A New Proof and an Application of Dergiades' Theorem

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In this article we'll present a new proof of Dergiades' Theorem, and we'll use this theorem to prove that the orthological triangles with the same orthological center are homological triangles.

Theorem 1 (Dergiades)

Let $C_1(O_1,R_1)$, $C_2(O_2,R_2)$, $C_3(O_3,R_3)$ three circles which pass through the vertexes B and C, C and A, A and B respectively of a given triangle ABC. We'll note D,E,F respectively the second point of intersection between the circles (C_1) and (C_3), (C_3) and (C_2), (C_1) and (C_2). The perpendiculars constructed in the points D,E,F on AD, BE respectively CF intersect the sides BC, CA, AB in the points X, Y, Z. Then the points X, Y, Z are collinear

Proof

To prove the collinearity of the points X, Y, Z, we will use the reciprocal of the Menelaus Theorem (see *Fig. 1*).

We have

$$\frac{XB}{XC}=\frac{Aria\Delta XDB}{Aria\Delta XDC}=\frac{DB\cdot \sin\widehat{XDB}}{DC\cdot \sin\widehat{XDC}}=\frac{DB\cdot \cos\widehat{ADB}}{DC\cdot \cos\widehat{ADC}}$$

Similarly we find

$$\frac{YC}{YA}=\frac{EC\cdot \cos\widehat{BEC}}{EA\cdot \cos\widehat{BEA}}$$

$$\frac{ZA}{ZB}=\frac{FA\cdot \cos\widehat{CFA}}{FB\cdot \cos\widehat{CFB}}$$

From the inscribed quadrilaterals $ADEB; BEFC; ADFC$, we can observe that

$$\sphericalangle ADB\equiv\sphericalangle BEA;\ \sphericalangle BEC\equiv\sphericalangle CFB;\ \sphericalangle CFA\equiv\sphericalangle ADC$$

Consequently,

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB} \qquad (1)$$

On the other side $DB=2R_3\sin\widehat{BAD}$; $EA=2R_3\sin\widehat{ABE}$; $DC=2R_2\sin\widehat{CAD}$; $FA=2R_2\sin\widehat{ACF}$; $FB=2R_1\sin\widehat{BCF}$; $EC=2R_1\sin\widehat{CBE}$.

Using these relations in (1), we obtain

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB}=\frac{\sin\widehat{BAD}}{\sin\widehat{CAD}}\cdot\frac{\sin\widehat{CBE}}{\sin\widehat{ABE}}\cdot\frac{\sin\widehat{ACF}}{\sin\widehat{BCF}} \qquad (2)$$

According to one of Carnot's theorem, the common strings of the circles (C_1), (C_2), (C_3) are concurrent, that is $AD\cap BE\cap CF=\{P\}$ (the point P is the radical center of the circles (C_1), (C_2), (C_3)).

