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The least eigenvalues of the signless Laplacian of non-bipartite graphs with fixed diameter

Min Zhu, Yihao Guo, Fenglei Tian, Lingfei Lu

College of Science, China University of Mining and Technology,
Xuzhou, Jiangsu, P. R. China

Abstract. Let $\zeta_n(d)$ ($\mu_n(d)$) be the set of connected non-bipartite (unicyclic) graphs with n vertices and diameter d. In this paper, we first determine the graph whose least eigenvalue of the signless Laplacian attains the minimum in $\mu_n(d)$, then by by the eigenvalue interlacing property, the problem of determining the minimizing graph in $\zeta_n(d)$ can be transformed to that of determining the minimizing graph in $\mu_n(d)$. Thus we obtain a lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph in terms of the diameter d.

Keywords: non-bipartite graph; signless Laplacian;Least eigenvalue; diameter

1 Introduction

Let G be a simple graph with vertices $1,2,\cdots,n$, of degrees $d_1,d_2,\cdots,d_n$, respectively. Let $A(G)$ be the (0, 1)-adjacency matrix of G, and let $D(G)$ be the diagonal matrix $diag(d_1,d_2,\cdots,d_n)$. The matrix $L(G)=D(G)-A(G)$ is the *Laplacian* of G, while $Q(G)=D(G)+A(G)$ is called the *signless Laplacian* of G. We call the eigenvalues of $Q(G)$ the $Q-$eigenvalues of graph G, it is known that $Q(G)$ is nonnegative, symmetric and positive semidefinite. So its eigenvalues are all nonnegative real numbers and can be arranged as $q_1(G)\ge q_2(G)\ge\cdots\ge q_n(G)\ge 0$. The least $Q-$eigenvalue is

The corresponding author. *E-mail address*: 741856964@qq.com

$q_n(G)$, and the eigenvectors corresponding to $q_n(G)$ are called *the first* $Q-$*eigenvectors* of G. For the properties of the least $Q-$eigenvalue, we refer the readers to [1-6]. A graph is called *minimizing* in a class of graphs if its least $Q-$eigenvalue attains the minimum among all graphs in the class. Denote by $\zeta_n(d)$ ($\mu_n(d)$) the set of connected non-bipartite (unicyclic) graphs with n vertices and diameter d. Let $\mu_n(g,d)$ denote the set of unicyclic graphs of order n with odd girth g and diameter d, ($d \geq \frac{g-1}{2}$).

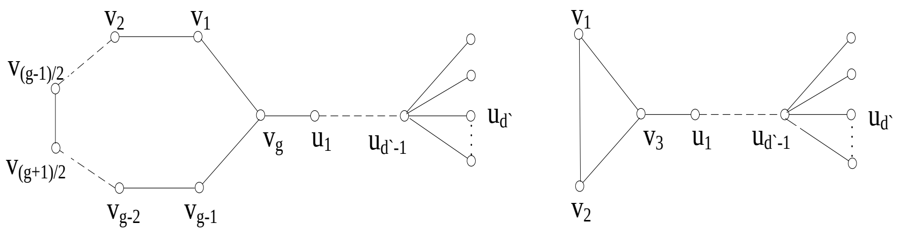

Fig.1

If G is connected, then $q_n(G)=0$ if and only if G is bipartite. So, connected non-bipartite graphs are considered here. The investigation on the lower bound of the least $Q-$eigenvalue of a graph is an important topic in the theory of $Q-$ spectra. M. Desai, V. Rao discuss the relationship between the least $Q-$eigenvalue and the bipartiteness of graphs in [8]. Cardoso et al. [3] and Fan et al. [10] investigate the least $Q-$eigenvalue of non-bipartite unicyclic graphs. Liu et al. [11] give some bounds for the clique number and independence number of graphs in terms of the least $Q-$eigenvalue. Lima et al. [7] survey the known results and present some new ones for the least $Q-$eigenvalue. Wang et al. [13] investigated how the least $Q-$eigenvalue of a graph changes by relocating a bipartite branch

The corresponding author. *E-mail address*: 741856964@qq.com

from one vertex to another vertex, and minimized the least $Q-$eigenvalue among the connected graphs of fixed order which contain a given non-bipartite graph as an induced subgraph. Fan et al. [14] determine the minimizing graph of non-partite graphs in terms of the number of pendant vertices.

