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3-prime near-rings with involutions

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Abstract. The objective of this paper is to introduce the concept of "involution" in 3-prime near-rings, and to prove some theorems involving this mapping. Moreover, an example is given to illustrate that the restriction imposed on the hypothesis of theorems were not superfluous.

Keywords: 3-prime near-rings, semigroup ideal, involution.

1 Definitions and terminology

A left near-ring is a set N with two operation + and . such that $(N,+)$ is a group (not necessarily abelian) and $(N,\ .\ )$ is a semigroup satisfying the left distributive law $x.(y+z)=x.y+x.z$ for all $x,y,z\in N$. A left near-ring N is called zero-Symmetric if $0.x=0$ for all $x\in N$ (recalling that left distributive yields $x.0=0$). A near-ring N is said to be 3-prime if $xNy=\{0\}$ for all $x,y\in N$ implies $x=0$ or $y=0$. In this paper, we will use the word near-ring to mean zero-Symmetric left near-ring. A nonempty subset I of N will be called a semigroup ideal if $IN\subseteq I$ and $NI\subseteq I$. A normal subgroup I of $(N,+)$ is called an ideal of N if $IN\subseteq I$ and $x(y+i)-xy\in I$ for all $i\in I,x,y\in N$. An additive

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mapping $\sigma : N \to N$ is called an involution if $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in N$. Recently, many researchers have studied commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, involutions acting on appropriate subsets of the rings (see, for example, [2], [3], [4]). Being motivated by their invaluable research, it is natural then to ask whether we can apply the involution as to study the structure of a 3-prime near-ring. In this paper, we would like to study the structure of a 3-prime near-ring as having an involution. More precisely, we shall prove that a 3-prime near-ring which admits an involution must be a ring.

2 Main Results

In order to prove our main theorems, we need the following lemma.

Lemma 1 [1, Lemma 1.3 (i)] Let I be a nonzero semigroup ideal of a 3-prime near-ring N and $a \in N$. If $aI = \{0\}$ or $Ia = \{0\}$, then $a = 0$.

Theorem 1 Let N be a 3-prime near-ring. If N admits an involution σ, then N is a ring.

Proof.
Suppose that σ is an involution of N, to show that N is a ring, it is enough to show that the multiplicative law is right distributive and that the additive law is abelian.

i) By defining σ, we have for all $x, y, z \in N$

$$\begin{aligned}
(x+y)z &= \sigma^2(x+y)\sigma^2(z) \\
&= \sigma\big(\sigma(x+y)\big)\sigma\big(\sigma(z)\big) \\
&= \sigma\big(\sigma(z)\sigma(x+y)\big) \\
&= \sigma\big(\sigma(z)\big(\sigma(x)+\sigma(y)\big)\big) \\
&= \sigma\big(\sigma(z)\sigma(x)+\sigma(z)\sigma(y)\big) \\
&= \sigma\big(\sigma(xz)+\sigma(yz)\big) \\
&= \sigma^2(xz)+\sigma^2(yz) \\
&= xz+yz.
\end{aligned}$$

And therefore ,

$$(x+y)z = xz + yz \text{ for all } x,y,z \in N. \qquad (1)$$

Hence, we get the right distributivity of the multiplicative law .

ii) Let $x,y,k \in N$, by applying (1) , we have

$$(k+k)(x+y) = k(x+y) + k(x+y)$$
$$= kx + ky + kx + ky \text{ for all } x,y,k \in N.$$

On the other hand ,

$$(k+k)(x+y) = (k+k)x + (k+k)y$$
$$= kx + kx + ky + ky \text{ for all } x,y,k \in N.$$

By comparing both expressions, we obtain

$$kx + ky = ky + kx \text{ for all } x,y,k \in N.$$

This is reduced to

$$k((x+y)-(y+x)) = 0 \text{ for all } x,y,k \in N$$

it means that

$$N((x+y)-(y+x)) = \{0\} \text{ for all } x,y \in N. \qquad (2)$$

Once again the Lemma 1, equation (2) shows that

$$x + y = y + x \text{ for all } x,y \in N.$$

Therefore, the additive law of N is abelian. Consequently N is a ring.

Theorem 2 Let N be a 3-prime near-ring and I is a semigroup ideal of N stable by the additive law. If I admits an involution σ , then N is a ring.

Proof.

a) By the hypothesis given, likewise and taking into account the first step of the proof of the previous theorem, we arrive at

$$(x+y)z = xz + yz \text{ for all } x,y,z \in I. \qquad (3)$$

Replacing x and y by xn and xm, respectively, in (3) we obtain

$$(xn+xm)z = xnz + xmz \text{ for all } x,z \in I, n,m \in N.$$

Which can be rewritten as

$$x(n+m)z = x(nz+mz) \text{ for all } x,z \in I, n,m \in N$$

this means that

$$x((n+m)z-(nz+mz)) = 0 \text{ for all } x,z \in I, n,m \in N.$$

Hence,

$$I\left(\left(n+m\right)z-\left(nz+mz\right)\right)=\{0\} \text{ for all } z\in I, n, m\in N. \tag{4}$$

By applying Lemma 1, equation (4) shows that

$$\left(n+m\right)z=nz+mz \text{ for all } z\in I, n, m\in N. \tag{5}$$

Now, replacing z by tz in (5), we obtain

$$\left(n+m\right)tz=ntz+mtz \text{ for all } z\in I, n, m, t\in N. \tag{6}$$

As $\left(ntz+mtz\right)=\left(nt+mt\right)z$ by (5), then equation (6) becomes

$$\left(n+m\right)tz=\left(nt+mt\right)z \text{ for all } z\in I, n, m, t\in N. \tag{7}$$

On the other hand, by (5) we have for all $z\in I, n, m, t\in N$

$$\left(\left(n+m\right)t-\left(nt+mt\right)\right)z=\left(n+m\right)tz+\left(-\left(nt+mt\right)\right)z \tag{8}$$

by combining (7) and (8), we find that

$$\begin{aligned}\left(\left(n+m\right)t-\left(nt+mt\right)\right)z &= \left(nt+mt\right)z+\left(-\left(nt+mt\right)\right)z\\ &= \left(\left(nt+mt\right)+\left(-\left(nt+mt\right)\right)\right)z\\ &= 0\end{aligned}$$

So that,

$$\left(\left(n+m\right)t-\left(nt+mt\right)\right)I=\{0\} \text{ for all } n, m, t\in N.$$

Applying Lemma 1, the last equation yields

$$\left(n+m\right)t=nt+mt \text{ for all } n, m, t\in N.$$

Hence, the multiplicative law of N is right distibutive.

b) To complete the demonstration of this theorem, it remains to show that the additive law is abelian. In this case, we use the same proof as that of step (ii) of the preceding theorem.

Corollary 1 Let N be a 3-prime near-ring and U is a nonzero ideal of N. If U admits an involution σ, then N is a ring.

The following example shows the necessity of the 3-primeness in the previous theorems.

Example
Let S be a noncommutative near-ring. We define N and $\sigma : N \to N$ by :

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$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}, \quad \sigma \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that N is not 3-prime and σ is an involution of N, but N is not a ring.

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