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Special Curves of 4D Galilean Space

Mehmet Bektaş[1] Mahmut Ergüt[1] Alper Osman Öğrenmiş[1,2]

[1] Fırat University, Faculty of Science, Department of Mathematics,
23119 Elazığ, Türkiye

mbektas@firat.edu.tr, mergut@firat.edu.tr, ogrenmisalper@gmail.com

Abstract. Special curves and their characterizations are one of the main area of mathematicians and physicians.
In the present paper we define Mannheim curves for 4-dimensional Galilean space and investigate some characterization of it.

Keywords: Galilean Space, Mannheim Curves, Frenet Formula

1 Introduction

In classical differential geometry, there are many works related with Bertrand and Mannheim curves [1-5]. We can see in most studies, properties of Bertrand and Mannheim curves which asserts the existence of a linear relation between curvatures.
In recent years, mathematicians have begun to investigate curves and surfaces in Galilean space [6-10].
A space curve in Euclidean 3-space is called 'mannheim curve' if and only if for some β constants, it satisfies the following relation

$$k_1 = \beta(k_1^2 + k_2^2)$$

where k_1 and k_2 are curvature and torsion, respectively.
Our work is organized as follows: In section 2, some basic properties of Galilean space are given which will be used in the later sections. In section 3, we give some properties of Mannheim curves in 4D Galilean space.

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[2] The corresponding author.

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2 Preliminaries

The Galilean space is a 3D complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points).
The study of mechanics of plane-parallel motions reduces to the study of a geometry of 3D space with coordinates $\{x, y, t\}$ are given by the motion formula [11]. This geometry is called 3D Galilean geometry. Differential geometry of the Galilean space G_3 has been largely developed in [7]. In [11], is explained that 4D Galilean geometry, which studies all properties invariant under motions of objects in space, is even more complex.
In addition it is stated that this geometry can be described more precisely as the study of those properties of 4D space with coordinates which are invariant under the general Galilean transformations as follows:

$$
\begin{aligned}
x' &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x + (\sin\beta\cos\alpha - \cos\gamma\cos\beta\sin\alpha)y \\
&\quad + (\sin\gamma\sin\alpha)z + (v\cos\delta_1)t + a \\
y' &= -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)x + (-\sin\beta\sin\alpha + \cos\gamma\cos\beta\cos\alpha)y \\
&\quad + (\sin\gamma\cos\alpha)z + (v\cos\delta_2)t + b \\
z' &= (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y + (\cos\gamma)z + (v\cos\delta_3)t + c \\
t' &= t + d
\end{aligned}
$$

with $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1.$
Some fundamental properties of curves in 4D Galilean space, is given for the purpose of the requirements in the next section
A curve in G_4 $(I \subset \mathbb{R} \to G_4)$ is given as follows

$$\alpha(t) = (x(t), y(t), z(t), w(t)),$$

where $x(t), y(t), z(t), w(t) \in C^4$ (smooth functions) and $t \in I$. Let α be a curve in G_4, which is parameterized by arclength $t = s$, and given in the following coordinate form

$$\alpha(s) = (s, y(s), z(s), w(s)).$$

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In affine coordinates the Galilean scalar product between two points $P_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}), \quad i = 1,2$ is defined by

$$g(P_1, P_2) = \begin{cases} |x_{21} - x_{11}| & \text{if } x_{11} \neq x_{21}, \\ \sqrt{(x_{22} - x_{12})^2 + (x_{23} - x_{13})^2 + (x_{24} - x_{14})^2} & \text{if } x_{11} = x_{21}. \end{cases}$$

For the vectors $a = (a_1, a_2, a_3, a_4,)$, $b = (b_1, b_2, b_3, b_4,)$ and $c = (c_1, c_2, c_3, c_4,)$, Galilean cross product in G_4 is defined as follows:

$$a \wedge b \wedge c = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

where e_i are the standard basis vectors.

In this paper, we denote the inner product of two vectors a, b in the sense of Galilean by the notation $< a, b >_G$.

Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength s in G_4. For a α Frenet curve, the Frenet formulas can be given as following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \\ \mathbf{e}' \end{bmatrix} = \begin{bmatrix} o & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \\ \mathbf{e} \end{bmatrix}.$$

We can know that $\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{e}$ are mutually orthogonal vector fields satisfying equations

$$< \mathbf{t}, \mathbf{t} >_G = < \mathbf{n}, \mathbf{n} >_G = < \mathbf{b}, \mathbf{b} >_G = < \mathbf{e}, \mathbf{e} >_G = 1$$
$$< \mathbf{t}, \mathbf{n} >_G = < \mathbf{t}, \mathbf{b} >_G = < \mathbf{t}, \mathbf{e} >_G = < \mathbf{n}, \mathbf{b} >_G = < \mathbf{n}, \mathbf{e} >_G = < \mathbf{b}, \mathbf{e} >_G = 0.$$

3 Mannheim Curves in Galilean Space G4

In [5], Mannheim curves for Euclidean 4-space are generalized. In this paper, we have investigated generalization of the curves in 4D Galilean space G_4 .

Definition 3.1. A special curve α in G_4 is called a generalized Mannheim curve if there exists a special Frenet curve α^* in G_4 such that the first normal line at each point of α is included in the plane generated by the second and the third normal line of α^* at the corresponding point under Ψ. Here we denote by Ψ a bijection from α to α^*. Then the curve α^* is called the generalized Mannheim mate curve of α.

A generalized Mannheim mate curve α^* is given by the map $\alpha^* : I^* \to G_4$ which satisfies the following equation

$$\alpha^*(t) = \alpha(t) + \gamma(t)\mathbf{n}(t), \quad t \in I. \tag{3.1}$$

Here we denote a smooth function on I by $\gamma(t)$. The parameter should not be an arclength of α^* . The arclength of α^* defined by

$$t^* = \int_0^t \left\| \frac{d\alpha^*(t)}{dt} \right\| dt$$

Where t^* is the arclength of α^*. For a smooth function $f : I \to I^*$ is given by $f(t) = t^*$, we have

$$f'(t) = \frac{dt^*}{dt} = \left\| \frac{d\alpha^*(t)}{dt} \right\| = 1$$

for $\forall t \in I$. The representation of curve α^* with arclength parameter t^* is

$$\alpha^* : I^* \to G_4, \; t^* \to \alpha^*(t^*).$$

For the bijection $\Psi : \alpha \to \alpha^*$ defined by $\Psi(\alpha(t)) = \alpha^*(f(t))$, the reparameterization of α^* is given by the following equation

$$\alpha^*(f(t)) = \alpha(t) + \gamma(t)\mathbf{n}(t), \quad t \in I$$

where $\gamma(t)$ is a smooth function on I. Then we obtain

$$\frac{d\alpha^*(f(t))}{dt} = \frac{d\alpha^*(t^*)}{dt}\Big|_{t^*=f(t)} = t^*(f(t)), \quad t \in I.$$

Theorem 3.1. If a special Frenet curve α in G_4 is a generalized Mannheim curve, the first curvature function κ and second curvature function τ satisfies the following equation

$$\kappa(t) = \gamma\tau^2(t) \quad t \in I \tag{3.2}$$

here we denote a constant number with γ.

Proof. In the following scheme we show α as a generalized Mannheim curve and α^* as a generalized Mannheim mate curve of α.

$$\begin{array}{ccccc} & & \alpha & & \alpha^* \\ & & \vdots & & \vdots \\ f & : & I & \rightarrow & I^* . \\ & & \downarrow & & \downarrow \\ \Psi & : & G_4 & \rightarrow & G_4 \end{array}$$

We define a smooth function f by $f(t) = \int \left\| \frac{d\alpha^*(t)}{dt} \right\| dt = t^*$ is the arclength parameter of α^*. In addition Ψ is a bijection that is defined by $\Psi(\alpha(t)) = \alpha^*(f(t))$. Then the curve α^* is reparameterized as following form

$$\alpha^*(f(t)) = \alpha(t) + \gamma(t)\mathbf{n}(t), \quad t \in I \tag{3.3}$$

where $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{e}\}$ and $\{\mathbf{t}^*, \mathbf{n}^*, \mathbf{b}^*, \mathbf{e}^*\}$ are orthogonal vector fields in G_4 along α and α^*, respectively.

