Volume Enclosed by Subdivision Surfaces

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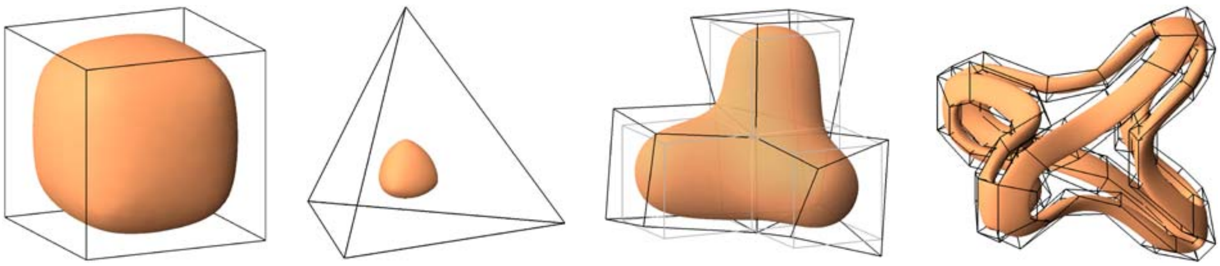


Figure: Doo-Sabin subdivision of the unit cube defines a surface that encloses a volume of $\frac{6241}{9920}$. The tetrahedron with all edges of length 1 generates a Loop subdivision surface with volume $\frac{44\,192\,429\,513\,855\,101}{6\,865\,302\,375\,425\,894\,400\,\sqrt{2}}$. The unperturbed and perturbed control mesh define different Catmull-Clark subdivision surfaces that enclose identical volume. In the fourth example, the volume of the initial mesh contracts by a factor of 0.677115... to the volume enclosed by the Catmull-Clark subdivision surface. ■

Abstract

We present a framework to derive the coefficients of trilinear forms that compute the exact volume enclosed by subdivision surfaces. The coefficients depend only on the local mesh topology, such as the valence of a vertex, and the subdivision rules. The input to the trilinear form are the initial control points of the mesh.

Our framework allows us to explicitly state volume formulas for surfaces generated by the popular subdivision algorithms Doo-Sabin, Catmull-Clark, and Loop. The trilinear forms grow in complexity as the vertex valence increases. In practice, the explicit formulas are restricted to meshes with a certain maximum valence of a vertex.

The approach extends to higher order momentums such as the center of gravity, and the inertia of the volume enclosed by subdivision surfaces.

The first author dedicates this work to the memory of Andrew Ladd, Nik Sperling, and Leif Dickmann. The article and additional resources are available at *www.hakenberg.de*. The first author was partially supported by personal savings accumulated during his visit to the Nanyang Technological University as a visiting research scientist in 2012-2013. He'd like to thank everyone who worked to make this opportunity available to him.

Introduction

A subdivision scheme is a mesh refinement procedure. Starting with an initial mesh, the repeated application of the subdivision scheme results in an increasingly dense mesh. The sequence of meshes converges to a piecewise smooth surface. Due to these properties, subdivision is a popular technique to design and represent surfaces in computer graphics.

[Doo/Sabin 1978] and [Catmull/Clark 1978] introduced the first subdivision schemes intended for the refinement of quad meshes. In the limit, large parts of the surface have piecewise polynomial parameterization. Later, [Loop 1987] designed a subdivision scheme for triangular meshes. The smoothness characteristics of the limit surface produced by the schemes are well-understood, see [Reif 1995].

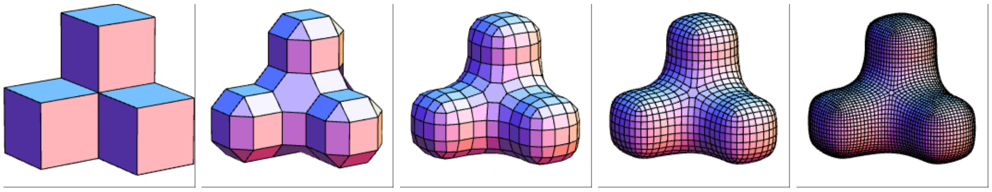


Figure: Four iterations of the Doo-Sabin subdivision scheme applied to an initial mesh of 4 unit cubes glued together. We prove that the limit surface encloses a volume of $\frac{10\,357\,799\,098\,161+2\,535\,566\,756\sqrt{5}}{3\,238\,292\,736\,000}$. ■

When the surface is generated from a closed, orientable mesh, the enclosed volume is a well-defined concept. A simple formula for the enclosed volume by the limit surface was not known previously. [Peters/Nasri 1997] only describe an approximation of the volume. Moreover, their framework requires "*regular submeshes to have a polynomial parametrization*". Volumes defined by the Loop scheme are not covered by their approach.

For the three subdivision schemes mentioned above, our article proves that the exact volume enclosed by the limit surface is a trilinear form with the *x*-, *y*-, *z*-vertex coordinates of the initial mesh as input. We provide universal trilinear forms that apply locally, and add up to the encosed volume globally. The evaluation is computationally efficient.

Possible applications of our new formula are 1) the design of surfaces to enclose a specific volume, 2) deformation of surfaces subject to volume preservation.

The article is structured as follows: We motivate the general volume formula. The regular and non-regular mesh topologies require separate treatment. Then, the volume formulas for Doo-Sabin, Catmull-Clark, and Loop are computed. We sketch how our framework applies to two other well-known subdivision schemes. Finally, we discuss the extension to moments of higher degree.

Framework

We begin with the well-known formula for the volume of piecewise linear surfaces. Our more complicated formula for volumes enclosed by subdivision surfaces is of identical prototype. The volume enclosed by a closed, orientable triangular mesh M is

$$\mathrm{vol}(M) = \sum_{t\in M} \frac{1}{6} \det\begin{pmatrix} \mathrm{px}_1 & \mathrm{px}_2 & \mathrm{px}_3 \\ \mathrm{py}_1 & \mathrm{py}_2 & \mathrm{py}_3 \\ \mathrm{pz}_1 & \mathrm{pz}_2 & \mathrm{pz}_3 \end{pmatrix}$$

$$= \frac{1}{6} \sum_{t\in M} \mathrm{px}_1(\mathrm{py}_2\,\mathrm{pz}_3 - \mathrm{py}_3\,\mathrm{pz}_2) + \mathrm{px}_2(\mathrm{py}_3\,\mathrm{pz}_1 - \mathrm{py}_1\,\mathrm{pz}_3) + \mathrm{px}_3(\mathrm{py}_1\,\mathrm{pz}_2 - \mathrm{py}_2\,\mathrm{pz}_1)$$

where px_i denotes the *x*-coordinate, py_i the *y*-coordinate, and pz_i the *z*-coordinate of the vertex $i \in \{1, 2, 3\}$ of the oriented triangle $t \in M$ in the mesh. The formula is a trilinear form that we write as

$$\mathrm{vol}(M) = \sum_{f\in M} \sum_{i,j,k}^{m(f)} \left(A_{i,j,k} - A_{i,k,j}\right) \mathrm{px}_i\,\mathrm{py}_j\,\mathrm{pz}_k$$

The tensor A has dimensions $3\times3\times3$. The facet $f \in M$ is an oriented triangle of the mesh. The surface corresponding to the triangle f is completely determined by $m(f) = 3$ control points. Throughout the article we use the abbreviation $\sum_{i,j,k}^{m} X(i, j, k) := \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} X(i, j, k)$.

We show that for stationary subdivision schemes with certain additional properties, the enclosed volume of the subdivision surface is also determined by a collection of trilinear forms. We require the subdivision surface to be partitioned by facets $f \in M$, where the surface corresponding to a facet f is completely determined by a set of control points $(\mathrm{px}_i, \mathrm{py}_i, \mathrm{pz}_i)$ for $i = 1, 2, \ldots, m(f)$ in the neighborhood of facet f. In other words: The coordinate $(\mathrm{px}_i, \mathrm{py}_i, \mathrm{pz}_i)$ is from the set of initial control points.

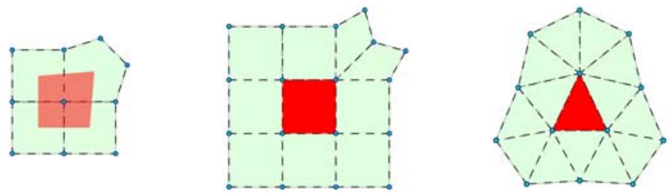


Figure: Shaded in red, a facet in a Doo-Sabin, Catmull-Clark, and Loop subdivision mesh together with the control points that define the surface across the facet. ■

Example: After two rounds of subdivision with Doo-Sabin's algorithm, each vertex v has valence 4. We associate a facet f to each vertex v: The facet is the quad spanned by the midpoints of the 4 faces adjacent to v. $m(f)$ is the number of vertices in the faces adjacent to v. ■

Example: For Catmull-Clark, the facet f is a quad of the one-time subdivided initial mesh. For Loop, the facet f is a triangle of the one-time subdivided initial mesh. $m(f)$ is the number of vertices in the one-ring of f. ■

For a general mesh M there is not a single trilinear form A that applies to all facets across the mesh. Instead, we have to provide a trilinear form $A^{\tau(f)}$ for each possible mesh topology $\tau(f)$ around a facet $f \in M$. An example for the characterization $\tau(f)$ is the valence of a non-regular vertex of f.

