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THERMAL STRESSES IN A THIN ANNULAR DISC

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Abstract. This paper is concern with direct steady-state thermoelastic problem to determine the temperature, displacement and stress functions of a thin annular disc of thickness $2h$, occupying the space $D: a \le r \le b, -h \le z \le h$ with the stated boundary conditions. The finite Hankel transform techniques have been used.

Keywords: Annular disc, steady-state problem, direct thermoelastic problem Hankel Transform.

1 Introduction

Grysa and Cialkowski [1], Grysa and Koalowski [2] have studied one-dimensional transient thermoelastic Problems derived the heating temperature and heat flux on the surface of an isotropic infinite slab. Khobragade and Wankhede [3] have studied the inverse steady-state thermoelastic problem to determine the temperature, displacement and stress function on the outer curved surface of a thin annular disc occupying the space $D: a \le r \le b, 0 \le z \le h$. The direct problems of thin circular plate have been studied by Roychaudhari [5] and Wankhede [7].

In the present problem an attempt is made to study the direct steady-state thermoelastic problem to detrermine the temperature , displacement and stress functions of a thin annular disc of thickness $2h$, occupying the space $D: a \le r \le b, -h \le z \le h$ with the known boundary conditions. The finite Hankel transform techniques have been used to find the solution of the problem.

2 Result Required

Consider If $f(x)$ satisfies the Dirichlet's conditions in the interval $(0, a)$ then its finite Hankel transform in that range is defined to be,

$$\overline{f}_{\mu}(\xi_i) = \int_0^a x f(x) J_{\mu}(x\xi_i) dx \tag{2.1}$$

where ξ_i is the root of the transcendental equation

$$J_{\mu}(a\xi_i) = 0 \tag{2.2}$$

then, at any point of $(0, a)$ at which the function $f(x)$ is continuous,

$$f(x) = \frac{2}{a^2} \sum_i \overline{f}_{\mu}(\xi_i) \frac{J_{\mu}(x\xi_i)}{[J'_{\mu}(a\xi_i)]^2} \tag{2.3}$$

where the sum is taken over all the positive roots of the equation (2.2)

properties of Hankel transform in (2.1)

If $f(x)$ satisfies the Dirichlet's conditions in the interval $(0, a)$ then,

1. Finite Hankel transform of $\frac{\partial f}{\partial x}$ i. e,

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$$H\left[\frac{\partial f}{\partial x}\right] = \int_0^a \frac{\partial f}{\partial x} xJ_\mu(x\xi_i)dx$$

$$= \frac{\xi_i}{2\mu}\left[(\mu-1)H_{\mu+1}\{f(x)\} - (\mu+1)H_{\mu-1}\{f(x)\}\right]$$

2. $$H_\mu\left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}\right] = \frac{\xi_i}{2}\left[-H_{\mu-1}\{\frac{\partial f}{\partial x}\} + H_{\mu+1}\{\frac{\partial f}{\partial x}\}\right]$$

If $f(x)$ satisfies the Dirichlet's conditions in the range $b \le x \le a$ and if its finite Hankel transform in that range is defined to be

$$\begin{aligned} H[f(x)] &= \overline{f}_\mu(\xi_i) \\ &= \int_b^a f(x)\left[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)\right]dx \end{aligned} \tag{2.4}$$

where ξ_i is the root of the transcendental equation

$$\left[J_\mu(\xi_i b)G_\mu(\xi_i a) - J_\mu(\xi_i a)G_\mu(\xi_i b)\right] = 0 \tag{2.5}$$

then at which the function is continuous

$$f(x) = \sum_i \frac{2\xi_i^2 J_\mu^2(b\xi_i)\overline{f}_\mu(\xi_i)}{J_\mu^2(a\xi_i) - J_\mu^2(b\xi_i)}\left[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)\right] \tag{2.6}$$

Now property of hankel transform defined in (2.4)

$$\begin{aligned} \int_a^b \left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}\right] & \left[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)\right]dx \\ &= -\xi_i^2 \overline{f}_\mu(\xi_i) + a\left[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)\right]_{x=a} \\ &\quad + b\left[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)\right]_{x=b} \\ &= -\xi_i^2 \overline{f}_\mu(\xi_i) \end{aligned}$$

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3 Statement of the Problem

Consider a thin circular plate of thickness $2h$ occupying the space $D: a \le r \le b, -h \le z \le h$. The differential equation governing the displacement function $U(r,z)$ as in Nowaki [4] is,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \tag{3.1}$$

$$\text{with} U_r = 0 \,\text{at}\, r = a \,\&\, r = b \tag{3.2}$$

ν and a_t are the poisson's ratio and the linear coefficient of thermal expansion of the material of plate and T is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3.3}$$

subject to the boundary conditions

$$\left[T(r,z) + \frac{\partial T(r,z)}{\partial z}\right]_{z=h} = f(r) \tag{3.4}$$

$$\left[T(r,z) + \frac{\partial T(r,z)}{\partial z}\right]_{z=-h} = 0 \tag{3.5}$$

$$T(a,z) = 0 \tag{3.6}$$

$$T(b,z) = 0 \tag{3.7}$$

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r} \tag{3.8}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \tag{3.9}$$

where μ is the lames constant while each of the stress functions σ_{rz}, σ_{zz}, $\sigma_{\theta z}$ are zero within the plate in the plane state of stress.

The equations (3.1) to (3.9) constitute the mathematical formulation of the problem under consideration.