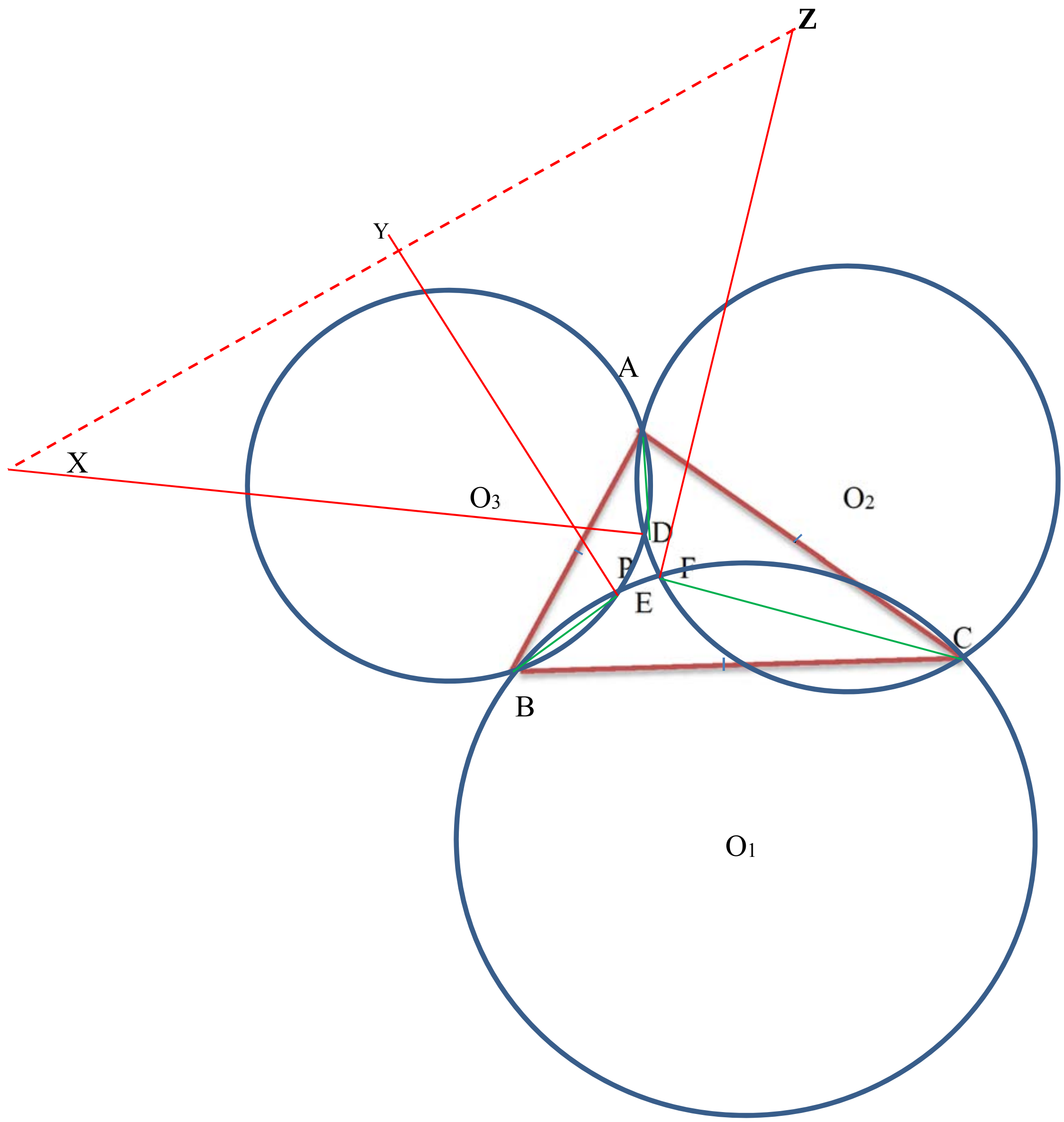

Fig 1.

In triangle ABC, the cevians AD, BE, CF being concurrent, we can use for them the trigonometrically form of the Ceva's theorem as follows

$$\frac{\sin \widehat{BAD}}{\sin \widehat{CAD}} \cdot \frac{\sin \widehat{CBE}}{\sin \widehat{ABE}} \cdot \frac{\sin \widehat{ACF}}{\sin \widehat{BCF}} = 1 \qquad (3)$$

The relations (2) and (3) lead to

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = 1$$

Relation, which in conformity with Menelaus theorem proves the collinearity of the points X, Y, Z.

Definition 1

Two triangles ABC and *A'B'C'* are called orthological if the perpendiculars constructed from *A* on *B'C'*, from B on *C'A'* and from *C* on *A'B'* are concurrent. The concurrency point of these perpendiculars is called the orthological center of the triangle ABC in rapport to triangle *A'B'C'*.

Theorem 2 (The theorem of orthological triangle of J. Steiner)

If the triangle ABC is orthological with the triangle *A'B'C'*, then the triangle *A'B'C'* is also orthological in rapport to triangle ABC.

For the proof of this theorem we recommend [1].

Observation

A given triangle and its contact triangle are orthological triangles with the same orthological center. Their common orthological center is the center of the inscribed circle of the given triangle.

Definition 3

Two triangles ABC and *A'B'C'* are called homological if and only if the lines AA', BB', CC' are concurrent. The congruency point is called the homological center of the given triangles.

Theorem 3 (Desargues – 1636)

If ABC and *A'B'C'* are two homological triangles, then the lines $(BC, B'C');\ (CA, C'A');\ (AB, A'B')$ are concurrent respectively in the points X, Y, Z, and these points are collinear. The line that contains the points X, Y, Z is called the homological axis of the triangles ABC and $A'B'C'$.

For the proof of Desargues theorem see [3].

Theorem 4

Two orthological triangles that have a common orthological center are homological triangles.

Lemma 1

Let ABC and $A'B'C'$ two orthological triangles. The orthogonal projections of the vertexes B and C on the sides $A'C'$ respectively $A'B'$ are concyclic.