Differentiating both sides of equation (3.3) with respect to t, we get

$$\mathbf{t}^*(f(t)) = \mathbf{t}(t) + \gamma'(t)\mathbf{n}(t) + \gamma(t)\tau(t)\mathbf{b}(t). \tag{3.4}$$

On the other hand, since the first normal line at the each point of α is lying in the plane generated by the second and the third normal line of

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α^* at the corresponding points under bijection Ψ, the vector field $\mathbf{n}(t)$ is obtained as follows

$$\mathbf{n}(t) = g(t)\mathbf{b}^*(f(t)) + h(t)\mathbf{e}^*(f(t))$$

where g and h are some smooth functions on $I \subset \mathbb{R}$. Taking into account of the following equation

$$< \mathbf{t}^*(f(t)), g(t)\mathbf{b}^*(f(t)) + h(t)\mathbf{e}^*(f(t)) >_G = 0$$

and using (3.4), we have $\gamma'(t) = 0$. Then we decompose the equation (3.4) as follows

$$f'(t)\mathbf{t}^*(f(t) = \mathbf{t}(t) + \gamma\tau(t)\mathbf{b}(t), \tag{3.5}$$

that is

$$\mathbf{t}^*(f(t) = \mathbf{t}(t) + \gamma\tau(t)\mathbf{b}(t) \tag{3.6}$$

where $f'(t) = 1$.

Differentiating both sides of the equation (3.6) with respect to $t \in I$, we obtain

$$\begin{aligned} \kappa^*(f(t))\mathbf{n}^*(f(t)) = (\kappa(t) - \gamma\tau^2(t))\mathbf{n}(t) + (\gamma\tau(t))'\,\mathbf{b}(t) \\ + \gamma\tau(t)\sigma(t)\mathbf{e}(t). \end{aligned} \tag{3.7}$$

Using

$$< \mathbf{n}^*(f(t)), g(t)\mathbf{b}^*(f(t)) + h(t)\mathbf{e}^*(f(t)) >_G = 0,$$

the coefficient of $\mathbf{n}(t)$ in equation (3.7) vanishes, that is,

$$\kappa(t) - \gamma\tau^2(t) = 0.$$

Then the proof is completed.

Theorem 3.2. Let α be a special Frenet curve such that its non-constant first and second curvature functions satisfy the following equation

$$\kappa(t) = \gamma\tau^2(t)$$

for all $t \in I \subset \mathbb{R}$. If the special Frenet curve α^* given by the following form

$$\alpha^*(t) = \alpha(t) + \gamma\mathbf{n}(t)$$

then α^* is a generalized Mannheim mate curve of α.

Proof. The arclength parameter of α^* is given by the equation

$$t^* = \int_0^t \left\| \frac{d\alpha^*(t)}{dt} \right\| dt, \quad t \in I.$$

Let us assume that

$$\kappa(t) = \gamma\tau^2(t),$$

then we obtain $f^{'}(t) = 1$, $t \in I$.

Differentiating the equation $\alpha^*(f(t)) = \alpha(t) + \gamma\mathbf{n}(t)$ with respect to t the we get

$$f^{'}(t)\mathbf{t}^*(f(t) = \mathbf{t}(t) + \gamma\tau(t)\mathbf{b}(t).$$

Then we can see

$$\mathbf{t}^*(f(t) = \mathbf{t}(t) + \gamma\tau(t)\mathbf{b}(t), \quad t \in I. \tag{3.8}$$

Differentiating the last equation with respect to t is

$$\kappa^*(f(t))\mathbf{n}^*(f(t)) = (\kappa(t) - \gamma\tau^2(t))\mathbf{n}(t) + (\gamma\tau(t))^{'}\mathbf{b}(t) + \gamma\tau(t)\sigma(t)\mathbf{e}(t). \tag{3.9}$$

From the assumption, we obtain

$$\kappa(t) - \gamma\tau^2(t) = 0.$$

Then, the coefficient of $\mathbf{n}(t)$ in the equation (3.9) is zero. One can see from the equation (3.8) that $\mathbf{t}^*(f(t)$ is a linear combination of $\mathbf{t}(t)$ and $\mathbf{b}(t)$. In addition, from equation (3.9), $\mathbf{n}^*(f(t))$ is given by linear combination of $\mathbf{b}(t)$ and $\mathbf{e}(t)$. On the otherhand, α^* is a special Frenet curve that the vector $\mathbf{n}(t)$ which satisfies the following linear combination of $\mathbf{t}^*(f(t)$ and $\mathbf{n}^*(f(t))$.

Therefore, the first normal line α lies in the plane generated by the second and third normal line of α^* at the corresponding points under the Ψ bijection which is defined by

$$\Psi(f(t)) = \alpha^*(f(t)).$$

The proof is completed.

Remark 3.1. In 4D Galilean space G_4, a special Frenet curve α with curvature functions κ and τ satisfying $\kappa(t)=\gamma\tau^2(t)$, it is not clear that a smooth curve α^* given by (3.1) is a special Frenet curve. The reverse of Theorem 3.1 is still a great puzzle for the authors.

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