted by of $ID_1^3=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a} \;\middle|\; \vec{a},\vec{a} \in IR_1^3\right\}$.

A dual vector $\vec{A}=\vec{a}+\varepsilon\vec{a} \in ID_1^3$ is called

A dual space-like vector if $\vec{a}$ is spacelike vector ,

A dual time-like vector if $\vec{a}$ is timelike vector,

A dual null(light-like) vector if $\vec{a}$ is lightlike vector .

For $\vec{A}\neq 0$, the norm $\left\|\vec{A}\right\|$ of $\vec{A}=\vec{a}+\varepsilon\vec{a} \in ID_1^3$ is defined by

$$\left\|\vec{A}\right\|=\sqrt{\left|\left\langle\vec{A},\vec{A}\right\rangle\right|}=\left\|\vec{a}\right\|+\varepsilon\frac{\left\langle\vec{a},\vec{a}\right\rangle}{\left\|\vec{a}\right\|},\quad \left\|\vec{a}\right\|\neq 0.$$

The dual Lorentzian cross-product of $\vec{A},\ \vec{B}\in ID_1^3$ is defined as

$$\vec{A}\wedge\vec{B}=\vec{a}\wedge\vec{b}+\varepsilon\left(\vec{a}\wedge\vec{b} \;+\vec{a} \;\wedge\vec{b}\right)$$

with the Lorentzian cross-product $\vec{a}\ ,\ \vec{b}\in IR_1^3$

$$\vec{a}\wedge\vec{b}=\left(a_3b_2-a_2b_3,a_1b_3-a_3b_1,a_1b_2-a_2b_1\right),[17].$$

Dual Frenet trihedron of the differentiable curve M in dual space ID_1^3 and instantaneous dual rotation vector have given in [1,16]. The dual angle between $\vec{A}$ and $\vec{B}$ is $\Phi=\varphi+\varepsilon\varphi^*$,such that

$$\begin{cases}\sinh\Phi=\sinh\left(\varphi+\varepsilon\varphi^*\right)=\sinh\varphi+\varepsilon\varphi^*\cosh\varphi\\ \cosh\Phi=\cosh\left(\varphi+\varepsilon\varphi^*\right)=\cosh\varphi+\varepsilon\varphi^*\sinh\varphi.\end{cases}$$

The dual Lorentzian sphere and the dual hyperbolic sphere of 1 radius in IR_1^3 are defined by

$$S_1^2=\left\{A=a+\varepsilon a_0 \mid \|A\|=(1,0);a,a_0\in IR_1^3,\text{ and } a \text{ is spacelike}\right\}$$

$$H_0^2=\left\{A=a+\varepsilon a_0 \mid \|A\|=(1,0);a,a_0\in IR_1^3,\text{ and } a \text{ is timelike}\right\}$$

respectively [15].

2 Preliminaries

Lemma 1 1: Let X and Y be nonzero Lorentz orthogonal vektors in ID_1^3. If X is timelike, then Y is spacelike, [11].

Lemma 2.2: Let X,Y be positive (negative) timelike vectors in ID_1^3.Then $\langle X,Y\rangle\le\|X\|\|Y\|$ with equality if and only if X and Y are linearly dependent, [11].

Lemma 2.3

i) Let X and Y be pozitive (negative) timelike vectors in ID_1^3. Then we ha $\langle X,Y\rangle\le\|X\|\|Y\|$, there is a unique non negative dual number $\Phi(X,Y)$ such that $\langle X,Y\rangle=\|X\|\|Y\|\cosh\Phi(X,Y)$ where $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y.

ii) Let X and Y be spacelike vectors in ID_1^3 that span a spacelike vector subspace.

Then we have $\left|\langle X,Y\rangle\right| \le \|X\|\|Y\|$. Hence, there is a unique dual number $\Phi(X,Y)$ between 0 and π such that $\langle X,Y\rangle = \|X\|\|Y\|\cos\Phi(X,Y)$ where $\Phi(X,Y)$ is the Lorentzian spacelike dual angle between X and Y.

***iii*)** Let X and Y be spacelike vectors in ID_1^3 that span a timelike vector subspace. Then wehave $\left|\langle X,Y\rangle\right| \ge \|X\|\|Y\|$. Hence, there is a unique positive dual number $\Phi(X,Y)$ such that $\langle X,Y\rangle = \|X\|\|Y\|\cosh\Phi(X,Y)$ where $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y.