Proof

We note with E, F the orthogonal projections f the vertexes B and C on $A'C'$ respectively $A'B'$ (see Fig. 2). Also, we'll note O the common orthological center of the orthological triangles ABC and $A'B'C'$ and $\{B''\} = EO \cap AC$, $\{C''\} = FO \cap AB$. In the triangle $A'B''C''$, O being the intersection of the heights constructed from B'', C'', is the orthocenter of this triangle, consequently, it results that $A'O \perp B''C''$. On the other side $A'O \perp BC$; we obtain, therefore that $B''C'' \parallel BC$. Taking into consideration that EF and $B''C''$ are antiparallel in rapport to $A'B'$ and $A'C'$, we obtain that EF is antiparallel with BC, fact that shows that the quadrilateral $BCFE$ is inscribable.

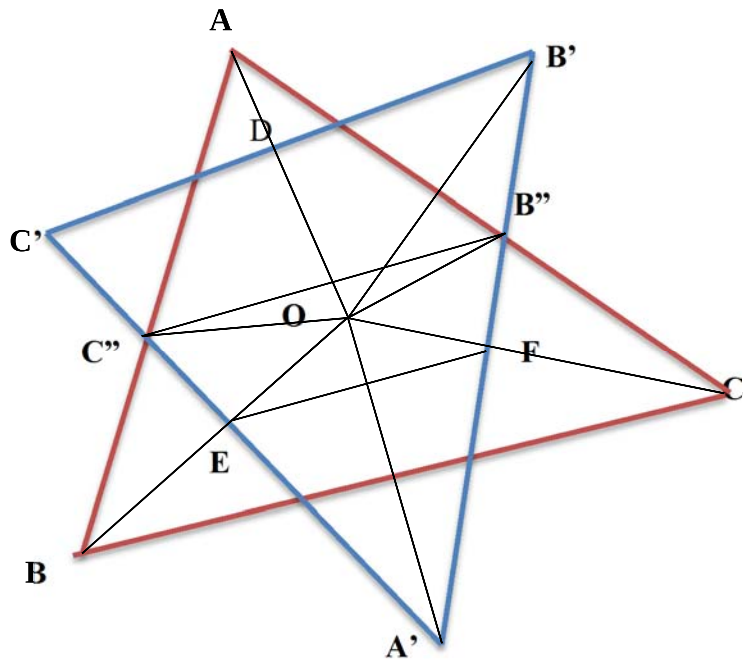

Fig. 2

Observation

If we denote with D the projection of A on $B'C'$, similarly, it will result that the points A, D, F, C respectively A, D, E, B are concyclic.

Proof of Theorem 4

The quadrilaterals $BCFE$, $CFDA$, $ADEB$ being inscribable, it result that their circumscribed circles satisfy the Dergiades theorem (Fig. 2). Applying this theorem it results that the pairs of lines $(BC, B'C')$; $(CA, C'A')$; $(AB, A'B')$ intersect in the collinear points X, Y, Z, respectively. Using the reciprocal theorem of Desargues, it result that the lines AA', BB', CC' are concurrent and consequently the triangles ABC and $A'B'C'$ are homological.

Observations

1. Triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to quadrilaterals $BCFE$, $CFDA$, $ADEB$ and the triangle ABC are orthological triangles.
   The orthological centers are the points P - the radical center of the circles $(O_1), (O_2), (O_3)$ and O - the center of the circumscribed circle of the triangle ABC.
2. The triangles $O_1O_2O_3$ and DEF (formed by the projections of the vertexes A, B, C on the sides of the triangle $A'B'C'$) are orthological. The orthological centers are the center of the circumscribed circle to triangle DEF and P the radical center of the circles $(O_1), (O_2), (O_3)$.

Indeed, the perpendiculars constructed from O_1, O_2, O_3 on EF, FD, DA respectively are the mediators of these segments and, therefore, are concurrent in the center of the circumscribed circle to triangle DEF, and the perpendiculars constructed from D, E, F on the sides of the triangle $O_1O_2O_3$ are the common strings AD, BE, CF, which, we observed above, are concurrent in the radical center P of the circles with the centers in O_1, O_2, O_3.

References

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[2] Barbu, C. – Teoreme fundamentale din geometria triunghiului – Ed. Unique, Bacău, 2008

[3] Smarandache, F. and Pătraşcu, I. – The Geometry of Homological Triangles – The Education Publisher, Inc. Columbus, Ohio, U.S.A. 2012.