In this paper, we first show that $U_n(g,d)$ (see Fig.1) is the unique minimizing graph in $\mu_n(g,d)$, and then determine that $U_n(3,d)$ is the unique minimizing graph in $\mu_n(d)$. At last, by the eigenvalue interlacing property (see following Lemma 2.6), the problem of determining the minimizing graph in $\zeta_n(d)$ can be transformed to that of determining the minimizing graph in $\mu_n(d)$.

2. Preliminaries

We first introduce some notations. Let C_n and P_n to denote the cycle and the path, on n vertices, respectively. We also use $P=v_1,v_2,\cdots v_n$ to denote a path on vertices $v_1,v_2,\cdots v_n$ with edges v_iv_{i+1} for $i=1,2,\cdots,n-1$. Let $N_G(v)$ be the set of the neighborhood of the vertex v in graph G. Let G be a graph, G is called *trivial* if it contains only one vertex; otherwise, it is called *nontrivial*. Graph G is called *unicyclic* if it is connected and has the same number of vertices and edges (or G contains exactly one cycle). The *girth* of G is the minimum of the lengths of all cycles in G. A *pendant vertex* of G is a vertex of degree 1. A path $P=v_0,v_1,\cdots v_{t-1},v_t$ in G is called a *pendant path* if $d_{v_0}\ge 3$, $d_{v_1}=d_{v_2}=\cdots=d_{v_{t-1}}=2$ and $d_{v_t}=1$. If $t=1$, then v_0v_1 is a pendant edge of G.

Let $x=(x_1,x_2,\cdots x_n)'$ be a column vector, and let G be a graph on vertices $V(G)=\{v_1,v_2,\cdots v_n\}$. The vector x can be viewed as a function defined

The corresponding author. *E-mail address*: 741856964@qq.com

on $V(G)$; that is, any vertex v_i is given by the value $x_i = x_{v_i}$. Thus the quadratic form $x'Qx$ can be written as

$$x'Qx = \sum_{uv \in E(G)} [x_u + x_v]^2.$$

One can find that q is a $Q-$eigenvalue of G corresponding to an eigenvector x if and only if $x \neq 0$ and

$$[q - d_v]x_v = \sum_{u \in N_G(v)} x_u,$$

for each $v \in V(G)$. In addition, for an arbitrary unit vector x,

$$q_n(G) \leq x'Q(G)x,$$

with equality if and only if x is a first $Q-$eigenvector of G.

Let G_1 and G_2 be two vertex-disjoint graphs, and let $v \in G_1$, $u \in G_2$. The *coalescence* of G_1 and G_2 with respect to v and u, denoted by $G_1 \bullet G_2$, is obtained from G_1 and G_2 by identifying v with u and forming a new vertex. Let G be a connected graph, and let v be a cut vertex of G. Then G can be expressed in the form $G = H(v) \bullet F(v)$, where H and F are subgraphs of G both containing v. Here, we call H (or F) *a branch of G with root* v. With respect to a vector x defined on G, the branch H is called a *zero branch* if $x_v = 0$ for all $v \in V(H)$; otherwise, H is called a *nonzero branch*.

Let $G = G_1(v_2) \bullet G_2(u)$, $G^* = G_1(v_1) \bullet G_2(u)$, where v_1 and v_2 are two distinct vertices of G_1 and u is a vertex of G_2. We say that G^* is obtained from G by relocating G_2 from v_2 to v_1. Then, we give some lemmas that will be used in the proof of our result.

Lemma 2.1 ([13]) Let H be a bipartite branch of a connected graph G with root u. Let x be a first $Q-$eigenvector of G.

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(1) If $x_u = 0$, then H is a zero branch of G with respect to x.

(2) If $x_u \neq 0$, then $x_p \neq 0$ for every vertex p of H. Furthermore, for every vertex p of H, $x_p x_u$ is positive or negative, depending on whether p is or is not in the same part of bipartite graph H as u; consequently, $x_p x_q < 0$ for each edge $pq \in E(H)$.

Lemma 2.2 ([13]) Let G be a connected non-bipartite graph, and let x be a first $Q-$eigenvector of G. Let T be a tree with root u, which is a nonzero branch with respect to x. Then $|x_q| < |x_p|$ whenever p and q are vertices of T such that q lies on the unique path from u to p.

Lemma 2.3 ([13]) Let G_1 be a connected graph containing at least two vertices v_1, v_2, and let G_2 be a connected bipartite graph containing a vertex u. Let $G = G_1(v_2) \bullet G_2(u)$ and $G^* = G_1(v_1) \bullet G_2(u)$. If there exists a first $Q-$ eigenvector of G such that $|x_{v_1}| \geq |x_{v_2}|$, then $q_n(G^*) \leq q_n(G)$, with equality only if $|x_{v_1}| = |x_{v_2}|$ and $d_{G_2(u)} x_u = -\sum_{v \in N_{G_2(u)}} x_v$.