***iv*)** Let X be a spacelike vector and Y a positive timelike vector in ID_1^3. Then there is a unique nonnegative dual number $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y, such that $\langle X,Y\rangle = \|X\|\|Y\|\sinh\Phi(X,Y)$,[11].

Let $\{T,N,B\}$ be the dual Frenet trihedron of the differentiable curve M in the dual space ID_1^3 and $T = t + \varepsilon t^*$, $N = n + \varepsilon n^*$ and $B = b + \varepsilon b^*$ be the tangent, the principal normal and the binormal vector of M, respectively. Depending on the causal character of the curve M, we have an instantaneous dual rotation vector:

1) Let M be a unit speed timelike dual space curve with dual curvature $\kappa = k_1 + \varepsilon k_1^*$ and dual torsion $\tau = k_2 + \varepsilon k_2^*$. The Frenet vectors T, N and B of M are timelike vector, spacelike vectors, spacelike vector, respectively, such that

$$T \wedge N = -B \quad , \quad N \wedge B = T \quad , \quad B \wedge T = -N. \tag{2.1}$$

From here ,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [18]. \tag{2.2}$$

(2.2) leaves the real and dual components

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

$$\begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ k_1^* & 0 & -k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}$$

The Frenet instantaneous rotation vector W of the timelike curve is given by

$$W = \tau T - \kappa B \text{,[14].} \tag{2.3}$$

(2.3) leaves the real and dual components

$$\begin{cases} w = k_2 t - k_1 b, \\ w^* = k_2^* t + k_2 t^* - k_1^* b - k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a Lorentzian timelike dual angle between the spacelike binormal unit vector B and the Frenet instantaneous dual rotation vector W. The $C = c + \varepsilon c^*$ is a unit dual vector in direction of W :

a) If $|\kappa| > |\tau|$, W is a spacelike vector. In this station, we can write

$$\begin{cases} \kappa = \|W\| \cosh\Phi \\ \tau = \|W\| \sinh\Phi \end{cases} , \quad \|W\|^2 = \langle W, W\rangle = \kappa^2 - \tau^2 \tag{2.4}$$

and

$$C = \sinh\Phi T - \cosh\Phi B. \tag{2.5}$$

b) If $|\kappa| < |\tau|$, W is a timelike vector. In this station, we can write

$$\begin{cases} \kappa = \|W\| \sinh\Phi \\ \tau = \|W\| \cosh\Phi \end{cases} , \quad \|W\|^2 = -\langle W, W\rangle = -\left(\kappa^2 - \tau^2\right) \tag{2.6}$$

and

$$C = \cosh\Phi T - \sinh\Phi B\ . \tag{2.7}$$

2) Let M be a unit speed dual spacelike space curve with spacelike binormal. The Frenet vectors T, N, B of M are spacelike vector, timelike vector, spacelike vector, respectively, such that

$$T \wedge N = -B \ , \ N \wedge B = -T \ , \ B \wedge T = N. \tag{2.8}$$

From here,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \text{,[18].} \tag{2.9}$$

(2.9) leaves the real and dual components

$$\begin{bmatrix} \mathrm{t}' \\ \mathrm{n}' \\ \mathrm{b}' \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{k}_1 & 0 \\ \mathrm{k}_1 & 0 & \mathrm{k}_2 \\ 0 & \mathrm{k}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t} \\ \mathrm{n} \\ \mathrm{b} \end{bmatrix},$$

$$\begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}$$

and the Frenet instantaneous rotation vector for the spacelike curve is given by

$$W = -\tau T + \kappa B, [14]. \tag{2.10}$$

(2.10) leaves the real and dual components

$$\begin{cases} \overline{w} = -k_2 t + k_1 b, \\ \overline{w^*} = -k_2^* t - k_2 t^* + k_1^* b + k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a dual angle between the B and the W. If B and W spacelike vectors that span a spacelike vector subspace, we can write

$$\begin{cases} \kappa = \|W\| \cos\Phi \\ \tau = \|W\| \sin\Phi \end{cases}, \quad \|W\|^2 = \langle W, W \rangle = \kappa^2 + \tau^2 \tag{2.11}$$