4 Solution of the Problem

Applying finite Hankel transform stated in [6] to (3.3) to (3.5) and using (3.6) and (3.7) one obtains,

$$\frac{d^2 \overline{T}}{dz^2} - \lambda_n^2 \overline{T} = 0 \tag{4.1}$$

$$\left[\overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz} \right]_{z=h} = \overline{f}(\lambda_n) \tag{4.2}$$

$$\left[\overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz} \right]_{z=-h} = 0 \tag{4.3}$$

where $\overline{T}$ denotes the finite Hankel transform of T and λ_n is the Hankel transform parameter. The equation (4.1) is a second order differential equation whose solution is given by

$$\overline{T}(\lambda_n, z) = Ae^{\lambda_n z} + Be^{-\lambda_n z} \tag{4.4}$$

where A and B are constants.

Using (4.2) and (4.3) in (4.4) we obtain the values of A and B substituting this values in (4.4) and then inversion of finite Hankel transform lead to

$$\begin{aligned} T(r,z) \quad &= 2\sum_{n=1}^{\infty}\lambda_n^2\overline{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b)-J_0^2(\lambda_n a)}\right] \\ &\times\left[\frac{\sinh(\lambda_n(z+h))+\lambda_n\cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h)+2\lambda_n\cosh(2\lambda_n h)}\right] \\ &\times\left[J_0(r\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_0(r\lambda_n)\right] \end{aligned} \tag{4.5}$$

where $\overline{f}(\lambda_n)=\int_a^b rf(r)\left[J_0(r\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_0(r\lambda_n)\right]dr$ and λ_n are the root of the transcendental equation

$$\left[J_0(a\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_0(a\lambda_n)\right]=0$$

Equation (4.5) is the desired solution of the given problem.

5 Determination of Thermoelastic Displacement

Substituting this values of $T(r,z)$ from (4.5) in (3.1) one obtains the thermoelastic displacement function $U(r,z)$ as

$$\begin{aligned} U(r,z) \quad &= -2(1+\nu)a_t\sum_{n=1}^{\infty}\overline{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b)-J_0^2(\lambda_n a)}\right] \\ &\times\left[\frac{\sinh(\lambda_n(z+h))+\lambda_n\cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h)+2\lambda_n\cosh(2\lambda_n h)}\right] \\ &\times\left[J_0(r\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_0(r\lambda_n)\right] \end{aligned} \tag{5.1}$$

6 Determination of Stress Function

For Using (5.1) in (3.8) and (3.9) the stress functions are obtained as

$$\begin{aligned}\sigma_{rr} &= \frac{4\mu(1+\nu)a_t}{r}\sum_{n=1}^{\infty}\lambda_n \overline{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b)-J_0^2(\lambda_n a)}\right] \\ &\times\left[\frac{\sinh(\lambda_n(z+h))+\lambda_n\cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h)+2\lambda_n\cosh(2\lambda_n h)}\right] \\ &\times\left[J_1(r\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_1(r\lambda_n)\right]\end{aligned} \tag{6.1}$$

$$\begin{aligned}\sigma_{\theta\theta} &= 4\mu(1+\nu)a_t\sum_{n=1}^{\infty}\lambda_n^2 \overline{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b)-J_0^2(\lambda_n a)}\right] \\ &\times\left[\frac{\sinh(\lambda_n(z+h))+\lambda_n\cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h)+2\lambda_n\cosh(2\lambda_n h)}\right] \\ &\times\left[J_1'(r\lambda_n)G_0(b\lambda_n)-J_0(b\lambda_n)G_1'(r\lambda_n)\right]\end{aligned} \tag{6.2}$$

7 Conclusion

In this paper, we discussed completely the direct steady state problem of thermoelastic deformation of a thin annular disc of thickness $2h$, with homogeneous boundary condition of the third kind is maintained on the lower plane surface while on upper plane surface, it is maintained at $f(r)$, which is known function of r and the temperature is maintained at zero on curved surfaces of the annular disc.

The finite Hankel transform technique is used to obtain the numerical results. The temperature , displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

References

1. Grysa, K. and Cialkowski, M. J.: On A Certain Inverse Problem of Temperature and Thermal Stress Fields, Acta Mechanica, vol. 36, pp. 169-185 (1980)
2. Grysa, K. and Kozlowski, Z.: One Dimensional Problem of Temperature and Heat Flux Determination at the Surfaces of A Thermoelastic Slab Part-I, The Analytical Solutions, Nuclear Engineering and Design, vol. 74(1), pp. 1-14 (1983)
3. khobragade N. W. and Wankhede P. C.,: An inverse steady-state thermoelastic problem of thin annular disc, Bull. of pure and appl.sci, vol. 19E(No. 1), pp. 551-555 (2001).
4. Nowaki W.,: The state of stress in a thick circular plate due to temperature field, Bull. Acad. Polo. Sci. Ser. Tech. 5, pp. 227(1957).
5. Roychaudhary S. K.,: A note on quasi-static stresses in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face, Bull. Acad. Sci. Ser. Tech. 20, pp. 20-21(1972).
6. Sneddon, I. N.,: The use of integral transforms, McGraw-Hill Book Co., chap. 3, pp. 174-180 (1974)
7. Wankhede P. C.,: On quasi-static thermal stresses in a circular plate. Indian J. of pure and Appl. Maths, 13, pp.1273-1277 (1982).