We state the general formula for the volume defined by subdivision of mesh M as

$$\mathrm{vol}(M) = \sum_{f\in M} \mathrm{vol}(f) = \sum_{f\in M} \sum_{i,j,k}^{m(f)} \left(A_{i,j,k}^{\tau(f)} - A_{i,k,j}^{\tau(f)}\right) \mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k = \sum_{f\in M} \sum_{i,j,k}^{m(f)} Y_{i,j,k}^{\tau(f)}\, \mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k$$

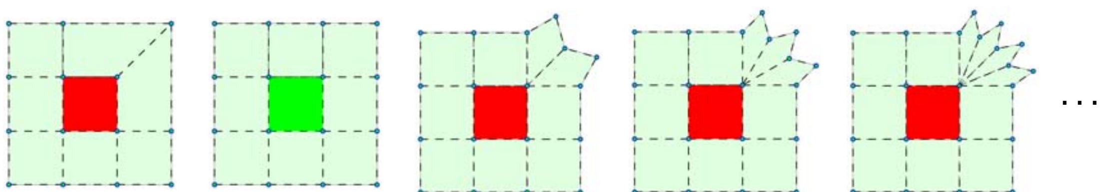


Figure: Different topologies $\tau(f) \in \{3, 4, 5, 6, 7\}$ around a facet f in a Catmull-Clark mesh. Each expression $\mathrm{vol}(f)$ involves a different trilinear form $Y_{i,j,k}^{\tau(f)}$. Green indicates the regular case. ■

We show that the coefficients $Y_{i,j,k}^{\tau(f)} := A_{i,j,k}^{\tau(f)} - A_{i,k,j}^{\tau(f)}$ depend only on the subdivision rules that determine the surface corresponding to the topology $\tau(f)$ of facet f. $A^{\tau(f)}$ are obtained by solving a system of linear equations. Once the trilinear forms $Y^{\tau(f)}$ are established for a subdivision scheme, the formula applies to any closed, orientable mesh M.

The guiding principle to obtain the coefficients is that the volume formula has to be invariant under one round of subdivision of the mesh

$$\mathrm{vol}(M) = \mathrm{vol}(S(M)),$$

since that operation does not change the limit surface. The careful choice of the partition with the facets allows to reduce the equation to

$$\mathrm{vol}(f) = \sum_h \mathrm{vol}(f_h)$$

where f_h denotes the collection of facets that are the result of subdividing facet f. We define the real valued trilinear form that maps a facet as $u(f) := \sum_{i,j,k}^{m(f)} A_{i,j,k}^{\tau(f)}\, \mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k$ with $(\mathrm{px}_i, \mathrm{py}_i, \mathrm{pz}_i)$ as the points that determine the surface over facet f. The transpose of u is defined as $u^\top(f) := \sum_{i,j,k}^{m(f)} A_{i,k,j}^{\tau(f)}\, \mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k$ where A has two indices swapped. We impose the relation

$$u(f) = \sum_h u(f_h)$$

that implies $u(M) := \sum_{f\in M} u(f) = u(S(M))$. Therefore, $\sum_{f\in M} u(f) - u^\top(f)$ is a candidate for the volume enclosed by the subdivision surface defined by mesh M. In fact, the value of $\mathrm{vol}(f) = u(f) - u^{\top}(f)$ is not meaningful unless added up globally over all facets f of a closed, orientable mesh M.

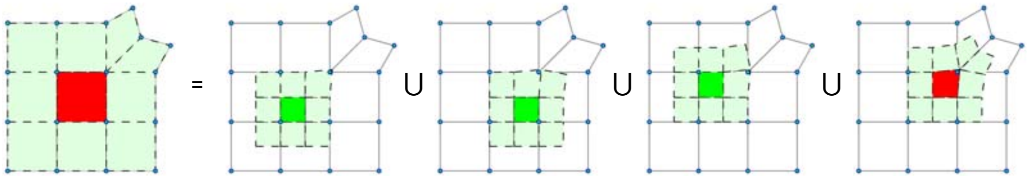


Figure: Catmull-Clark subdivision of a facet with a vertex of valence 5 into 3 regular facets, and one facet with a valence 5 vertex. ■

Remark: One round of subdivision with either of the schemes Doo-Sabin, Catmull-Clark, and Loop partitions a facet f into 4 facets f_h for $h \in \{1, 2, 3, 4\}$. If f is regular, all f_h are regular. If f has a non-regular one-ring, the new partition contains 3 regular facets f_1, f_2, f_3, and one non-regular facet f_4 that has the same topology type as f, i.e. $\tau(f) = \tau(f_4)$. ■

A surface subdivision scheme S typically partitions a facet f into 4 smaller facets $S(f) \to \{f_1, f_2, f_3, f_4\}$ in the refined mesh, which is what we assume henceforth to keep the notation reasonable.

Subdivision of the control points of facet f to the control points of f_h is a coordinatewise, linear mapping that we express as the matrix S^h with dimensions $m(f_h) \times m(f)$ for $h \in \{1, 2, 3, 4\}$. We write explicitly

$$u(f_h) = \sum_{a,b,c}^{m(f_h)} A_{a,b,c}^{\tau(f_h)} \left(\sum_{i=1}^{m(f)} S_{a,i}^h \,\mathrm{px}_i\right) \left(\sum_{j=1}^{m(f)} S_{b,j}^h \,\mathrm{py}_j\right) \left(\sum_{k=1}^{m(f)} S_{c,k}^h \,\mathrm{pz}_k\right) \qquad \text{for } h \in \{1, 2, 3, 4\}.$$

Considering all products $\mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k$ as a basis, we obtain a total of $m(f)^3$ equations from $u(f) = \sum_h u(f_h)$,

$$A_{i,j,k}^{\tau(f)} = \sum_{a,b,c}^{m(f_1)} A_{a,b,c}^{\tau(f_1)} S_{a,i}^1 S_{b,j}^1 S_{c,k}^1 + \sum_{a,b,c}^{m(f_2)} A_{a,b,c}^{\tau(f_2)} S_{a,i}^2 S_{b,j}^2 S_{c,k}^2 + \sum_{a,b,c}^{m(f_3)} A_{a,b,c}^{\tau(f_3)} S_{a,i}^3 S_{b,j}^3 S_{c,k}^3 + \sum_{a,b,c}^{m(f_4)} A_{a,b,c}^{\tau(f_4)} S_{a,i}^4 S_{b,j}^4 S_{c,k}^4$$

for all $i, j, k = 1, 2, \ldots, m(f)$. The equations help to solve for unknown coefficients $A_{i,j,k}^{\tau(f)}$. We emphasize on two scenarios that are relevant to derive the volume forms for Doo-Sabin, Catmull-Clark, as well as Loop subdivision surfaces.

To enumerate the triple index (i, j, k) in a linear fashion, we write $\sharp(i, j, k) := i + (j-1)\, m(f) + (k-1)\, m(f)^2$.

Corollary 1: Let the facets f, and f_h be regular, $\tau(f) = \tau(f_h)$ for all $h \in \{1, 2, 3, 4\}$. If the coefficients $A_{i,j,k}^{\tau(f)}$ are unknown, the equations become the linear system $(E - I).x = 0$ where

$$E_{\sharp(i,j,k), \sharp(a,b,c)} = \sum_{h=1}^{4} S_{a,i}^h S_{b,j}^h S_{c,k}^h,$$

and I is the identity matrix. x is the vector with $x_{\sharp(i,j,k)} = A_{i,j,k}^{\tau(f)}$. A solution x is an element in the nullspace of the matrix $E - I$. ■

Corollary 2: Let the two facets f and f_4 be non-regular, $\tau(f) = \tau(f_4)$, the facets f_1, f_2, f_3 regular, and $A^{\tau(f_1)} = A^{\tau(f_2)} = A^{\tau(f_3)}$ known. If the coefficients $A_{i,j,k}^{\tau(f)}$ are unknown, the equations are the linear system $(F - I).x = b$ where

$$F_{\sharp(i,j,k), \sharp(a,b,c)} = S_{a,i}^4 S_{b,j}^4 S_{c,k}^4,$$

and I is the identity matrix. The vector b contains all known quantities from the rhs

$$b_{\sharp(i,j,k)} = -\left(\sum_{a,b,c}^{m(f)} A_{a,b,c}^{\tau(f_1)} S_{a,i}^1 S_{b,j}^1 S_{c,k}^1 + \sum_{a,b,c}^{m(f)} A_{a,b,c}^{\tau(f_2)} S_{a,i}^2 S_{b,j}^2 S_{c,k}^2 + \sum_{a,b,c}^{m(f)} A_{a,b,c}^{\tau(f_3)} S_{a,i}^3 S_{b,j}^3 S_{c,k}^3\right).$$

The vector x with $x_{\sharp(i,j,k)} = A_{i,j,k}^{\tau(f)}$ is a solution to the system of linear equations. ■

The matrix F in Corollary 2 defines a tensor product subdivision scheme. If the matrix S^4 has an eigenvalue 1 with multiplicity 1, and all other eigenvalues have absolute value < 1, then F also has an eigenvalue 1 with multiplicity 1, and all other eigenvalues have absolute value < 1. That means the matrix $F - I$ has 1-dimensional nullspace. Equivalently, $\text{rank}(F-I) = m(f)^3 - 1$. For Doo-Sabin, Catmull-Clark, and Loop this is precisely the case for all non-regular topologies. We capture an important consequence in a Lemma.