Lemma 2.4 ([13]) Let G_1 be a connected non-bipartite graph containing two vertices v_1, v_2, and let P be a nontrivial path with u as an end vertex. Let $G = G_1(v_2) \bullet P(u)$, and let $G^* = G_1(v_1) \bullet P(u)$. If there exists a first $Q-$ eigenvector x of G such that $|x_{v_1}| > |x_{v_2}|$ or $|x_{v_1}| = |x_{v_2}| > 0$, then $q_n(G^*) \leq q_n(G)$.

Lemma 2.5 ([14]) Let $U_n(g,d)$ be the graph with some vertices labeled as in Fig.1, where $v_1, v_2, \cdots v_g$ are the vertices of the unique cycle C_g labeled in an

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anticlockwise way. Let x be a first $Q-$eigenvector of $U_n(g,d)$. Then, the following hold:

(1) $x_{v_i} = x_{v_{g-i}}$ for $i = 1,2,\cdots,\frac{g-1}{2}$.

(2) $x_{v_{\frac{g-1}{2}}} x_{v_{\frac{g+1}{2}}} > 0$, and $x_{v_v} x_{v_w} < 0$ for every edges vw of $U_n(g,d)$ except $v_{\frac{g-1}{2}} v_{\frac{g+1}{2}}$.

(3) $\left|x_{v_g}\right| > \left|x_{v_1}\right| > \left|x_{v_2}\right| > \cdots > \left|x_{v_{\frac{g-1}{2}}}\right| > 0$.

Lemma 2.6 ([3]) Let G be a graph of order n containing an edge e. Let $q_1, q_2, \cdots q_n$ ($q_1 \geq q_2 \geq \cdots \geq q_n$) and $s_1, s_2, \cdots s_n$ ($s_1 \geq s_2 \geq \cdots \geq s_n$) be the $Q-$eigenvalues of G and $G-e$. Then

$$0 \leq s_n \leq q_n \leq \cdots s_2 \leq q_2 \leq s_1 \leq q_1.$$

3. Characterization of the extremal graph

Lemma 3.1 Let U be the minimizing graph in $\mu_n(g,d)$ and P be a diameter-path of U, then P must encounters the unique cycle C, and $|V(P) \cap V(C)| = \frac{g+1}{2}$.

Proof. Let $P = u_1, u_2, \cdots u_d, u_{d+1}$ be a diameter-path of U and $C = v_1, v_2, \cdots v_g, v_1$ be the unique cycle of U. Suppose that $|V(P) \cap V(C)| = \phi$. Since U is connected, then suppose that there exists a shortest path $v_g, w_1, w_2, \cdots, w_s, u_k$ connecting C and P, where $w_1, w_2, \cdots, w_s \in V(U) \setminus \{V(P) \cup V(C)\}$. Let x be an eigenvector of $Q(U)$ corresponding to $q_n(U)$ and define graph U_1

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$$U_1 = \begin{cases} U - \sum_{w \in N(u_k) \setminus \{w_s\}} wu_k + \sum_{w \in N(u_k) \setminus \{w_s\}} wv_g, & if \left| x_{v_g} \right| \geq \left| x_{u_k} \right|; \\ U - \sum_{w \in N(v_g) \setminus \{w_1\}} wv_g + \sum_{w \in N(v_g) \setminus \{w_1\}} wu_k, & f \left| x_{v_g} \right| < \left| x_{u_k} \right|. \end{cases}$$

Then in either case (indeed, they are isomorphic, without loss of generality, we choose U_1 be the graph with vertices labels as the latter), P is still a diameter-path of U_1, $U_1 \in \mu_n(g,d)$ and $|V(P) \cap V(C)| = 1$. And by Lemma 2.2, $q_n(U) \geq q_n(U_1)$, a contradiction. Hence, $|V(P) \cap V(C)| \neq \phi$, so, the diameter-path P encounters the cycle C in U.