and

$$C = -\sin\Phi T + \cos\Phi B . \tag{2.12}$$

3) Let M be a unit speed dual spacelike space curve. The Frenet vectors T , N and B of M are spacelike vector, timelike vector and spacelike vector, respectively, such that

$$T \wedge N = B \ , \ N \wedge B = -T \ , \ B \wedge T = -N. \tag{2.13}$$

From here

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [18]. \tag{2.14}$$

(2.14) leaves the real and dual components

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

$$\begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ -k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}$$

and the Frenet instantaneous dual rotation vector W of the spacelike curve is given by

$$W = \tau T - \kappa B, [14]. \tag{2.15}$$

(2.15) leaves the real and dual components

$$\begin{cases} w = k_2 t - k_1 b, \\ w^* = k_2^* t + k_2 t^* - k_1^* b - k_1 b^*. \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a Lorentzian timelike dual angle between the B and W:

a) If $|\kappa| < |\tau|$, W is a spacelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \sinh\Phi \\ \tau = \|W\| \cosh\Phi \end{cases}, \quad \|W\|^2 = \langle W, W\rangle = \tau^2 - \kappa^2 \tag{2.16}$$

and

$$C = \cosh\Phi T - \sinh\Phi B. \tag{2.17}$$

b) If $|\kappa| > |\tau|$, W is a timelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \cosh\Phi \\ \tau = \|W\| \sinh\Phi \end{cases}, \quad \|W\|^2 = -\langle W, W\rangle = -\left(\tau^2 - \kappa^2\right) \tag{2.18}$$

and

$$C = \sinh\Phi T - \cosh\Phi B. \tag{2.19}$$

3 The Dual Involutes of The Spacelike Curve with Timelike Binormal in ID_1^3

Defination 3.1: Let $M_1 : I \to ID_1^3$ $M_1 = M_1(s)$ be the unit speed dual spacelike curve with timelike binormal and $M_2 : I \to ID_1^3$ $M_2 = M_2(s)$ be the unit speed dual curve. If tangent vector of curve M_1 is ortogonal to tangent vector of M_2, M_1 is called evolute of curve M_2 and M_2 is called involute of M_1 .Thus the dual involute – evolute curve couple is denoted by (M_2, M_1). So the tangent vector of M_2 is a spacelike curve with spacelike or timelike binormal. In this station (M_2, M_1) is called "the spacelike – spacelike involut – evolut dual curve couple".

Theorem 3.1: Let (M_2, M_1) be the spacelike – spacelike involute – evolute dual curve couple. Let $\{T,N,B\}$ and $\{V_1,V_2,V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively. The dual distance between M_1 and M_2 at the corresponding points is

$$d\left(M_1(s),M_2(s)\right)=|c_1-s|+\varepsilon c_2,\quad c_1,c_2=\text{constant}.$$

Proof: If M_2 is the dual involute of M_1, we can write

$$M_2(s)=M_1(s)+\lambda T(s),\quad \lambda=\lambda_1+\varepsilon\lambda_1^*\in ID \tag{3.1}$$

Differentiating (3.1) with respect to s we have

$$V_1\frac{ds^*}{ds}=(1+\lambda')T+\lambda\kappa N$$

where s and s^* are arc parameter of M_1 and M_2, respectively. Since thedirection of T is orthogonal to the direction of V_1, we obtain

$$\lambda'=-1.$$

From here, it can be easily seen

$$\lambda=(c_1-s)+\varepsilon c_2 \tag{3.2}$$

Furthermore, the dual distance between the points $M_1(s)$ and $M_2(s)$

$$d\left(M_1(s),M_2(s)\right)=\sqrt{\left|\left\langle\lambda T(s),\lambda T(s)\right\rangle\right|}$$
$$=|\lambda_1|+\varepsilon\lambda_1^*\ .$$

Since $\lambda_1=(c_1-s),\quad \lambda_1^*=c_2$, we have

$$d\left(M_1(s),M_2(s)\right)=|c_1-s|+\varepsilon c_2. \tag{3.3}$$

Theorem 3.2: Let (M_2, M_1) be the spacelike – spacelike involut – evolute dual curve couple. Let $\{T,N,B\}$ and $\{V_1,V_2,V_3\}$ be the dual Frenet frames of M_1 and M_2,respectively, Since the dual curvature of M_2 is $P=p+\varepsilon p^*$, we have

$$P^2=\mp\frac{(k_1^2-k_2^2)}{(c_1-s)^2k_1^2}\mp\left[\frac{2k_2\left(k_1^*k_2-k_1k_2^*\right)}{(c_1-s)^2k_1^3}-\frac{2c_2\left(k_1^2-k_2^2\right)}{(c_1-s)^3k_1^2}\right]$$

where the dual curvature of M_1 is $\kappa=k_1+\varepsilon k_1^*$.

