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Transient Thermoelastic Behavior of Semi-infinite Cylinder by Using Marchi-Zgrablich and Fourier Transform Technique

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Abstract. The present work deals with transient thermoelastic problem of a semi infinite hollow cylinder to determine the temperature, displacement and thermal stresses with the stated conditions. The transient heat conduction equation with stated conditions is solved by using Marchi-Zgrablich transform and Fourier Sine transform simultaneously and the results for temperature distribution, heat flux distribution, displacement and thermal stress functions are obtained in terms of infinite series of Bessel's function. These results solved for special case by using Math-Cad 2007 software and presented graphically by using Origin software.

Keywords: Semi-infinite hollow cylinder, Transient heat conduction, Thermal stresses, Marchi-Zgrablich and Fourier Sine transform.

1 Introduction

The Thermoelastic problems are one of the most frequently encountered problems by scientists. The wide varieties of problems that are covered under conduction also make it one of the most researched and thought about problems in the field of engineering and technology. This kind of problems can be solved by various methods. These problems consist of determination of unknown temperature, heat flux, displacement and thermal stress functions of solids when the conditions of temperature and displacement and stress are known at the some points of the solid under consideration.

Sirakowski and Sun [1] studied the direct problems of finite length hollow cylinder and determined an exact solution. Grysa and Cialkowski [2] and Grysa and Kozlowski [3] discussed one dimensional transient thermoelastic problems derived the heating temperature and the heat flux on the surface of an isotropic infinite slab. Further Deshmukh and Wankhede [4] studied an axisymmetric inverse steady state problem of thermoelastic deformation to determine the temperature, displacement and stress functions on the outer curved surface of finite length hollow cylinder. Recently

Walde and Khobragade [5] obtained the solution for transient thermoelastic problem of finite length hollow cylinder.

In this work we consider semi-infinite hollow cylinder. The transient heat conduction equation is solved by using finite Marchi-Zgrablich integral transform as in Marchi and Zgrablich [6] and Fourier Sine transform as defined in Sneddon [7] simultaneously and the results for temperature distribution, unknown heat flux, displacement and thermal stresses are obtained in terms of infinite series of Bessel's function and it is solved for special case by using Mathcad-2007 software and illustrated graphically by using Origin software.

2 The Mathematical Model

Consider a hollow cylinder occupying space D as define $D:\left\{(x,y,z)\middle| a\le r=\sqrt{x^2+y^2}\le b, 0\le z\le \infty\right\}$ and as shown in figure 1 below.

The thermoelastic displacement function is governed by the Poisson's equation as in [5]

$$\nabla^2\phi=\frac{(1+\nu)}{(1-\nu)}a_t T \tag{1}$$

with $\phi=0$ at $r=a$ and $r=b$ (2)

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$, ν and a_t are the Poisson's ratio and linear coefficient of thermal expansion of the material of the cylinder and T is the temperature of the cylinder satisfying the differential equation as in Ozisik [8]

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial T}{\partial t} \tag{3}$$

subject to the initial condition

$$T(r,z,t)\big|_{t=0}=0 \tag{4}$$

and the boundary conditions

$$\left[T(r,z,t)+\frac{\partial T(r,z,t)}{\partial r}\right]_{r=a}=0 \tag{5}$$

$$\left[T(r,z,t)+\frac{\partial T(r,z,t)}{\partial r}\right]_{r=b}=0 \tag{6}$$

$$\left[T(r,z,t)+\frac{\partial T(r,z,t)}{\partial z}\right]_{z=0}=0 \tag{7}$$

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$$\left[T(r,z,t)+\frac{\partial T(r,z,t)}{\partial z}\right]_{z=\infty}=0 \tag{8}$$

where k is the thermal diffusivity of the material of the cylinder.

The radial and axial displacements U and W satisfying the uncoupled thermoelastic equations are

$$\nabla^2 U-\frac{U}{r^2}+(1+2\nu)^{-1}\frac{\partial e}{\partial r}=2\frac{(1+\nu)}{(1-2\nu)}a_t\frac{\partial T}{\partial r} \tag{9}$$

$$\nabla^2 W+(1+2\nu)^{-1}\frac{\partial e}{\partial z}=2\frac{(1+\nu)}{(1-2\nu)}a_t\frac{\partial T}{\partial z} \tag{10}$$

where $e=\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{\partial W}{\partial z}$ is the volume dilation and

$$U=\frac{\partial \phi}{\partial r} \tag{11}$$

$$W=\frac{\partial \phi}{\partial z} \tag{12}$$

The stress functions are given by

$$\tau_{rz}(a,z,t)=0, \tau_{rz}(b,z,t)=0, \tau_{rz}(r,z,0)=0, \tag{13}$$

and

$$\sigma_r(a,z,t)=p_1,\ \sigma_r(b,z,t)=-p_0,\ \ \sigma_z(r,0,t)=0, \tag{14}$$

where p_1 and p_0 are the surface pressures assumed to be uniform over the boundaries of the cylinder. The boundary conditions for the stress functions (13) and (14) are expressed in terms of the displacement components by the following relations:

