3D Green's function for scattering integrals

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A three dimensional Green's function is derived mathematically by using the two dimensional Green's function, which is obtained as a result of the modified theory of physical optics' algorithm. The integration of the three dimensional Green's function leads to the two dimensional one when there spatial symmetry in the scatterer's geometry and the incident wave. In order to obtain the three dimensional function, the two dimensional Green's function is mathematically transformed into an infinite integral according to z'. The derived Green's function is generalized and expressed in a scattering integral for soft and hard surfaces.

The scattering process of waves is generally investigated by the methods of physical optics (PO) or Kirchhoff's diffraction integral. For continuous surfaces, e.g. whole plane, these methods yield the exact solution. However the PO and Kirchhoff's diffraction integral will give incorrect wave expressions if the scatterer has discontinuities like edges [1]. In 2004, we showed that it was possible to obtain the exact solution of the diffraction problem of waves by a perfectly conducting half-plane with the PO method [2]. The new theory, modified theory of physical optics (MTPO), is based on three axioms. We directly obtained a novel two dimensional (2D) Green's function by the application of these axioms to the edge diffraction problem. The function was in the same structure with the one that was heuristically introduced by Gori [3]. The 2D Green's function is directional, since it has its maximum radiation in one direction and zero emission in the opposite side. However most of the scattering problems are three dimensional (3D). For this reason there is a request in the literature for the extension of MTPO into 3D form [4]. The first step in the extension of the improved PO theory is the determination of a 3D Green's function that satisfies the Helmholtz equation in the spherical coordinates. The motivation of this letter is to mathematically derive a 3D directional Green's function that will yield directly to the 2D one of MTPO for spatially symmetric problems. In the literature, 3D Huygens' sources exist [5]. These are derived by the radiation integrals of electric and magnetic dipoles that are places perpendicularly according to each other [6]. However, the radiated waves from these 3D Huygens' sources do not satisfy the Helmholtz equation. A time factor $\exp(j\omega t)$ is suppressed throughout the paper. ω is the angular frequency.

We consider the diffraction problem of waves by a soft (total field is equal to zero on the surface) or hard (normal derivative of the total field is equal to zero on the surface) half-plane. The geometry is given in Fig. 1. The half-screen is located at the plane y=0. The problem is symmetric according to z. The coordinates with primes show the integration point. P and Q are the points of observation and integration (scattering) respectively. α is the angle of incidence at the integration point. β shows the angle of scattering. The total field can be written as

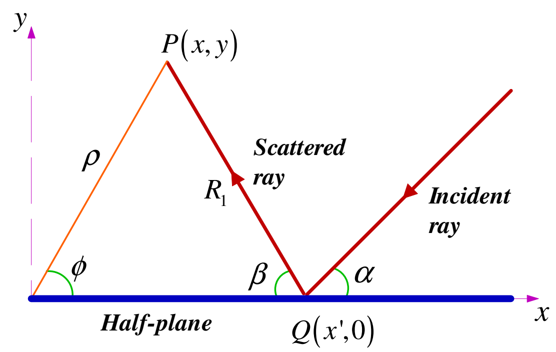

Fig. 1. (Color online) Scattering of waves by a perfectly conducting half-plane.

$$u_t(P) = u_i(P) + \frac{ke^{j\frac{\pi}{4}}}{\sqrt{2\pi}} \int_0^{\infty} u_i(Q)\left(\sin\frac{\beta-\alpha}{2} \mp \sin\frac{\beta+\alpha}{2}\right) \times \frac{e^{-jkR_1}}{\sqrt{kR_1}}dx' \tag{1}$$

where u_i is the incident wave [2, 7]. Plus and minus signs are valid for hard and soft surfaces respectively. k represents the wavenumber. R_1 is the distance between the points of observation and scattering and equal to

$$R_1 = \sqrt{(x-x')^2 + y^2}\,. \tag{2}$$

Equation (1) gives the exact scattered waves by the perfectly conducting half-plane. The form of the 2D Green's functions, in Eq. (1), can be given by

$$G_2 = \cos\frac{\phi}{2}\frac{e^{-jk\rho}}{\sqrt{k\rho}} \tag{3}$$

for ρ and ϕ are the polar coordinates. Q is at the origin for G_2, in Eq. (3). G_2 satisfies the Helmholtz equation as

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G_2}{\partial\rho}\right) + \frac{\partial^2 G_2}{\partial\phi^2} + k^2\rho^2 G_2 = 0 \tag{4}$$

in the cylindrical coordinates.