Proof: Differentiating (3.1), with respect to s, we get

$$\frac{dM_2}{ds^*}\frac{ds^*}{ds}=\frac{dM_1}{ds}+\frac{d\lambda}{ds}T+\lambda\frac{dT}{ds},$$

$$V_1\frac{ds^*}{ds}=\lambda\kappa N\cdot$$

From here , we can write

$$V_1=N \tag{3.4}$$

and $\frac{ds^*}{ds}=\lambda\kappa$. By differentiating the last equation and using (2.14), we obtain

$$\frac{dV_1}{ds^*}\frac{ds^*}{ds}=\frac{dN}{ds}=-\kappa T+\tau B,$$

$$PV_2=\frac{1}{\lambda\kappa}\left(-\kappa T+\tau B\right)$$

From here,we have

$$P^2=\mp\frac{\left(\kappa^2-\tau^2\right)}{\lambda^2\kappa^2} \tag{3.5}$$

From the fact that $P=p+\varepsilon p^*$, $\lambda=\lambda_1+\varepsilon\lambda_1^*$, $\kappa=k_1+\varepsilon k_1^*$ and $\tau=k_2+\varepsilon k_2^*$ we get

$$P^2=\mp\frac{\left(k_1^2-k_2^2\right)}{\lambda_1^2k_1^2}\mp\left[\frac{2k_2\left(k_1^*k_2-k_1k_2^*\right)}{\lambda_1^2k_1^3}-\frac{2\lambda_1^*\left(k_1^2-k_2^2\right)}{\lambda_1^3k_1^2}\right].$$

From here , by using $\lambda_1=\left(c_1-s\right)$, $\lambda_2=c_2$, we obtain

$$P^2=\mp\frac{\left(k_1^2-k_2^2\right)}{\left(c_1-s\right)^2k_1^2}\mp\left[\frac{2k_2\left(k_1^*k_2-k_1k_2^*\right)}{\left(c_1-s\right)^2k_1^3}-\frac{2c_2\left(k_1^2-k_2^2\right)}{\left(c_1-s\right)^3k_1^2}\right] \tag{3.6}$$

Theorem 3.3: Let (M_2, M_1) be the spacelike – spacelike involute – evolute dual curve couple. Let $\{T,N,B\}$ and $\{V_1,V_2,V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively. The dual torsion $\tau=k_2+\varepsilon k_2^*$ of M_1 and the dual torsion $Q=q+\varepsilon q^*$ of M_2 is the following equation

$$Q=\frac{k_1k_2'-k_1'k_2}{\left|k_1^2-k_2^2\right|k_1\left|c_1-s\right|}+\varepsilon\left[\frac{k_1\left(k_1k_2^{*\prime}-k_1'k_2^*\right)+k_2\left(k_1^*k_1'-k_1^{*\prime}k_1\right)}{\left|k_1^2-k_2^2\right|\left|c_1-s\right|k_1^2}\right].$$

Proof: By differentiating (3.1) three time with respect to s ,we get

$$M_2'=\lambda\kappa N,$$
$$M_2''=-\lambda\kappa^2T+\left(\lambda\kappa'-\kappa\right)N+\lambda\kappa\tau B,$$
$$M_2'''=\left(2\kappa^2-3\lambda\kappa\kappa'\right)T+\left(\lambda\kappa\tau^2-\lambda\kappa^3-2\kappa'+\lambda\kappa''\right)N$$
$$+\left(-2\kappa\tau+2\lambda\kappa'\tau+\lambda\kappa\tau'\right)B.$$

The vectorel product of M_2' and M_2'' is

$$M_2'\wedge M_2''=-\lambda^2\kappa^2\tau T+\lambda^2\kappa^3B=\lambda^2\kappa^2\left(-\tau T+\kappa B\right) \tag{3.7}$$