Lemma 1: Let matrix S^4 have eigenvalue 1 with multiplicity 1, and all other eigenvalues absolute value <1. Any element w from the nullspace of matrix $F-I$ is mapped to 0 when performing the skew operation $w_{\sharp(i,j,k)} - w_{\sharp(i,k,j)}$ for all $i, j, k = 1, 2, \ldots, m(f)$.

Proof: Let v be the right eigenvector to eigenvalue 1 of S^4 with $v.S^4 = v$. Then, the eigenvector w to eigenvalue 1 of F with $F.w = w$ is $w_{\sharp(a,b,c)} = v_a\, v_b\, v_c$, since

$$\sum_{a,b,c}^{m(f)} F_{\sharp(i,j,k),\sharp(a,b,c)}.w_{\sharp(a,b,c)} = \sum_{a,b,c}^{m(f)} S^4_{a,i}\, S^4_{b,j}\, S^4_{c,k}\, v_a\, v_b\, v_c$$
$$= \sum_{a=1}^{m(f)} v_a\, S^4_{a,i} \cdot \sum_{b=1}^{m(f)} v_b\, S^4_{b,j} \cdot \sum_{c=1}^{m(f)} v_c\, S^4_{c,k} = v_i\, v_j\, v_k = w_{\sharp(i,j,k)}$$

Naturally, $w_{\sharp(i,j,k)} - w_{\sharp(i,k,j)} = v_i\left(v_j\, v_k - v_k\, v_j\right) = 0$. ■

Because of the 1-dimensional nullspace, $A^{\tau(f)}$ for non-regular $\tau(f)$ are not uniquely determined by Corollary 2. However, the Lemma asserts that $Y^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{i,j,k} - A^{\tau(f)}_{i,k,j}$ follow uniquely.

A particular solution $A^{\tau(f)}$ exists as an integral expression. We denote this special trilinear form with $\overline{A}^{\tau(f)}$. The concept has been presented before, for instance in [Peters/Nasri 1997]. We put the derivation as follows: The divergence theorem in three dimensions states that for a smooth vector field $G : \mathbb{R}^3 \to \mathbb{R}^3$ and a compact subset $V \subset \mathbb{R}^3$ with piecewise smooth boundary ∂V and surface normal $\vec{n}$

$$\int_V \text{div}\, G\, dV = \int_{\partial V} G \cdot \vec{n}\, d(\partial V)$$

Selecting $G(x, y, z) = (x, 0, 0)$ results in $\text{div}\, G = 1$ and

$$\text{vol}(V) = \int_V 1\, dV = \int_{\partial V} (x, 0, 0) \cdot \vec{n}\, d(\partial V) = \int_{\partial V} x\, n_x\, d(\partial V)$$

We parameterize the subdivision surface corresponding to a facet f with $\phi : D \to \mathbb{R}^3$ as

$$\phi(s, t) = \begin{pmatrix} \sum_{i=1}^{m(f)} \text{px}_i\, B_i(s, t) \\ \sum_{i=1}^{m(f)} \text{py}_i\, B_i(s, t) \\ \sum_{i=1}^{m(f)} \text{pz}_i\, B_i(s, t) \end{pmatrix}$$

where $D \subset \mathbb{R}^2$, and $B_i : D \to \mathbb{R}$ denotes the basis function characteristic to the subdivision scheme over the facet contributed by control point $i = 1, 2, \ldots, m(f)$.

Example: When the facet $f \in \mathcal{M}$ is of quad type, we choose the unit square $D = [0, 1] \times [0, 1]$ as the domain. When f is triangular, $D = \left\{(s, t) \in \mathbb{R}^2 : 0 \le s, t \wedge s + t \le 1\right\}$ is the canonic choice. ■

We use the substitution $\alpha(s, t) := \sqrt{\left|\det\left(\text{d}\phi(s, t)^T.\text{d}\phi(s, t)\right)\right|}$, where

$$\text{d}\phi(s, t) = \begin{pmatrix} \sum_{i=1}^{m(f)} \text{px}_i\, \partial_s B_i(s, t) & \sum_{i=1}^{m(f)} \text{px}_i\, \partial_t B_i(s, t) \\ \sum_{i=1}^{m(f)} \text{py}_i\, \partial_s B_i(s, t) & \sum_{i=1}^{m(f)} \text{py}_i\, \partial_t B_i(s, t) \\ \sum_{i=1}^{m(f)} \text{pz}_i\, \partial_s B_i(s, t) & \sum_{i=1}^{m(f)} \text{pz}_i\, \partial_t B_i(s, t) \end{pmatrix}$$

Let the piecewise smooth boundary ∂V be the subdivision surface. Then, the integral becomes

$$\int_{\partial V} x\, n_x\, d(\partial V)$$
$$= \sum_{f \in \mathcal{M}} \int_D \sum_{i,j,k}^{m(f)} \text{px}_i\, B_i(s, t)\, \frac{\text{py}_j\, \partial_s B_j(s,t)\, \text{pz}_k\, \partial_t B_k(s,t) - \text{pz}_j\, \partial_s B_j(s,t)\, \text{py}_k\, \partial_t B_k(s,t)}{\alpha(s,t)}\, \alpha(s, t)\, ds\, dt$$
$$= \sum_{f \in \mathcal{M}} \sum_{i,j,k}^{m(f)} \text{px}_i\left(\text{py}_j\, \text{pz}_k - \text{py}_k\, \text{pz}_j\right) \int_D B_i(s, t)\, \partial_s B_j(s, t)\, \partial_t B_k(s, t)\, ds\, dt$$

By comparison of formulas, the coefficient $\overline{A}^{\tau(f)}_{i,j,k}$ substitutes the integral that remains in the expression, and satisfies the relation $u(f) = \sum_h u(f_h)$

$$\begin{aligned}
\overline{A}_{i,j,k}^{\tau(f)} &= \int_D B_i(s,t)\,\partial_s B_j(s,t)\,\partial_t B_k(s,t)\,ds\,dt \\
&= \frac{1}{4}\int_{2D} B_i\left(\frac{s}{2},\frac{t}{2}\right)\left(\partial_s B_j\right)\left(\frac{s}{2},\frac{t}{2}\right)\left(\partial_t B_k\right)\left(\frac{s}{2},\frac{t}{2}\right)ds\,dt \\
&= \int_{2D} B_i\left(\frac{s}{2},\frac{t}{2}\right)\partial_s B_j\left(\frac{s}{2},\frac{t}{2}\right)\partial_t B_k\left(\frac{s}{2},\frac{t}{2}\right)ds\,dt \\
&= \sum_{h=1}^4 \int_D \left(\sum_{a=1}^{m(f_h)} S_{a,i}^h\,B_a(s,t)\right)\left(\sum_{b=1}^{m(f_h)} S_{b,j}^h\,\partial_s B_b(s,t)\right)\left(\sum_{c=1}^{m(f_h)} S_{c,k}^h\,\partial_t B_c(s,t)\right)ds\,dt \\
&= \sum_{h=1}^4 \sum_{a,b,c}^{m(f_h)}\left(\int_D B_a(s,t)\,\partial_s B_b(s,t)\,\partial_t B_c(s,t)\,ds\,dt\right)S_{a,i}^h\,S_{b,j}^h\,S_{c,k}^h \\
&= \sum_{h=1}^4 \sum_{a,b,c}^{m(f_h)} \overline{A}_{a,b,c}^{\tau(f_h)}\,S_{a,i}^h\,S_{b,j}^h\,S_{c,k}^h
\end{aligned}$$

With $\overline{Y}_{i,j,k}^{\tau(f)} := \overline{A}_{i,j,k}^{\tau(f)} - \overline{A}_{i,k,j}^{\tau(f)}$ the divergence theorem asserts $\mathrm{vol}(M) = \sum_{f\in M}\sum_{i,j,k}^{m(f)} \overline{Y}_{i,j,k}^{\tau(f)}\,\mathrm{px}_i\,\mathrm{py}_j\,\mathrm{pz}_k$.