Then we continue to define graph U_i, ($i = 1,2,\cdots,\frac{g+1}{2}$)

$$U_i = \begin{cases} U_{i-1} - \{u_{k+i-1}u_{k+i}\} + \{v_{i-1}u_{k+i}\}, & if \left| x_{v_{i-1}} \right| \geq \left| x_{u_{k+i-1}} \right|; \\ U_{i-1} - \{v_{i-1}v_i\} + \{v_i u_{k+i-1}\}, & if \left| x_{v_{i-1}} \right| < \left| x_{u_{k+i-1}} \right|. \end{cases}$$

In the graph U_i, we can easily see that P is still a diameter-path of U_i, $U_i \in \mu_n(g,d)$ and $|V(P) \cap V(C)| = i$. And by Lemma 2.2, $q_n(U_1) \geq q_n(U_2) \geq q_n(U_3) \geq \cdots \geq q_n\left(U_{\frac{g+1}{2}}\right)$.

It doesn't continue to define graphs according to the above method, otherwise, it contradicts to that P is the diameter-path, so P must encounters the unique cycle C in U, and $|V(P) \cap V(C)| = \frac{g+1}{2}$.

Lemma 3.2 Among all graphs in $\mu_n(g,d)$, $U_n(g,d)$ is the unique minimizing graph.

Proof. Let G be a minimizing graph in $\mu_n(g,d)$, and let C_g be the unique cycle of G on vertices $v_1, v_2, \cdots v_g$. Graph G can be considered as one obtained from C_g by identifying each v_i with one vertex of some tree T_i of

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order n_i for each $i=1,2,\cdots,g$, where $\sum_{i=1}^{g} n_i = n$. Note that some trees T_i may be trivial.

Let x be a unit first $Q-$eigenvector of G, then there exists at least one i, ($1 \le i \le g$), such that $x_{v_i} \ne 0$. Otherwise, by Lemma 2.1(1), each T_i, ($1 \le i \le g$), is a zero branch of G with respect to x, and it follows that x is the zero vector, a contradiction. We claim that each nontrivial tree T_j is a nonzero branch with respect to x. Otherwise, there exists a nontrivial tree T_j attached at v_j, ($1 \le j \le g$), such that $x_{v_j} = 0$. By Lemma 2.3, relocating the tree T_j from v_j to v_i for some i for which $x_{v_i} \ne 0$, we obtain a graph in $\mu_n(g,d)$ with smaller least $Q-$eigenvalue, a contradiction. We also claim that there is only one nontrivial tree T. If not, there exist two nontrivial trees, say T_i, T_j attached at v_i, v_j, respectively. By Lemma 2.2 and 2.4, relocating the tree T_j from v_j to one vertex of tree T_i (if $\left|x_{v_i}\right| \ge \left|x_{v_j}\right|$), or relocating the tree T_i from v_i to to one vertex of tree T_j, (if $\left|x_{v_j}\right| \ge \left|x_{v_i}\right|$), we can obtain a graph in $\mu_n(g,d)$ with smaller least $Q-$eigenvalue, it contradicts to the minimum of G.

We assume that $P' = u_0, u_1, \cdots u_{d'}$ (let $u_0 = v_g$ and $d' = d - \frac{g-1}{2}$) is the diameter-path of the only nontrivial tree T. We claim that any vertex $x \in V\left(G \setminus \{C_g \cup P'\}\right)$ is a pendant vertex, and if exists, it attached to the unique vertex $u_{d'-1}$. First, we suppose that there exits a pendant path $P'' = w_0, w_1, \cdots w_s$ ($2 \le s \le d'$) attached to the path P'.

Case 1. $s = d'$, then we can see that $w_0 = u_0$.

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If $\left|x_{w_{s-1}}\right| \geq \left|x_{u_{d'-1}}\right|$, then replacing edge $w_{s-1}w_s$ by $u_{d'-1}w_s$, (otherwise, replacing $u_{d'-1}u_{d'}$ by $w_{s-1}u_{d'}$). We can obtain a new graph G', and $G' \in \mu_n(g,d)$, by lemma 2.4, it followed that G' has smaller least $Q-$eigenvalue, a contradiction.

Case 2. $2 \leq s < d'$, then $w_0 \in \{u_0, u_1, \cdots u_{d'-s}\}$, as the P' is the diameter-path of T. As the assumption that G is a minimizing graph and $\left|x_{u_{d'-s}}\right| \geq \left|x_{u_0}\right|$ (by lemma 2.2), by lemma 2.4, we can see that $w_0 = u_{d'-s}$. Then we compare $\left|x_{w_{s-1}}\right|$ with $\left|x_{u_{d'-1}}\right|$, by the same discussion as the Case 1, We can obtain a new graph G', and $G' \in \mu_n(g,d)$, by lemma 2.4, it followed that G' has smaller least $Q-$eigenvalue, a contradiction.