From here, we obtain

$$\left\|M_2'\wedge M_2''\right\|^2=\left|\lambda\right|^4\left|\kappa\right|^4\left|\tau^2-\kappa^2\right| \tag{3.8}$$

and

$$\det\left(M_2',M_2'',M_2'''\right)=\lambda^3\kappa^3\left(\kappa\tau'-\kappa'\tau\right). \tag{3.9}$$

Substituting by (3.8) and (3.9) values into $Q=\dfrac{\det\left(M_2',M_2'',M_2'''\right)}{\left\|M_2'\wedge M_2''\right\|^2}$, we get

$$Q=\frac{\left(\kappa\tau'-\kappa'\tau\right)}{\left|\lambda\right|\kappa\left|\tau^2-\kappa^2\right|} \tag{3.10}$$

and substituting by values $Q=q+\varepsilon q^*$, $\lambda=\lambda_1+\varepsilon\lambda_1^*$, $\kappa=k_1+\varepsilon k_1^*$ and $\tau=k_2+\varepsilon k_2^*$ into the last equation, we have

$$Q=\frac{k_1k_2'-k_1'k_2}{\left|\lambda_1\right|k_1\left|k_2^2-k_1^2\right|}+\varepsilon\left[\frac{k_1\left(k_1k_2^{*\prime}-k_1'k_2^*\right)+k_2\left(k_1^*k_1'-k_1^{*\prime}k_1\right)}{\left|\lambda_1\right|k_1^2\left|k_2^2-k_1^2\right|}\right]$$

By the fact that $\lambda_1=\left(c_1-s\right)$, we get

$$Q=\frac{k_1k_2'-k_1'k_2}{|c_1-s|k_1|k_2^2-k_1^2|}+\varepsilon\left[\frac{k_1\left(k_1k_2^{*\prime}-k_1'k_2^*\right)+k_2\left(k_1^*k_1'-k_1^{*\prime}k_1\right)}{|c_1-s|k_1^2|k_2^2-k_1^2|}\right]. \quad (3.11)$$

Theorem 3.4: Let (M_2, M_1) be the spacelike – spacelike involute – evolute dual curve couple. Let $\{T,N,B\}$ and $\{V_1,V_2,V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively and $\Phi=\varphi+\varepsilon\varphi^*$ be the Lorentzian dual spacelike angle between binormal vector B and W For (M_2, M_1) dual curve couple, the the following equations is obtained:

1)If W spacelike ,

$$\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\begin{bmatrix}0&1&0\\\sinh\Phi&0&-\cosh\Phi\\-\cosh\Phi&0&\sinh\Phi\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}$$

leaves the real and dual components

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0&1&0\\\sinh\varphi&0&-\cosh\varphi\\-\cosh\varphi&0&\sinh\varphi\end{bmatrix}\begin{bmatrix}t\\n\\b\end{bmatrix},$$

$$\begin{bmatrix}v^*_1\\v^*_2\\v^*_3\end{bmatrix}=\varphi^*\begin{bmatrix}0&1&0\\\cosh\varphi&0&-\sinh\varphi\\-\sinh\varphi&0&\cosh\varphi\end{bmatrix}\begin{bmatrix}t\\n\\b\end{bmatrix}+\begin{bmatrix}0&1&0\\\sinh\varphi&0&-\cosh\varphi\\-\cosh\varphi&0&\sinh\varphi\end{bmatrix}\begin{bmatrix}t^*\\n^*\\b^*\end{bmatrix}$$

2)If W timelike

$$\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\begin{bmatrix}0&0&0\\-\cosh\Phi&0&\sinh\Phi\\-\sinh\Phi&0&\cosh\Phi\end{bmatrix}\begin{bmatrix}T\\N\\N\end{bmatrix}$$

leaves the real and dual components

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0&0&0\\-\cosh\varphi&0&\sinh\varphi\\-\sinh\varphi&0&\cosh\varphi\end{bmatrix}\begin{bmatrix}t\\n\\b\end{bmatrix},$$

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$$\begin{bmatrix} v^*_1 \\ v^*_2 \\ v^*_3 \end{bmatrix} = \varphi^* \begin{bmatrix} 0 & 1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ -\cosh\varphi & 0 & \sinh\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}$$