The choice $G(x,y,z) = (0,y,0)$ results in $\overline{Y}_{j,k,i}^{\tau(f)}$, and $G(x,y,z) = (0,0,z)$ corresponds to $\overline{Y}_{k,i,j}^{\tau(f)}$. By construction, we have $-\overline{Y}_{i,k,j}^{\tau(f)} = -\overline{A}_{i,k,j}^{\tau(f)} + \overline{A}_{i,j,k}^{\tau(f)} = \overline{Y}_{i,j,k}^{\tau(f)}$, $-\overline{Y}_{j,i,k}^{\tau(f)} = \overline{Y}_{j,k,i}^{\tau(f)}$, and $-\overline{Y}_{k,j,i}^{\tau(f)} = \overline{Y}_{k,i,j}^{\tau(f)}$. The average of all 6 permutations of the volume formula, results in the alternating trilinear form

$$\hat{Y}_{i,j,k}^{\tau(f)} := \frac{1}{6}\left(\overline{Y}_{i,j,k}^{\tau(f)} - \overline{Y}_{i,k,j}^{\tau(f)} + \overline{Y}_{j,k,i}^{\tau(f)} - \overline{Y}_{j,i,k}^{\tau(f)} + \overline{Y}_{k,i,j}^{\tau(f)} - \overline{Y}_{k,j,i}^{\tau(f)}\right) \qquad \text{for all } \tau(f)$$

that also satisfies $\mathrm{vol}(M) = \frac{1}{6}\,6\,\mathrm{vol}(M) = \sum_{f\in M}\sum_{i,j,k}^{m(f)} \hat{Y}_{i,j,k}^{\tau(f)}\,\mathrm{px}_i\,\mathrm{py}_j\,\mathrm{pz}_k$. Additionally, any affine combination of volume forms also constitutes a valid volume formula. For instance,

$$Y^{\tau(f)} = (1-\beta)\,\hat{Y}^{\tau(f)} + \beta\,\overline{Y}^{\tau(f)} = \hat{Y}^{\tau(f)} + \beta\left(\overline{Y}^{\tau(f)} - \hat{Y}^{\tau(f)}\right) \qquad \text{for any } \beta\in\mathbb{R}.$$

When the basis functions B_i for $i = 1, 2, \ldots, m(f)$ are polynomials, the evaluation of the integrals is the straightforward way to obtain a solution to $u(f) = \sum_h u(f_h)$. Generally, the basis functions B_i do not have a closed form expression, albeit properties such as smoothness, and boundedness are known [Reif 1995]. Corollaries 1 and 2 allow us to study the solution space of volume forms regardless of the availability of the basis functions.

Applications

We apply Corollaries 1 and 2 to the Doo-Sabin, Catmull-Clark, and Loop subdivision schemes. We derive the solution space of the trilinear forms $A^{\tau(f)}$ that satisfy the equation $u(f) = \sum_h u(f_h)$. For each scheme, we treat the regular case first. The forms $Y^{\tau(f)}$ for non-regular valences follow uniquely according to Lemma 1. For Doo-Sabin and Catmull-Clark, we also identify $\overline{Y}^{\tau(f)}$ for comparison.

Since the number of coefficients $A_{i,j,k}^{\tau(f)}$ as well as the size of the matrix $F - I$ grow with the valence of the non-regular vertex, there is a limit to how many topologies $\tau(f)$ we can cover in practice.

For all three algorithms there is a unique set of volume forms $\hat{Y}^{\tau(f)}$ that are also alternating trilinear forms. The coefficients $\hat{Y}_{i,j,k}^{\tau(f)}$ are available for download, see [Hakenberg 2014].

Doo-Sabin

A surface generated by the Doo-Sabin subdivision scheme is parameterized by a partition of quad facets. A facet f is associated to a vertex v of the two-times subdivided initial mesh. To indicate the topology type of f, we choose $\tau(f)$ as the number of vertices in the non-regular face adjacent to vertex v, or $\tau(f) = 4$ in the regular case. The surface parameterized by facet f is determined by $m(f) = 5 + \tau(f)$ vertices.

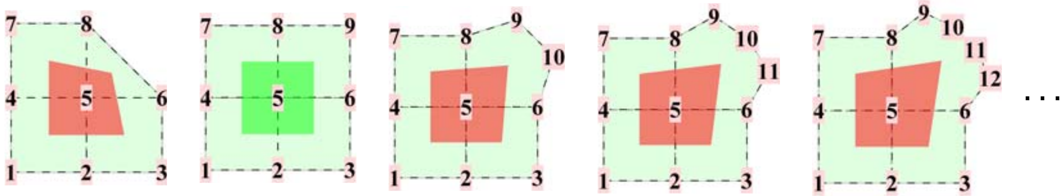


Figure: Facets of a Doo-Sabin mesh with $\tau(f) \in \{3, 4, \ldots, 7\}$, and $m(f) \in \{8, 9, \ldots, 12\}$, and indexing of the control points. ■

Regular facet

In the regular case $\tau(f) = 4$, the surface associated to f is determined by $m(f) = 9$ control points. We use Corollary 1 to obtain the coefficients $Y^4_{i,j,k} = A^4_{i,j,k} - A^4_{i,k,j}$. Besides the linear system $(E - I).x = 0$ provided for A^4, we impose the obvious symmetries: Rotational invariance is expressed as $A^4_{i,j,k} - A^4_{\rho(i),\rho(j),\rho(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 9$ where $\rho(1) = 3$, $\rho(2) = 6$, $\rho(3) = 9$, $\rho(4) = 2$, ..., $\rho(9) = 7$. Inversion of sign when inverting the order of vertices means $A^4_{i,j,k} + A^4_{\sigma(i),\sigma(j),\sigma(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 9$ where $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$, ..., $\sigma(9) = 7$ is a mirror operation.

Symbolic computation shows that the nullspace of the combined linear system is 3-dimensional. Of the 3 dimensions, a 1-dimensional subspace is projected to 0 when forming the skew trilinear form $Y^4_{i,j,k} = A^4_{i,j,k} - A^4_{i,k,j}$. We use another 1-dimensional subspace to calibrate the form Y^4 to match a known volume: For calibration we construct a closed quad mesh M with surface invariant under Doo-Sabin subdivision. We begin with a cube with all edges of length 1. Initially, the cube mesh has 8 vertices, each with valence 3. We linearly subdivide the quads of the mesh 2 times. Then, each vertex of the refined mesh is moved to the position of the cube vertex that is closest.

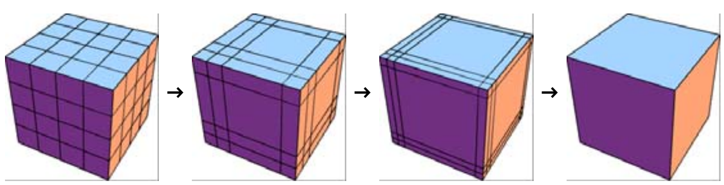


Figure: Construction of the degenerate cube mesh M: Linear subdivision followed by vertex collapse. ■

The degenerate cube mesh M has the following properties: 1) each of the 8 non-regular vertices are topologically surrounded by regular vertices, 2) the one-ring of each facet associated to a non-regular vertex is degenerated to a single vertex, thus the volume contribution of such a facet is 0 regardless of the choice of coefficients $A^3_{i,j,k}$, 3) the subdivision surface defined by Doo-Sabin is the cube of volume 1 that we started with.

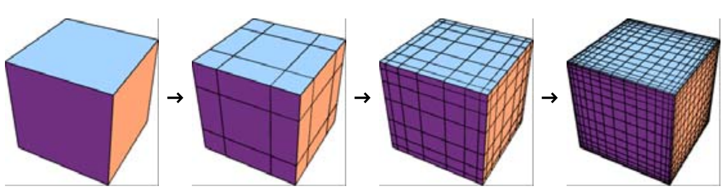


Figure: 3 rounds of Doo-Sabin subdivision of the degenerate mesh. ■

The remaining degree of freedom can be chosen arbitrarily, but affects $A^{\tau(f)}$ when f is non-regular. The 1-dimensional solution space is identical to $\hat{Y}^4 + \beta\left(\overline{Y}^4 - \hat{Y}^4\right)$ for $\beta \in \mathbb{R}$. The trilinear form $\overline{Y}^4$ is $\overline{Y}^4_{i,j,k} = \overline{A}^4_{i,j,k} - \overline{A}^4_{i,k,j}$. The coefficients $\overline{A}^4_{i,j,k}$ for all $i, j, k = 1, 2, \ldots, 9$ are obtained by evaluating the integral expression. The basis functions are the 9 polynomials in $(s, t) \in D = [0, 1] \times [0, 1]$

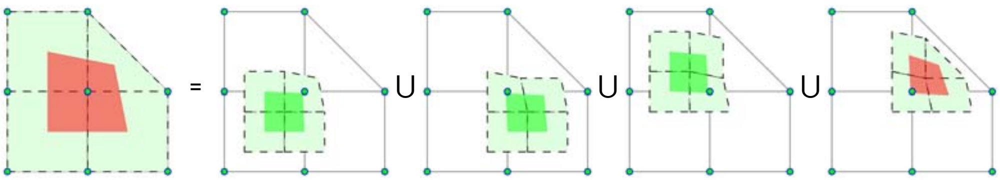
$B_1(s,t) = \frac{1}{4}(s-1)^2(t-1)^2$, $B_2(s,t) = \frac{1}{2}\left(\frac{1}{2}+s-s^2\right)(t-1)^2$, $B_3(s,t) = \frac{1}{4}s^2(t-1)^2$, $B_4(s,t) = \frac{1}{2}(s-1)^2\left(\frac{1}{2}+t-t^2\right)$, $B_5(s,t) = \left(\frac{1}{2}+s-s^2\right)\left(\frac{1}{2}+t-t^2\right)$, $B_6(s,t) = \frac{1}{2}s^2\left(\frac{1}{2}+t-t^2\right)$, $B_7(s,t) = \frac{1}{4}(s-1)^2 t^2$, $B_8(s,t) = \frac{1}{2}\left(\frac{1}{2}+s-s^2\right)t^2$, and $B_9(s,t) = \frac{1}{4}s^2 t^2$.