So, any vertex $x \in V(G \setminus \{C_g \cup P'\})$ is a pendant vertex.

Now, suppose that G contains at least one such star S_{u_k}, which has center u_k, ($k = 0,1,2,\cdots,d'-2$), as $\left|x_{u_{d'-1}}\right| \geq \left|x_{u_k}\right|$ (by lemma 2.2), denote by G' the graph

$$G - \sum_{w \in N_{S_{u_k}}(u_k)} wu_k + \sum_{w \in N_{S_{u_k}}(u_k)} wu_{d'-1}$$

and $G' \in \mu_n(g,d)$, by lemma 2.4, G' has smaller least $Q-$eigenvalue, a contradiction.

So, we can easily conclude that $U_n(g,d)$ is the unique minimizing graph.

Denote by $t_n(g,d)$ *the minimum of the least* $Q-$*eigenvalues of graphs in* $\mu_n(g,d)$*, that is, the least* $Q-$*eigenvalue of* $U_n(g,d)$.

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Lemma 3.3 $t_n(g,d)$ is strictly increasing with respect to odd g, ($g \geq 3$).

Proof. Let $U_n(g,d)$ with some vertices labeled as in Fig.1, and x be a unit first $Q-$eigenvector of $U_n(g,d)$. Suppose that $g \geq 5$, as $x_{v_1} = x_{v_{g-1}}$ by lemma 2.5, replacing edge $v_{g-2}v_{g-1}$ by edge $v_{g-2}v_1$, we obtain a new graph $G' \in \mu_n(g-2,d)$, which satisfies that $x'Q(G')x = x'Q(U_n(g,d))x = t_n(g,d)$. So, $q_n(G') \leq t_n(g,d)$, and hence, $t_n(g-2,d) \leq$
$q_n(G') \leq t_n(g,d)$. The result follows.

Corollary 3.4 Among all graphs in $\mu_n(d)$, $U_n(3,d)$ is the unique minimizing graph.

By the lemma 2.6 and lemma 3.1-3.4, we arrive at the main Theorem of this paper.

Theorem 3.5 Among all graphs in $\zeta_n(d)$, $U_n(3,d)$ is the unique minimizing graph.

Proof. Let G be a minimizing graph in $\zeta_n(d)$. Then G contains at least an induced odd cycle, say C_g. Let G' be a unicyclic spanning subgraph of G, which obtained by deleting an edge in every cycle except for C_g and maintain that $G' \in \mu_n(g,d)$. By lemma 2.6 and Corollary 3.4, we can see that

$$q_n(U_n(3,d)) = t_n(3,d) \leq t_n(g,d) \leq q_n(G') \leq q_n(G) \quad .$$

(3.1)

As G is a minimizing graph in $\zeta_n(d)$, all inequalities in (3.1) hold as equalities, by Lemma 3.2 and 3.3, which implies that $g = 3$, $G' = U_n(3,d)$ and $q_n(G) = q_n(U_n(3,d))$.

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Now, we prove that $G = U_n(3,d)$. Suppose that $E(G)\setminus E(U_n(3,d)) \neq \phi$. Recalling the definition of G' and $G' = U_n(3,d)$, the set $E(G)\setminus E(U_n(3,d))$ consists of some edges joining the vertices of C_3 and the vertices of T or some edges within the vertices of T. So, for each edge $uv \in E(G)\setminus E(U_n(3,d))$, if x is a first $Q-$eigenvector of $U_n(3,d)$, then by Lemma 2.2 and Lemma 2.5(3) we can see that $x_u + x_v \neq 0$.

Let x be a unit first $Q-$eigenvector of G. Then

$$q_n(G) = \sum_{uv\in E(G)} [x_u + x_v]^2$$
$$= \sum_{uv\in E(U_n(3,d))} [x_u + x_v]^2 + \sum_{uv\in E(G)\setminus E(U_n(3,d))} [x_u + x_v]^2$$
$$\geq \sum_{uv\in E(U_n(3,d))} [x_u + x_v]^2 \geq q_n(U_n(3,d))$$

Since $q_n(G) = q_n(U_n(3,d))$, x is also a first $Q-$ eigenvector of $U_n(3,d)$, so for each edge $uv \in E(G)\setminus E(U_n(3,d))$, $x_u + x_v = 0$, a contradiction. Hence, $E(G)\setminus E(U_n(3,d)) = \phi$, the result follows.

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