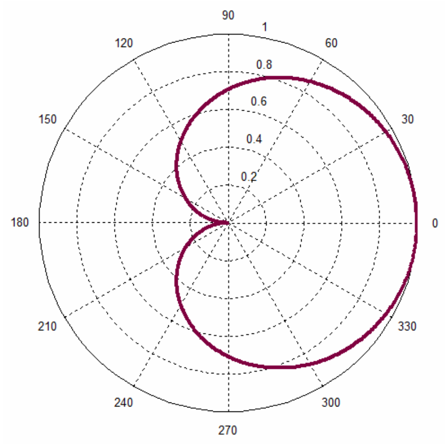

Fig. 2. (Color online) The variation of G_2 versus ϕ.

Figure 2 shows the variation of G_2 with respect to ϕ for a constant value of ρ. It can be seen that the maximum radiation occurs at 0^0. The radiated field is equal to zero at 180^0 which is in the opposite direction of 0^0. This behavior is similar to a cardioid shape [5].

Our aim is the derive the 3D Green's function G_3 by using the identity

$$\int_{-\infty}^{\infty} G_3(P,Q)\,dz' = G_2(P,Q)\,. \tag{5}$$

Note that the 2D Green's function of the classical PO and Kirchhoff's diffraction integral can be evaluated from the relation

$$\int_{-\infty}^{\infty} \frac{e^{-jkR}}{R}\,dz' = \frac{\pi}{j} H_0^{(2)}(kR_1) \tag{6}$$

where $H_0^{(2)}(x)$ is the second kind zero order Hankel function. R is

$$R = \sqrt{(x-x')^2 + y^2 + (z-z')^2}\,. \tag{7}$$

Equation (5) is inspired from Eq. (6). Now we will express G_2, given by Eq. (3), in terms of the integral of the 3D green's function G_3. Thus the form of G_3 will be determined, once the integral is constructed. The cylindrical wave factor, in G_2, satisfies the relation

$$\frac{e^{-jk\rho}}{\sqrt{k\rho}} = -\frac{2e^{-j\frac{\pi}{4}}}{k}\frac{dI}{d\rho} \tag{8}$$

where I represents the integral

$$I = \int_{e^{j\frac{\pi}{4}}\sqrt{k\rho}}^{\infty} e^{-\nu^2}\,d\nu\,. \tag{9}$$

We define the variable transform $\nu^2 = q$ for the integral. Thus I becomes

$$I = \frac{1}{2}\int_{jk\rho}^{\infty} \frac{e^{-q}}{\sqrt{q}}\,dq\,. \tag{10}$$

Equation (10) can be arranged as

$$I = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\left[\int_{jk\rho}^{\infty} e^{-q(1+\zeta^2)}\,dq\right]d\zeta \tag{11}$$

when the relation

$$\int_{-\infty}^{\infty} e^{-q\zeta^2}\,d\zeta = \sqrt{\frac{\pi}{q}} \tag{12}$$

is taken into account. The integral, in the brackets of Eq. (11), reads

$$\int_{jk\rho}^{\infty} e^{-q(1+\zeta^2)}\,dq = \frac{e^{-jk\rho(1+\zeta^2)}}{1+\zeta^2}\,. \tag{13}$$

As a result I yields the expression

$$I = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-jk\rho(1+\zeta^2)}}{1+\zeta^2}\,d\zeta \tag{14}$$

when Eq. (13) is used in Eq. (11). The 2D Green's function can be evaluated from the equation

$$G_2(\rho,\phi) = -\frac{e^{-j\frac{\pi}{4}}}{k\sqrt{\pi}}\cos\frac{\phi}{2}\frac{d}{d\rho}\left(\int_{-\infty}^{\infty} \frac{e^{-jk\rho(1+\zeta^2)}}{1+\zeta^2}\,d\zeta\right), \tag{15}$$