Example:

$$\overline{A}^4_{8,1,2} = \int_{[0,1]^2} B_8(s,t)\,\partial_s B_1(s,t)\ \partial_t B_2(s,t)\,ds\,dt$$
$$= \int_{[0,1]^2} \frac{1}{2}\left(\frac{1}{2}+s-s^2\right)t^2 \cdot \frac{1}{2}(s-1)(t-1)^2 \cdot \left(\frac{1}{2}+s-s^2\right)(t-1)\,ds\,dt = \frac{3}{3200},$$
$$\hat{A}^4_{8,1,2} = \frac{11}{19\,200}. \blacksquare$$

Non-regular facet

We use Corollary 2 to obtain solutions for $Y^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{i,j,k} - A^{\tau(f)}_{i,k,j}$ for valences $\tau(f) \in \{3, 5, 6, \ldots, 12\}$. For $\tau(f) \in \{3, 5, 6, 8, 10, 12\}$ the Doo-Sabin subdivision weights are rational, or involve a single square root. For these topologies we establish $Y^{\tau(f)}$ in symbolic form. The trilinear forms for valences $\tau(f) \in \{7, 9, 11\}$ are obtained numerically.



Example: For valence $\tau(f) = 3$, the facet decomposition is determined by $m(f) = m(f_4) = 8$ initial control points. The matrix S^h maps the control points of facet f to the control points of f_h for $h \in \{1, 2, 3, 4\}$ during one round of subdivision. The matrices are

$$S^1 = \frac{1}{16}\begin{pmatrix} 9 & 3 & 0 & 3 & 1 & 0 & 0 & 0 \\ 3 & 9 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 9 & 3 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 9 & 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 9 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 9 & 3 & 0 & 0 \\ 0 & 0 & 0 & 9 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 & 9 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \frac{32}{3} & \frac{8}{3} & 0 & \frac{8}{3} \end{pmatrix} \quad S^2 = \frac{1}{16}\begin{pmatrix} 3 & 9 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 9 & 3 & 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 9 & 0 & 1 & 3 & 0 & 0 \\ 1 & 3 & 0 & 3 & 9 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 9 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 3 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 & 9 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \frac{32}{3} & \frac{8}{3} & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & \frac{8}{3} & \frac{32}{3} & 0 & \frac{8}{3} \end{pmatrix}$$

$$S^3 = \frac{1}{16}\begin{pmatrix} 3 & 1 & 0 & 9 & 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 9 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 9 & 3 & 0 & 0 \\ 0 & 0 & 0 & 9 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 & 9 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \frac{32}{3} & \frac{8}{3} & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 3 & 1 & 0 & 9 & 3 \\ 0 & 0 & 0 & 1 & 3 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & \frac{8}{3} & \frac{8}{3} & 0 & \frac{32}{3} \end{pmatrix} \quad S^4 = \frac{1}{16}\begin{pmatrix} 1 & 3 & 0 & 3 & 9 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 9 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 3 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 & 9 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & \frac{32}{3} & \frac{8}{3} & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & \frac{8}{3} & \frac{32}{3} & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 1 & 3 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & \frac{8}{3} & \frac{8}{3} & 0 & \frac{32}{3} \end{pmatrix}$$

The indexing means $S^4_{7,8} = \frac{9}{16}$. The eigenvalues of S^4 are $\left[1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{16}\right]$ in descending order. A right eigenvector to eigenvalue 1 is $v = [0, 0, 0, 0, 1, 1, 0, 1]$. Lemma 1 asserts that Y^3 follows uniquely from the choice of A^4. For instance, $\overline{Y}^3_{8,7,2} = \frac{9\,660\,923}{5\,782\,665\,600}$, and $\hat{Y}^3_{8,7,2} = \frac{3\,324\,559}{2\,891\,332\,800}$. $\blacksquare$

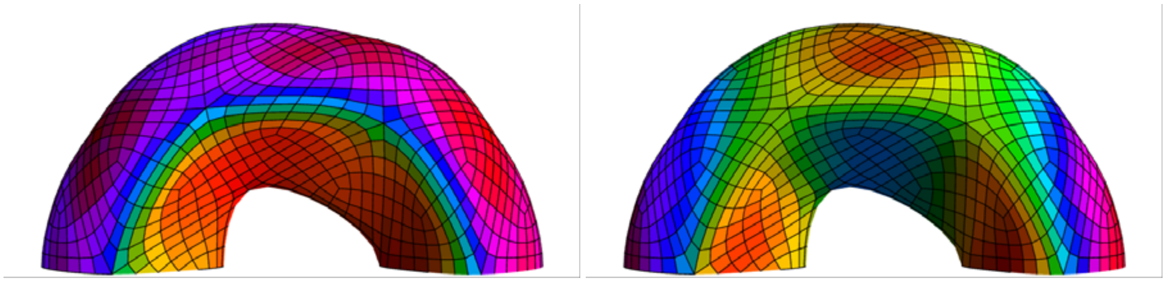


Figure: Facets of a Doo-Sabin mesh colored based on their volume contribution vol(f). Left uses the alternating forms $\hat{Y}^{\tau(f)}$, and right visualizes the difference $\overline{Y}^{\tau(f)} - \hat{Y}^{\tau(f)}$. ■

Remark: Mirror symmetry implies $A^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{\sigma(i),\sigma(k),\sigma(j)}$ for all $i, j, k = 1, 2, \ldots, m(f)$, where σ maps the index to the counterpart opposite of the diagonal. When $\tau(f) = 5$ for instance, the map is $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 7$, ..., $\sigma(6) = 8$, ..., $\sigma(10) = 9$. Making use of this relation reduces the number of variables approximately by half. Moreover, for an index (i, j, k) with all $i, j, k \in \{1, 2, 3, 4, 7\}$, we have $A^{\tau(f)}_{i,j,k} = A^4_{i,j,k}$ for all $\tau(f)$. ■

Catmull-Clark

A surface defined by the Catmull-Clark subdivision scheme is parameterized by a partition of quad facets. A facet f corresponds to a quad of the one-time subdivided initial mesh. We choose $\tau(f)$ as the valence of the non-regular vertex of the quad, or $\tau(f) = 4$ for regular f. The number of control points in the one-ring of f is $m(f) = 8 + 2\,\tau(f)$.

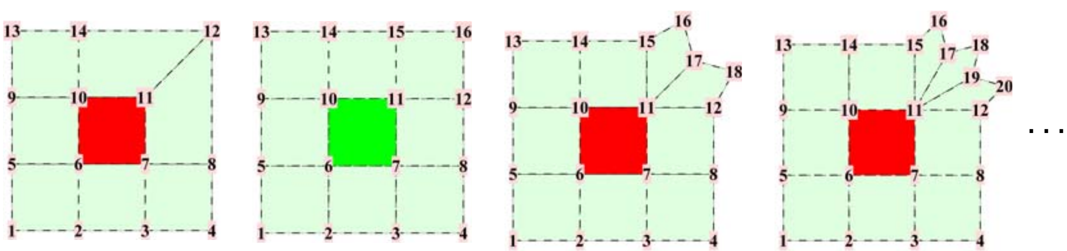


Figure: Catmull-Clark facet topologies $\tau(f) \in \{3, 4, 5, 6\}$, and indexing of the control points. ■

Regular facet

In the regular case $\tau(f) = 4$, the surface parameterized by f is determined by $m(f) = 16$ control points. We use Corollary 1 to obtain the coefficients $Y^4_{i,j,k} = A^4_{i,j,k} - A^4_{i,k,j}$. Besides the linear system $(E - I).x = 0$ provided for A^4, we impose the obvious symmetries: Rotational invariance is expressed as $A^4_{i,j,k} - A^4_{\rho(i),\rho(j),\rho(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 16$ where $\rho(1) = 4$, $\rho(2) = 8$, $\rho(3) = 12$, $\rho(4) = 16$, ..., $\rho(15) = 9$, $\rho(16) = 13$. Inversion of sign when inverting the order of vertices means $A^4_{i,j,k} + A^4_{\sigma(i),\sigma(j),\sigma(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 16$ where $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 2$, $\sigma(4) = 1$, ..., $\sigma(15) = 14$, $\sigma(16) = 13$ is a mirror operation.

Symbolic computation shows that the nullspace of the combined linear system is 3-dimensional. Of the 3 dimensions, a 1-dimensional subspace is projected to 0 when forming the skew trilinear form $Y^4_{i,j,k} = A^4_{i,j,k} - A^4_{i,k,j}$. We use another 1-dimensional subspace to calibrate the form Y^4 to match a known volume: For calibration we construct a closed quad mesh $\mathcal{M}$ with surface invariant under Catmull-Clark subdivision. We begin with a cube with all edges of length 1. Initially, the cube mesh has 8 vertices, each with valence 3. We linearly subdivide the quads of the mesh 3 times. Then, each vertex of the refined mesh is moved to the position of the cube vertex that is closest.