$$\sigma_r=(\lambda+2G)\frac{\partial U}{\partial r}+\lambda\left[\frac{U}{r}+\frac{\partial W}{\partial z}\right] \tag{15}$$

$$\sigma_z=(\lambda+2G)\frac{\partial W}{\partial z}+\lambda\left[\frac{\partial U}{\partial r}+\frac{U}{r}\right] \tag{16}$$

$$\sigma_\theta=(\lambda+2G)\frac{U}{r}+\lambda\left[\frac{\partial U}{\partial r}+\frac{\partial W}{\partial z}\right] \tag{17}$$

$$\tau_{rz}=G\left[\frac{\partial W}{\partial r}+\frac{\partial U}{\partial z}\right] \tag{18}$$

where $\lambda=\frac{2G\nu}{1-2\nu}$ is the Lame's constant, G is the shear modulus and U and W are the displacement components. The equations (1) to (18) constitute the mathematical formulation of the problem under consideration.

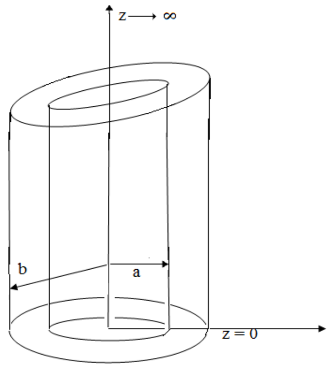

Figure 1: Geometry of Semi-infinite Cylinder.

3 The Solution

3.1 Determination Temperature Function $T(r,z,t)$

Applying finite Marchi-Zgrablich transform [6] to the equation (3), one obtains

$$-\mu_m^2\bar{T}+\frac{d^2\bar{T}}{dz^2}=\frac{1}{k}\frac{d\bar{T}}{dt} \tag{19}$$

Further applying Fourier Sine transform [7] to the equation (19), one obtains

$$\frac{d\bar{T}^*}{dt}+kp^2\bar{T}^*=0 \tag{20}$$

where $p^2=\left(\lambda_n^2+\mu_m^2\right)$

Equation (20) is the first order linear differential equation, whose solution is given by

$$\bar{T}^*=C_1.e^{-kp^2t} \tag{21}$$

where C_1 is constant.

Applying inversion of Fourier Sine transform [7] and Marchi-Zgrablich transform [6] to the equation (21), we obtain

$$T=\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)}{\mu_m}e^{-kp^2t} \tag{22}$$

where $S_p(k_1,k_2,\mu_m x)=J_p(\mu_m x)\left\{G_p(k_1,\mu_m a)+G_p(k_2,\mu_m b)\right\}$

$$-G_p(\mu_m x)\left\{J_p(k_1,\mu_m a)+J_p(k_2,\mu_m b)\right\}$$

3.2 Determination of Thermoelastic Displacement

Substituting the value of $T(r,z,t)$ from equation (22) in equation (1), one obtains the thermoelastic Displacement $\phi(r,z,t)$ function as

$$\phi(r,z,t)=\frac{(1+\nu)}{(1-\nu)}a_t\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{4\mu_m}e^{-kp^2 t} \tag{23}$$

Using equation (23) in equation (11) and (12), one obtains the radial and axial displacement U and W as

$$U=\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t}$$
$$\times\left[r^2\mu_m S_0^{'}(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)\right] \tag{24}$$

$$W=\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n\cos\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t} \tag{25}$$

3.3 Determination of Stress Functions

Using equations (24) and (25) in equations (15) to (18), the stress functions are obtained as

$$\sigma_r=\left(\lambda+2G\right)\left\{\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1}{\mu_m}e^{-kp^2 t}\right]\left(4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\right.\right.$$
$$+r^4\mu_m S_0(k_1,k_2,\mu_m r)S_0^{''}(k_1,k_2,\mu_m r)+r^4\mu_m S_0^2(k_1,k_2,\mu_m r)$$
$$\left.\left.+6r^2\mu_m S_0(k_1,k_2,\mu_m r)S_0^2(k_1,k_2,\mu_m r)+4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\right)\right\}$$
$$+\left(\lambda\right)\left\{\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1 r S_0(k_1,k_2,\mu_m r)}{\mu_m}e^{-kp^2 t}\right.\right.$$
$$\left.\times\left(r^2\mu_m S_0^{'}(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)\right)\right]$$
$$\left.-\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n^2\sin\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t}\right\} \tag{26}$$

$$\sigma_z=\left(\lambda+2G\right)\left[-\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n^2\sin\left(\lambda_n z\right)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t}\right]$$
$$+\left(\lambda\right)\left\{\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin\left(\lambda_n z\right)\frac{C_1}{\mu_m}e^{-kp^2 t}\right]\left(4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\right.\right.$$