Proof: 1) From the equation (3.4) and (3.8),we have,

$$\left\| M_2' \wedge M_2'' \right\| = \lambda^2 \kappa^2 \left\| W \right\|. \tag{3.12}$$

By using (3.7) and (3.12) and from the fact that $V_3 = \dfrac{M_2' \wedge M_2''}{\left\| M_2' \wedge M_2'' \right\|}$ we obtain

$$V_3 = -\frac{\tau}{\|W\|} T + \frac{\kappa}{\|W\|} B,$$

substituning (2.16) into the last equation, we obtain

$$V_3 = -\cosh\Phi T + \sinh\Phi B. \tag{3.13}$$

Since $V_2 = V_3 \wedge V_1$, it can beeasily seen that

$$V_2 = \sinh\Phi T - \cosh\Phi B. \tag{3.14}$$

Considering (3.4), (3.13) and (3.14) according to dual components, the following equations are obtained:

$$\begin{cases} V_1 = n + \varepsilon n^* \\ V_2 = (\sinh\varphi t - \cosh\varphi b) + \varepsilon\left[(\sinh\varphi t^* - \cosh\varphi b^*) + \varphi^* (\cosh\varphi t - \sinh\varphi b) \right] \\ V_3 = (-\cosh\varphi t + \sinh\varphi b) + \varepsilon\left[(-\cosh\varphi t^* + \sinh\varphi b^*) + \varphi^* (-\sinh\varphi t + \cosh\varphi b) \right] \end{cases} \tag{3.15}$$

written (3.15) in matrix form, the prof is completed.

2) From the equation (3.4) and (3.10), we have,

$$V_3 = -\frac{\tau}{\|W\|} T + \frac{\kappa}{\|W\|} B.$$

Substituning (2.18) into the last equation, we obtain

$$V_3 = -\sinh\Phi T + \cosh\Phi B, \tag{3.16}$$

$$V_2 = -\cosh\Phi T + \sinh\Phi B. \tag{3.17}$$

Considering (3.4), (3.16) and (3.17) according to dual components, the following equations are obtained:

$$\begin{cases} V_1 = n + \varepsilon n^* \\ V_2 = (-\cosh\varphi t + \sinh\varphi b) + \varepsilon\left[(-\cosh\varphi t^* + \sinh\varphi b^*) + \varphi^*(-\sinh\varphi t + \cosh\varphi b)\right] \\ V_3 = (-\sinh\varphi t + \cosh\varphi b) + \varepsilon\left[(-\sinh\varphi t^* + \cosh\varphi b^*) + \varphi^*(-\cosh\varphi t + \sinh\varphi b)\right] \end{cases} \tag{3.18}$$

Theorem 3.5: Let (M_2, M_1) be the spacelike – spacelike involute – evolute dual curve couple. $W = w + \varepsilon w^*$ and $\overline{W} = \overline{w} + \varepsilon \overline{w}^*$ be the dual Frenet instantaneous rotation vectors of M_1 and M_2 respectively. Thus,

1)If $\overline{W}$ spacelike,

$$\overline{W} = \frac{1}{|\lambda|\kappa}(\Phi' N - W).$$

2)If $\overline{W}$ timelike,

$$\overline{W} = \frac{1}{|\lambda|\kappa}(\Phi' N + W).$$

Proof: 1) From (2.10), we can write $\overline{W} = -QV_1 + PV_3$. Using the (3.4), (3.5), (3.10) and (3.13) the equations, we have

$$\overline{W} = -\frac{\kappa\tau' - \kappa'\tau}{|\lambda|\kappa|\kappa^2 - \tau^2|} N + \frac{\sqrt{|\tau^2 - \kappa^2|}}{|\lambda|\kappa}(-\cosh\Phi T + \sinh\Phi B)$$

Substituning (2.16) into the last equation, we obtain

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|} N - W\right)$$

and then, we get

$$\overline{W} = \frac{1}{|\lambda|\kappa}(\Phi' N - W). \tag{3.19}$$

Considering (3.19) according to dual components and substituting $\lambda_1 = (c_1 - s)$ into (3.19), we leaves the real and dual components

$$\begin{cases} \overline{w} = \dfrac{\varphi' n - w}{|c_1 - s| k_1}, \\ \overline{w^*} = \dfrac{\varphi' n^* + \varphi^{*\prime} n - w^*}{|c_1 - s| k_1} - \dfrac{k_1^* (\varphi' n - w)}{|c_1 - s| k_1^2}. \end{cases} \tag{3.20}$$