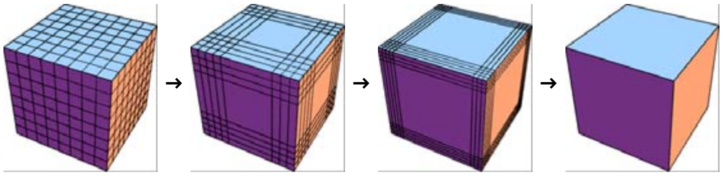


Figure: Construction of the degenerate cube mesh M: Linear subdivision followed by vertex collapse. ■

The degenerate cube mesh M has the following properties: 1) each of the 8 non-regular vertices are topologically surrounded by regular vertices, 2) the one-ring of each facet adjacent to a non-regular vertex is degenerated to a single vertex, thus the volume contribution of such a facet is 0 regardless of the choice of coefficients $A^3_{i,j,k}$, 3) the subdivision surface defined by Catmull-Clark is the cube of volume 1 that we started with.

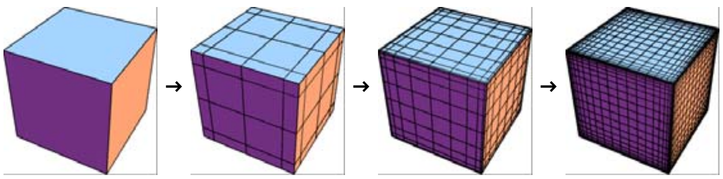


Figure: 3 rounds of Catmull-Clark subdivision of the degenerate mesh. ■

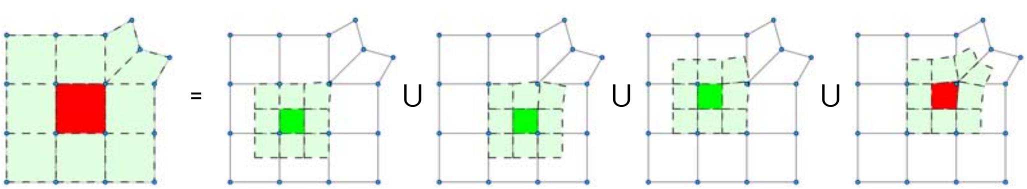
The remaining degree of freedom can be chosen arbitrarily, but affects $A^{\tau(f)}$ when f is non-regular. The 1-dimensional solution space is identical to $\hat{Y}^4 + \beta\left(\overline{Y}^4 - \hat{Y}^4\right)$ for $\beta \in \mathbb{R}$. The trilinear form $\overline{Y}^4$ is $\overline{Y}^4_{i,j,k} = \overline{A}^4_{i,j,k} - \overline{A}^4_{i,k,j}$. The coefficients $\overline{A}^4_{i,j,k}$ for $i,\ j,\ k = 1,\ 2,\ \ldots,\ 16$ are obtained by evaluating the integral expression. The basis functions are the 16 polynomials in $(s,\ t) \in D = [0,\ 1] \times [0,\ 1]$

$B_1(s,t) = \frac{1}{36}(s-1)^3(t-1)^3$, $B_2(s,t) = -\frac{1}{36}\left(4-6s^2+3s^3\right)(t-1)^3$, $B_3(s,t) = -\frac{1}{36}\left(1+3s+3s^2-3s^3\right)(t-1)^3$,
$B_4(s,t) = -\frac{1}{36}s^3(t-1)^3$, $B_5(s,t) = -\frac{1}{36}(s-1)^3\left(4-6t^2+3t^3\right)$, $B_6(s,t) = \frac{1}{36}\left(4-6s^2+3s^3\right)\left(4-6t^2+3t^3\right)$,
$B_7(s,t) = \frac{1}{36}\left(1+3s+3s^2-3s^3\right)\left(4-6t^2+3t^3\right)$, $B_8(s,t) = \frac{1}{36}s^3\left(4-6t^2+3t^3\right)$, $B_9(s,t) = -\frac{1}{36}(s-1)^3\left(1+3t+3t^2-3t^3\right)$,
$B_{10}(s,t) = \frac{1}{36}\left(4-6s^2+3s^3\right)\left(1+3t+3t^2-3t^3\right)$, $B_{11}(s,t) = \frac{1}{36}\left(1+3s+3s^2-3s^3\right)\left(1+3t+3t^2-3t^3\right)$,
$B_{12}(s,t) = \frac{1}{36}s^3\left(1+3t+3t^2-3t^3\right)$, $B_{13}(s,t) = -\frac{1}{36}(s-1)^3t^3$, $B_{14}(s,t) = \frac{1}{36}\left(4-6s^2+3s^3\right)t^3$, $B_{15}(s,t) = \frac{1}{36}\left(1+3s+3s^2-3s^3\right)t^3$,
and $B_{16}(s,t) = \frac{1}{36}s^3t^3$.

Example: $\overline{A}^4_{5,1,6} = \frac{3103}{48\,771\,072}$, and $\hat{A}^4_{5,1,6} = \frac{61\,595}{877\,879\,296}$. ■

Non-regular facet

The forms $Y^{\tau(f)}$ follow uniquely from the choice of A^4 by Lemma 1. The subdivision weights are rational. We obtain symbolic solutions for $Y^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{i,j,k} - A^{\tau(f)}_{i,k,j}$ for valences $\tau(f) \in \{3,\ 5,\ 6,\ 7\}$. Due to the growing size of the matrix $F - I$, and limited computational resources, we establish Y^8 only numerically.



Example: The facet decomposition for valence $\tau(f) = 5$ is determined by $m(f) = m(f_4) = 18$ initial control points. The matrices S^1, S^2, S^3 have dimension 16×18. S^4 is a 18×18 matrix with eigenvalues $[1,\ 0.549\ldots,\ \ldots,\ 0.015\ldots]$ in descending order. Lemma 1 applies. For instance,

$\overline{Y}^5_{11,5,3} = \frac{555\,157\,620\,972\,704\,545\,156\,972\,240\,729\,805\,393\,939\,048\,761\,580\,669}{981\,547\,964\,721\,533\,145\,894\,588\,000\,817\,830\,293\,184\,287\,837\,041\,075\,200}$, and

$\hat{Y}^5_{11,5,3} = \frac{17\,370\,080\,190\,337\,845\,057\,804\,377\,871\,061\,038\,705\,914\,533}{87\,765\,797\,398\,722\,937\,794\,631\,199\,843\,811\,365\,617\,667\,174\,400}$. ■

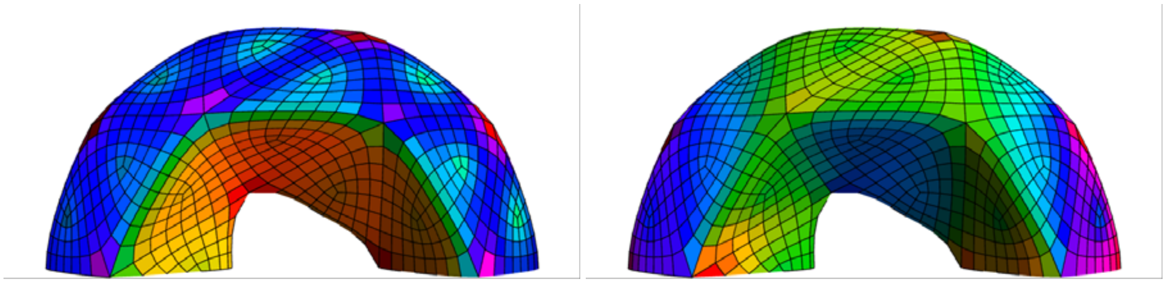


Figure: Facets of a Catmull-Clark mesh colored based on their volume contribution vol(f). Left uses the alternating forms $\hat{Y}^{\tau(f)}$, and rigtht visualizes the difference $\overline{Y}^{\tau(f)} - \hat{Y}^{\tau(f)}$. ■

Remark: Mirror symmetry implies $A^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{\sigma(i),\sigma(k),\sigma(j)}$ for all i, j, $k = 1, 2, \ldots, m(f)$ where σ maps the index to the counterpart opposite of the diagonal. When $\tau(f) = 5$ for instance, the map is $\sigma(1) = 1$, $\sigma(2) = 5$, $\sigma(3) = 9$, $\sigma(4) = 13$, ..., $\sigma(17) = 17$, $\sigma(18) = 16$. Making use of this relation reduces the number of variables approximately by half. Moreover, for an index (i, j, k) with all $i, j, k \in \{1, 2, 3, 4, 5, 9, 13\}$, we have $A^{\tau(f)}_{i,j,k} = A^4_{i,j,k}$ for all $\tau(f)$. ■

Loop

A subdivision surface generated by the Loop algorithm is parameterized by a partition of triangular facets. A facet f corresponds to a triangle of the one-time subdivided initial mesh. As index $\tau(f)$ we choose the valence of the non-regular vertex of f, or $\tau(f) = 6$ when f is regular. The vertices in the one-ring around f completely define the subdivision surface associated to f. Their cardinality is $m(f) = 6 + \tau(f)$.