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$$+6r^2\mu_m S_0(k_1,k_2,\mu_m r)S_0^2(k_1,k_2,\mu_m r)+4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\Big)$$

$$+\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin(\lambda_n z)\frac{C_1 r S_0(k_1,k_2,\mu_m r)}{\mu_m}e^{-kp^2 t}\right.$$

$$\left.\times\left(r^2\mu_m S_0^{'}(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)\right)\right]\Bigg\} \tag{27}$$

$$\sigma_\theta=(\lambda+2G)\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin(\lambda_n z)\frac{C_1 r S_0(k_1,k_2,\mu_m r)}{\mu_m}e^{-kp^2 t}\right.$$

$$\left.\times\left(r^2\mu_m S_0^{'}(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)\right)\right]$$

$$+(\lambda)\left\{\left[\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\sin(\lambda_n z)\frac{C_1}{\mu_m}e^{-kp^2 t}\right]\Big(4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\right.$$

$$+r^4\mu_m S_0(k_1,k_2,\mu_m r)S_0^{''}(k_1,k_2,\mu_m r)+r^4\mu_m S_0^2(k_1,k_2,\mu_m r)$$

$$+6r^2\mu_m S_0(k_1,k_2,\mu_m r)S_0^2(k_1,k_2,\mu_m r)+4r^3\mu_m S_0(k_1,k_2,\mu_m r)S_0^{'}(k_1,k_2,\mu_m r)\Big)$$

$$\left.-\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n^2\sin(\lambda_n z)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t}\right\} \tag{28}$$

$$\tau_{rz}=G\left\{\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n\cos(\lambda_n z)\frac{C_1\left(2rS_0(k_1,k_2,\mu_m r)+\mu_m S_0^{'}(k_1,k_2,\mu_m r)r^2\right)}{\mu_m}e^{-kp^2 t}\right.$$

$$+\frac{(1+\nu)}{(1-\nu)}\frac{a_t}{4}\sum_{m=1}^{\infty}\lambda_n\cos(\lambda_n z)\frac{C_1 S_0(k_1,k_2,\mu_m r)r^2}{\mu_m}e^{-kp^2 t}$$

$$\left.\times\left[r^2\mu_m S_0^{'}(k_1,k_2,\mu_m r)+2rS_0(k_1,k_2,\mu_m r)\right]\right\} \tag{29}$$

4 Special Case and Numerical Results

For the special case semi infinite cylinder is made up of steel material with inner radius $a=1$ unit and outer $b=2$ units. A numerical results for equations (22), (23), (26)-(29) are obtained by mathematical software Mathcad and are depicted graphically as shown in following figure 2-7.

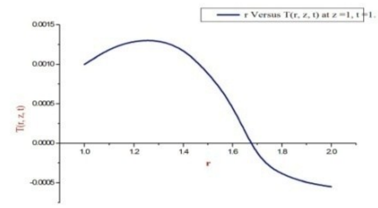

Figure2. Temperature distribution

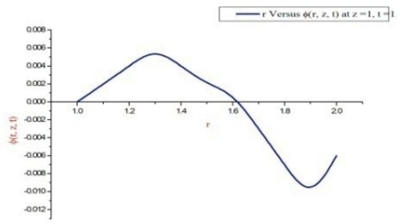

Figure3. Displacement distribution

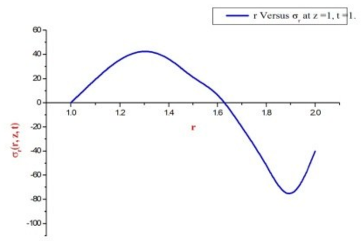

Figure4. Radial stress distribution

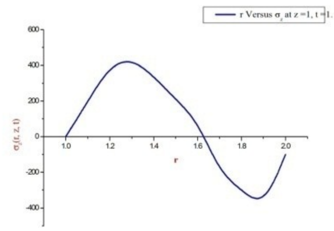

Figure5. Normal stress distribution

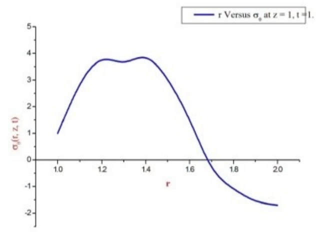

Figure6. Tangential stress distribution

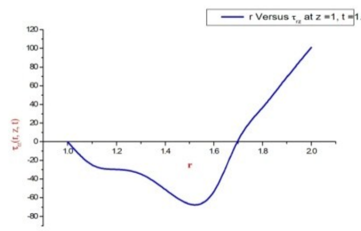

Figure7. τ_{rz} distribution

5 Conclusion

In this paper, the temperature distribution, displacement and stress functions have been investigated with the help of integral transform techniques. The expressions are represented graphically.

Figure 2 show the temperature variation in the cylinder along radial directions.

Figure 3 show the displacement develops in the cylinder along radial directions.

Figure 4-7 show the stresses occur in the semi infinite cylinder along the radial directions.

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The results that are obtained can be applied to the design of useful structures and can be use in study of oil transport.

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