which is the combination of Eqs. (3) and (8). The relation

$$G_2(\rho,\phi) = \frac{e^{j\frac{\pi}{4}}}{\sqrt{\pi}}\cos\frac{\phi}{2}\int_{-\infty}^{\infty} e^{-jk\rho(1+\zeta^2)}\,d\zeta \tag{16}$$

will be obtained if Eq. (14) is used in Eq. (15). In this step, we will transform the integral, in Eq. (16), into the type, in Eq. (6). With this aim, the variable transform

$$\rho(1+\zeta^2) = R \tag{17}$$

is defined. R is expressed by

$$R = \sqrt{\rho^2 + z'^2}\,. \tag{18}$$

For the sake of simplicity, z is accepted as zero. The differential of Eq. (17) leads to the expression

$$d\zeta = \frac{1}{2\sqrt{\rho(R-\rho)}}\frac{z'}{R}\,dz'\,. \tag{19}$$

Equation (16) can be represented as

$$G_2(\rho,\phi) = \frac{e^{j\frac{\pi}{4}}}{2\sqrt{\pi}}\cos\frac{\phi}{2}\int_{-\infty}^{\infty} \frac{z'}{\sqrt{\rho(R-\rho)}}\frac{e^{-jkR}}{R}\,dz' \tag{20}$$

after the variable transform. As a result, the 3D Green's function is found to be

$$G_3(\rho,\phi,z') = \frac{e^{j\frac{\pi}{4}}}{2\sqrt{\pi}}\cos\frac{\phi}{2}\frac{z'}{\sqrt{\rho(R-\rho)}}\frac{e^{-jkR}}{R} \tag{21}$$

when Eq. (5) and (20) are compared. We used the operations, introduced by Clemmow, in order to transform Eq. (9) into Eq. (14) [8]. We can write G_3 as

$$G_3(\rho,\phi,z)=\frac{e^{j\frac{\pi}{4}}}{2\sqrt{\pi}}\cos\frac{\phi}{2}\frac{z}{\sqrt{\rho(r-\rho)}}\frac{e^{-jkr}}{r} \qquad (22)$$

by taking z instead of z'. This means that the integration point (x',y',z') is at the origin. In this case, R becomes r of the spherical coordinates (r,θ,ϕ). It is apparent that Eq. (22) represents a 3D Huygens' source. The relations

$$z=r\cos\theta \qquad (23)$$

and

$$\rho=r\sin\theta \qquad (24)$$

can be defined in the spherical coordinates. Equation (22) can be rewritten as

$$G_3(\rho,\phi,z)=\frac{e^{j\frac{\pi}{4}}}{2\sqrt{\pi}}\cos\frac{\phi}{2}\frac{\cos\theta}{\sqrt{\sin\theta(1-\sin\theta)}}\frac{e^{-jkr}}{r} \qquad (25)$$

in this coordinate system. G_3 can be arranged as

$$G_3(r,\gamma,\phi)=\frac{e^{j\frac{\pi}{4}}}{2\sqrt{\pi}}\cos\frac{\phi}{2}\frac{\sin\gamma}{\sqrt{\cos\gamma(1-\cos\gamma)}}\frac{e^{-jkr}}{r} \qquad (26)$$

where γ is $(\pi/2)-\theta$. The 3D Huygens' source reads

$$G_3(r,\theta,\phi)=\frac{e^{j\frac{\pi}{4}}}{\sqrt{2\pi}}\cos\frac{\phi}{2}\frac{\cos\frac{\pi-2\theta}{4}}{\sqrt{\sin\theta}}\frac{e^{-jkr}}{r} \qquad (27)$$

after some trigonometric manipulations. The Helmholtz equation

$$\nabla^2 u+k^2u=0 \qquad (28)$$

can be expressed as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+k^2r^2u=0 \qquad (29)$$

in the spherical coordinates. The Helmholtz equation becomes

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{\partial^2 v}{\partial\phi^2}=0 \qquad (30)$$

when Eq. (27) is used in Eq. (29). v is equal to

$$v=\cos\frac{\phi}{2}\frac{\cos\frac{\pi-2\theta}{4}}{\sqrt{\sin\theta}}. \qquad (31)$$

Equation (30) yields

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)-\frac{w}{4}=0 \qquad (32)$$

for w has the expression

$$w=\frac{\cos\frac{\pi-2\theta}{4}}{\sqrt{\sin\theta}}. \qquad (33)$$

It can be verified that the left side of Eq. (32) is equal to zero for w, defined in Eq. (33). Thus G_3 satisfies the Helmholtz equation in the spherical coordinates.