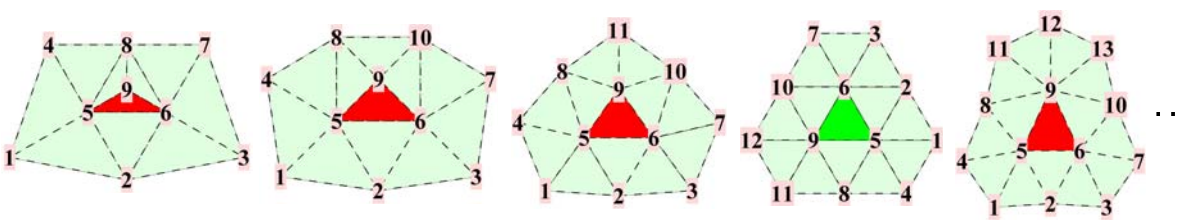


Figure: Indexing of vertices in the one-ring of a triangular facet f for $\tau(f) \in \{3, 4, 5, 6, 7\}$. ■

Regular facet

In the regular case $\tau(f) = 6$, the surface associated to f is determined by $m(f) = 12$ control points. We use Corollary 1 to obtain the coefficients $Y^6_{i,j,k} = A^6_{i,j,k} - A^6_{i,k,j}$. Besides the linear system $(E - I).x = 0$ provided for A^6, we impose the obvious symmetries: Rotational invariance is expressed as $A^6_{i,j,k} - A^6_{\rho(i),\rho(j),\rho(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 12$ where $\rho(1) = 7$, $\rho(2) = 10$, $\rho(3) = 12$, $\rho(4) = 3$, ..., $\rho(12) = 4$. Inversion of sign when inverting the order of vertices means $A^6_{i,j,k} + A^6_{\sigma(i),\sigma(j),\sigma(k)} = 0$ for all $i, j, k = 1, 2, \ldots, 12$ where $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 7$, ..., $\sigma(12) = 11$ is a mirror operation.

Symbolic computation shows that the nullspace of the combined linear system is 3-dimensional. Of the 3 dimensions, a 1-dimensional subspace is projected to 0 when forming the skew trilinear form $Y^6_{i,j,k} = A^6_{i,j,k} - A^6_{i,k,j}$. We use another 1-dimensional subspace to calibrate the form Y^6 to match a known volume: For calibration we construct a closed triangular mesh M with surface invariant under Loop subdivision. We begin with a tetrahedron with all edges of length 1. Initially, the tetrahedron mesh has 4 vertices, each with valence 3. We linearly subdivide the triangles of the mesh 3 times. Then, each vertex of the refined mesh is moved to the position of the tetrahedral vertex that is closest.

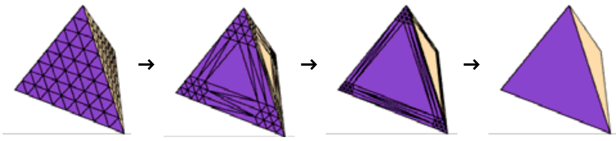


Figure: The construction of the degenerate mesh M: Linear subdivision followed by vertex collapse. ■

The degenerate tetrahedron mesh M has the following properties: 1) each of the 4 non-regular vertices are topologically surrounded by regular vertices, 2) the one-ring of each facet adjacent to a non-regular vertex is degenerated to a single vertex, thus the volume contribution of such a facet is 0 regardless of the choice of coefficients $A^3_{i,j,k}$, 3) the subdivision surface defined by Loop is the tetrahedron of volume $\frac{1}{6\sqrt{2}}$ that we started with.

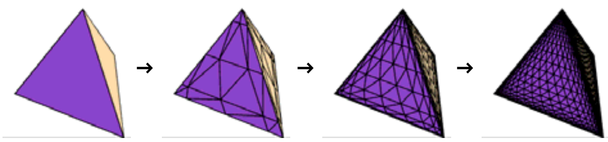


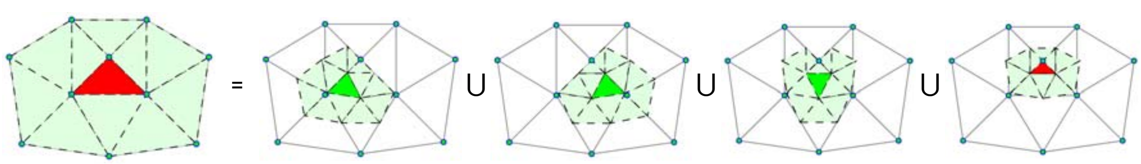
Figure: 3 rounds of Loop subdivision of the degenerate mesh M. ■

The remaining degree of freedom can be chosen arbitrarily, but affects the coefficients $A^{\tau(f)}$ when f is non-regular. The 12 basis functions do not have a closed form expression. We are unable to identify $\overline{A}^6$ that corresponds to the integral expression. The trilinear form $\hat{A}^6$ that is also alternating is a unique solution.

Example: $\hat{A}^6_{5,1,7} = \frac{1787}{119\,750\,400}$. ■

Non-regular facet

We yield the volume forms $Y^{\tau(f)}_{i,j,k} = A^{\tau(f)}_{i,j,k} - A^{\tau(f)}_{i,k,j}$ for non-regular valences $\tau(f) \in \{3, 4, 5, 7, 8, \ldots, 12\}$ from Corollary 2. The matrix $F - I$ has dimensions $m(f)^3 \times m(f)^3$. For instance, for a facet with valence 10, $m(f) = 6 + 10$, and the number of coefficients $A^{10}_{i,j,k}$ is 1000. Our specific choice $\hat{A}^6$ results in alternating forms $\hat{Y}^{\tau(f)}$ for all $3 \le \tau(f) \le 12$. Because of limited computational resources, the trilinear forms for $\tau(f) \in \{7, 9, \ldots, 12\}$ are derived only numerically.



Example: Subdivision of a non-regular facet f with $\tau(f) = 4$. S^4 has eigenvalues $\left[1, \frac{3}{8}, \frac{3}{8}, \frac{9}{64}, \frac{1}{8}, \ldots, \frac{1}{16}\right]$ in descending order. For instance,

$$\hat{Y}^4_{10,8,9} = \frac{22\,663\,731\,969\,915\,204\,014\,725\,535\,462\,759\,500\,947\,683}{18\,792\,363\,864\,674\,566\,857\,783\,671\,734\,314\,366\,207\,000\,000}. \blacksquare$$

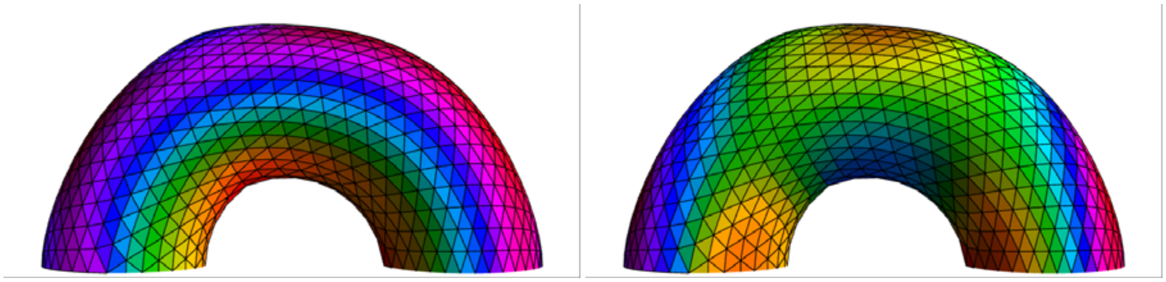
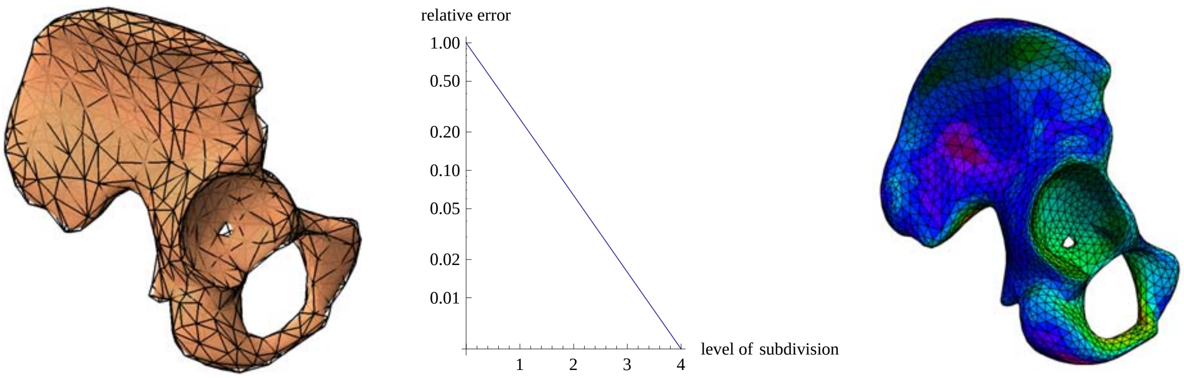


Figure: Facets in a Loop mesh colored based on their volume contribution vol(f). Left uses the alternating forms $\hat{Y}^{\tau(f)}$, and right visualizes the difference $Y^{\tau(f)} - \hat{Y}^{\tau(f)}$. ■



Example: The mesh of a hip bone has vertices with valences $\tau(f) \in \{3, 4, 5, 6, 7, 8, 9, 12\}$. We plot the approximation quality of the piecewise linear meshes defined at different levels of subdivision to the exact volume obtained by our new formula. 2 rounds of subdivision seem to achieve slightly more than 1 digit of decimal precision. Right: Variation of vol(f) across the mesh. ■

Remark: $A_{i,j,k}^{\tau(f)} = -A_{k,j,i}^{\tau(f)}$ for all $i, j, k = 1, 2, \ldots, m(f)$. And, mirror symmetry implies $A_{i,j,k}^{\tau(f)} = -A_{\sigma(i),\sigma(j),\sigma(k)}^{\tau(f)}$ for all $i, j, k = 1, 2, \ldots, m(f)$ where σ maps the index to the counterpart opposite of the diagonal. When $\tau(f) = 5$ for instance, the map is $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$, $\sigma(4) = 7$, ..., $\sigma(10) = 8$, $\sigma(11) = 11$. Making use of this relation reduces the number of variables approximately by factor 1/4. Moreover, for an index (i, j, k) with all $i, j, k \in \{1, 2, 3, 4, 7\}$, we have $A_{i,j,k}^{\tau(f)} = A_{i,j,k}^{4}$ for all $\tau(f)$. ■

Other schemes

Our framework makes it possible to obtain the trilinear forms that compute the volume enclosed by surfaces defined by two other well-known subdivision schemes:

The *Butterfly* algorithm by [Dyn et al. 1990] is an interpolatory subdivision scheme for triangular meshes. A facet is a triangle of the two-times subdivided initial mesh. The surface associated to a facet is determined by the control points in the two-ring of the triangle. For instance, a regular facet has $m(f) = 27$ control points.