2) From (2.15), we can write $\overline{W} = QV_1 - PV_3$. Using the (3.4), (3.5), (3.10) and (3.16) the equations, we have

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|} N - \sqrt{|\tau^2 - \kappa^2|}\left(-\sinh\Phi T + \cosh\Phi B\right)\right).$$

Substituning (2.18) into the last equation, we obtain

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|} N + W\right)$$

and then, we get

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\Phi' N + W\right). \tag{3.21}$$

Considering (3.21) according to dual components and substituting $\lambda_1 = (c_1 - s)$ into (3.21), we leaves the real and dual components

$$\begin{cases} \overline{w} = \dfrac{\varphi' n + w}{|c_1 - s| k_1}, \\ \overline{w^*} = \dfrac{\varphi' n^* + \varphi^{*\prime} n + w^*}{|c_1 - s| k_1} + \dfrac{k_1^* (\varphi' n + w)}{|c_1 - s| k_1^2} \end{cases} \tag{3.22}$$

Theorem 3.6: Let (M_2, M_1) be the spacelike – spacelike involute – evolute dual curve couple. $C = c + \varepsilon c^*$ and $\overline{C} = \overline{c} + \varepsilon \overline{c}^*$ be unit dual vector of W and $\overline{W}$, respectively. Thus,

1)If W spacelike, $\overline{C} = \dfrac{\Phi'}{\sqrt{|\tau^2 - \kappa^2 + \Phi'^2|}} N - \dfrac{\sqrt{|\tau^2 - \kappa^2|}}{\sqrt{|\tau^2 - \kappa^2 + \Phi'^2|}} C,$

2)If W timelike, $\overline{C}=\dfrac{-\Phi'}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}N+\dfrac{\sqrt{\left|\tau^2-\kappa^2\right|}}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}C$.

Proof: 1) From the fact that the unit dual vector of $\overline{W}$ is $\overline{C}=\dfrac{\overline{W}}{\left\|\overline{W}\right\|}$ we obtain

$$\overline{C}=\frac{\Phi'}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}N-\frac{\sqrt{\left|\tau^2-\kappa^2\right|}}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}C. \tag{3.23}$$

(3.23) leaves the real and dual components

$$\begin{cases}\overline{c}=\dfrac{\varphi'}{\sqrt{\left|k_2^2-k_1^2+\varphi'^2\right|}}n-\dfrac{\sqrt{\left|k_2^2-k_1^2\right|}}{\sqrt{\left|k_2^2-k_1^2+\varphi'^2\right|}}c,\\[2ex] \overline{c^*}=\dfrac{\varphi'n^*+\varphi^{*\prime}n}{\sqrt{\left|k_2^2-k_1^2+\varphi'^2\right|}}-\dfrac{\sqrt{\left|k_2^2-k_1^2\right|}c^*}{\sqrt{\left|k_2^2-k_1^2+\varphi'^2\right|}}\end{cases} \tag{3.24}$$

2) Substituning (3.21) into the equation (3.23) we obtain

$$\overline{C}=\frac{-\Phi'}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}N+\frac{\sqrt{\left|\tau^2-\kappa^2\right|}}{\sqrt{\left|\tau^2-\kappa^2+\Phi'^2\right|}}C\ . \tag{3.25}$$

(3.25) leaves the real and dual components

$$\begin{cases}\overline{c}=\dfrac{\varphi'}{\sqrt{\left|k_2^2-k_1^2+(\varphi')^2\right|}}n+\dfrac{\sqrt{\left|k_2^2-k_1^2\right|}}{\sqrt{\left|k_2^2-k_1^2+(\varphi')^2\right|}}c,\\[2ex] \overline{c^*}=\dfrac{\varphi'n^*+(\varphi^*)'n}{\sqrt{\left|k_2^2-k_1^2+(\varphi')^2\right|}}+\dfrac{\sqrt{\left|k_2^2-k_1^2\right|}c^*}{\sqrt{\left|k_2^2-k_1^2+(\varphi')^2\right|}}.\end{cases} \tag{3.27}$$

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