Now we will introduce the scalar diffraction integral for soft and hard surfaces with the aid of G_3. The geometry, in Fig. 3, is taken into account. The integration point is in the 3D space.

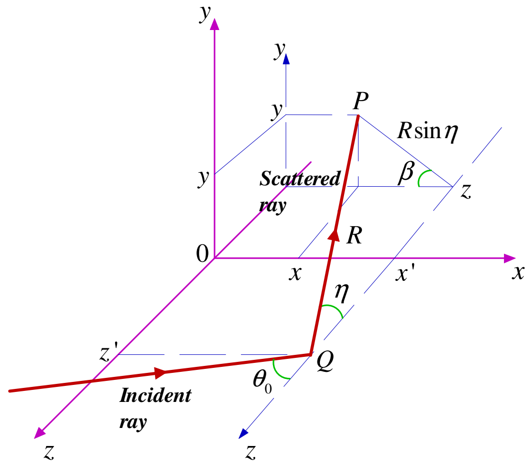

Fig. 3. (Color online)3D scattering geometry.

An arbitrary incident wave is hitting a perfectly conducting surface, located at $y=0$. The scattered ray travels from the integration point to the point of observation by following the path R. θ_0 (α) is the angle between the incident ray and the z (x) axis. The angle between the scattered ray and the z (x) axis is η (β). The 3D Green's function can be introduced by

$$G_{3\mp}(r,\theta,\phi)=\sin\frac{\beta\mp\alpha}{2}\frac{\cos\frac{\eta-\theta_0}{2}}{\sqrt{\cos(\eta-\theta_0)}}\frac{e^{-jkR}}{R}. \qquad (34)$$

The plus and minus signs are valid for hard and soft surfaces respectively. The trigonometric functions of η are determined in order to guarantee that the maximum value is taken at $\eta=\theta_0$. This criteria is related with the fact that the same functions reach their maximum value at $\theta=\pi/2$, in Eq. (27). The total field can be written as

$$u_t(P)=u_i(P)+u_s(P) \qquad (35)$$

where u_s is the scattered wave, which can be defined by

$$u_s(P)=\frac{jk}{2\pi}\iint_S u_i(Q)\left[G_{3-}(P,Q)-G_{3+}(P,Q)\right]dS' \qquad (36)$$

according to Eq. (1). The constant terms, in Eq. (27), are not considered at Eq. (34), because Eq. (27) is derived for a cylindrical wave factor with unit amplitude. The same constant terms come automatically from Eq. (1).

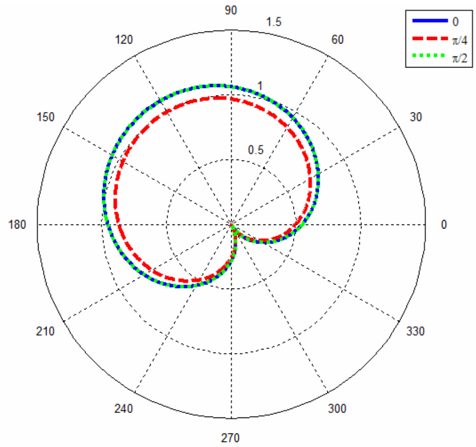

Fig. 4. (Color online)Variation of the G_{3+} with respect to β for different values of η.

Figure 4 shows the variation of G_{3+} versus β for different values of η. The shape of the Green's function does not change with the various values of η. However, the intensity of G_{3+} is affected. The maximum values of intensity occur equally when η is 0^0 and 90^0. The intensity of the Green's function decreases till η=45^0.

In this letter, we introduced a 3D Green's function, which satisfies the Helmholtz equation in the spherical coordinates, other than the spherical wave factor, used in PO and the diffraction integral of Kirchhoff for the first time in the literature to our knowledge. Our motivation was the extension of MTPO for the 3D problems. Our future work will be to relate the evaluation of the surface currents and 3D Green's function from the boundary conditions and the incident wave. However Eq. (36) gives an improved form of the Kirchhoff's diffraction integral for 3D scattering problems.

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