[Levin/Levin 2003] and [Schaefer/Warren 2005] define subdivision schemes for mixed triangle/quad meshes. The facets are the quads and triangles of the two-times subdivided initial mesh. For triangular facets away from mixed topologies, the trilinear forms derived for Loop apply. For quad facets away from mixed topologies, the trilinear forms derived for Catmull-Clark apply. For facets adjacent or close to tri-quad interfaces, additional trilinear forms need to be computed. The surface associated to these facets is determined by control points from more than just the one-ring. Different tri-quad configurations around non-regular vertices need to be investigated. Facets adjacent or close to non-regular vertices also have support larger than the one-ring.

Generalization

Requirements for convergence

We assume that the subdivision scheme S has the following property: For each initial mesh M, there is a constant C that is an upper bound for the coordinates (in absolute value) of the vertices in the sequence of meshes $S^n(M)$ at all levels $n = 1, 2, 3, \ldots$. This asserts compactness for the application of the divergence theorem. If the subdivision weights are non-negative, a valid choice is $C = \max_i \max\{|\mathrm{px}_i|, |\mathrm{py}_i|, |\mathrm{pz}_i|\}$.

In order to show that the formula corresponds to the exact volume of the subdivision surface, we make following argument: By construction, our formula satisfies

$$\mathrm{vol}(M) = \mathrm{vol}(S(M)) = \mathrm{vol}(S^n(M)) \qquad \text{for all } n = 1, 2, 3, \ldots$$

The number of non-regular vertices is constant throughout the iteration. We show that the contribution $\mathrm{vol}(f)$ of an non-regular facet $f \in S^n(M)$ converges to 0 as the level of subdivision increases $n \to \infty$. We require that the subdivision scheme S satisfies

$$\left|S^n(f)_i - S^n(f)_j\right| \le \lambda^n \left|f_i - f_j\right| \le \lambda^n 2\,C \qquad \text{for all } n\text{, and facets } f$$

for some fixed $0 < \lambda < 1$ that may depend on the initial mesh M. Here, f_i denotes the control point i in the one-ring of facet f; same for f_j.

$$\mathrm{vol}(f) = \sum_{i,j,k}^{m(f)} \overline{Y}_{i,j,k}^{\tau(f)} \mathrm{px}_i\, \mathrm{py}_j\, \mathrm{pz}_k = \sum_{i,j,k}^{m(f)} \overline{A}_{i,j,k}^{\tau(f)} \mathrm{px}_i \left(\mathrm{py}_j\, \mathrm{pz}_k - \mathrm{py}_k\, \mathrm{pz}_j\right)$$

The term $\mathrm{py}_j\, \mathrm{pz}_k - \mathrm{py}_k\, \mathrm{pz}_j$ is a component of a surface normal direction, but the component is not normalized. We expand the expression as

$$\mathrm{py}_j\, \mathrm{pz}_k - \mathrm{py}_k\, \mathrm{pz}_j = \mathrm{py}_j\left(\mathrm{pz}_k - \mathrm{pz}_j\right) + \left(\mathrm{py}_j - \mathrm{py}_k\right)\left(\mathrm{pz}_k - \mathrm{pz}_j\right) + \mathrm{pz}_j\left(\mathrm{py}_j - \mathrm{py}_k\right)$$

Together we have

$$\mathrm{vol}(f) = \sum_{i,j,k}^{m(f)} \overline{A}_{i,j,k}^{\tau(f)} \mathrm{px}_i\left(\mathrm{py}_j\left(\mathrm{pz}_k - \mathrm{pz}_j\right) + \left(\mathrm{py}_j - \mathrm{py}_k\right)\left(\mathrm{pz}_k - \mathrm{pz}_j\right) + \mathrm{pz}_j\left(\mathrm{py}_j - \mathrm{py}_k\right)\right), \text{ and}$$

$$\left|\mathrm{vol}(S^n(f))\right| \le \sum_{i,j,k}^{m(f)} \left|\overline{A}_{i,j,k}^{\tau(f)}\right| \underbrace{|\tilde{\mathrm{px}}_i|}_{\le C} \left(\underbrace{|\tilde{\mathrm{py}}_j|}_{\le C} \underbrace{\left|\tilde{\mathrm{pz}}_k - \tilde{\mathrm{pz}}_j\right|}_{\le \lambda^n 2C} + \underbrace{\left|\tilde{\mathrm{py}}_j - \tilde{\mathrm{py}}_k\right|}_{\le \lambda^n 2C} \underbrace{\left|\tilde{\mathrm{pz}}_k - \tilde{\mathrm{pz}}_j\right|}_{\le \lambda^n 2C} + \underbrace{|\tilde{\mathrm{pz}}_j|}_{\le C} \underbrace{\left|\tilde{\mathrm{py}}_j - \tilde{\mathrm{py}}_k\right|}_{\le \lambda^n 2C} \right)$$

where $\tilde{\mathrm{px}}_i$ denotes a coordinate of the one-ring of the facet after n rounds of subdivision. Then, the contribution of a non-regular facet f converges as $\lim_{n\to\infty} |\mathrm{vol}(S^n(f))| = 0$, and the volume formula is correct asserted by the divergence theorem.

Momentum of degree d

We derive the coefficients of the multilinear form that yield the momentum of degree d with respect to the x-axis. We choose the vector field $G : \mathbb{R}^3 \to \mathbb{R}^3$ as $G(x, y, z) = \left(x^{d+1}, 0, 0\right)$. Then, $\mathrm{div}\, G = (d+1)\, x^d$ and

$$(d+1) \int_V x^d\, dV = \int_{\partial V} \left(x^{d+1}, 0, 0\right) \cdot \vec{n}\, d(\partial V) = \int_{\partial V} x^{d+1}\, n_x\, d(\partial V)$$

Assuming a homogeneous mass distribution, the momentum of degree $d = 1$ corresponds to the center of gravity along the x-coordinate. The momentum of degree $d = 2$ is the first entry in the inertia tensor.

The momentum of degree d relative to the x-axis, and multiplied by the factor $(d+1)$ is the multilinear form

$$(d+1) \int_V x^d\, dV = \sum_{f \in M} \sum_{i_1,\ldots,i_{d+1},j,k}^{m(f)} \left(A_{i_1,\ldots,i_{d+1},j,k}^{\tau(f)} - A_{i_1,\ldots,i_{d+1},k,j}^{\tau(f)}\right) \mathrm{px}_{i_1} \ldots \mathrm{px}_{i_{d+1}}\, \mathrm{py}_j\, \mathrm{pz}_k$$

A particular solution is given by

$$\overline{A}^{\tau(f)}_{i_1,\ldots,i_{d+1},j,k} = \int_D B_{i_1}(s,t) \ldots B_{i_{d+1}}(s,t)\, \partial_s B_j(s,t)\, \partial_t B_k(s,t)\, ds\, dt$$

Corollaries 1 and 2 are adapted easily. Matrices E, and F have the entries

$$E_{\#(i_1,\ldots,i_{d+1},j,k),\#(a_1,\ldots,a_{d+1},b,c)} = \sum_{h=1}^{4} \overset{h}{S}{}^{a_1}_{i_1} \ldots \overset{h}{S}{}^{a_{d+1}}_{i_{d+1}}\, \overset{h}{S}{}^{b}_{j}\, \overset{h}{S}{}^{c}_{k}, \text{ and}$$

$$F_{\#(i_1,\ldots,i_{d+1},j,k),\#(a_1,\ldots,a_{d+1},b,c)} = \overset{4}{S}{}^{a_1}_{i_1} \ldots \overset{4}{S}{}^{a_{d+1}}_{i_{d+1}}\, \overset{4}{S}{}^{b}_{j}\, \overset{4}{S}{}^{c}_{k}.$$

The coefficients $A^{\tau(f)}_{i_1,\ldots,i_{d+1},j,k}$ depend only on the subdivision rules for facet topology $\tau(f)$. The number of coefficients is $m(f)^{d+3}$. That is humongous, and our straightforward approach is not practical for $1 \leq d$ as of 2014.

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