PROPERTIES OF THE ADJOIN CIRCLES TO A TRIANGLE

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In this article we'll present the properties of the radicale axes and the adjoin circles of a triangle.

Definition 1

Given a triangle ABC, we call the circle that passes through the vertexes C, A and it is tangent in the point A to the side AB, that it is an adjoin circle to the given triangle.

Observations

a) We note the circle from the above definition $C\bar{A}$.

b) To a triangle, in general, there are corresponding 6 different adjoin circles. If the given triangle is isosceles, it will have 5 different adjoin circles, and if the triangle is equilateral, there will be 3 different adjoin circles associated to it.

Theorem 1 (A. L. Crelle, 1816)

i). The adjoin circles $C\overline{A}$, $A\overline{B}$, $B\overline{C}$, of a random triangle ABC have a common point Ω with the property: $\angle\Omega AB \equiv \angle\Omega BC \equiv \angle\Omega CA$

ii). The adjoin circles $A\overline{C}$, $C\overline{B}$, $B\overline{A}$, of a random triangle ABC have a common point Ω' with the property: $\angle\Omega' AC \equiv \angle\Omega' BA \equiv \angle\Omega' CB$

Proof:

i). Let Ω a second point of intersection of the circles $C\overline{A}$ and $A\overline{B}$, (see fig.1).

We have:

$$\angle\Omega CA \equiv \angle\Omega AB \quad şi \quad \angle\Omega AB \equiv \angle\Omega BC.$$

Indeed, the first angles have as measure the half of the measure of the cord $A\Omega$, and those from the second congruence have as measure half from the measure of the cord $B\Omega$.

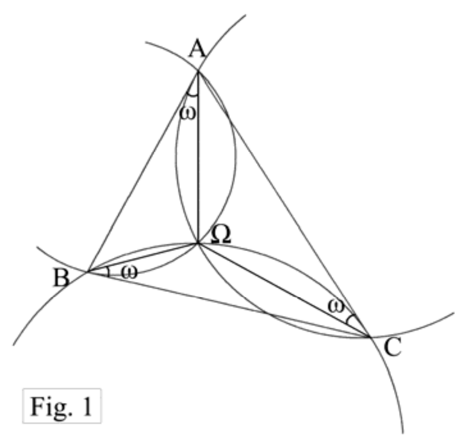


Fig. 1

We obtain that:

$$\angle\Omega AB \equiv \angle\Omega BC \equiv \angle\Omega CA.$$

The relation $\angle\Omega BC \equiv \angle\Omega CA$ show that the circumscribed circle to the triangle $B\Omega C$ is tangent in C to the side AC and it is, therefore, the adjoin circle $B\overline{C}$.

Observations:

a). Similarly it can be proved ii).

b). The point Ω is called the first point of Brocard[(1)], and Ω' is called the second point of Brocard.

c). The Brocard's point Ω is the radical center of the adjoin circles $C\overline{A}$, $A\overline{B}$, $B\overline{C}$, and the Brocard's point Ω' is the radical center of the adjoin circles $A\overline{C}$, $C\overline{B}$, $C\overline{A}$, of the triangle ABC.

Indeed, the Ω as well as Ω' have equal powers (null) in rapport to the triplet of the adjoin circles indicated, and therefore these are their radical centers.

Theorem 2

The Brocard's points Ω and Ω' are isogonal point in the triangle ABC

Proof

We'll note $m(\angle\Omega AB) = \omega$.

Applying the sinus theorem in the triangles $A\Omega B$ and $A\Omega C$ we obtain:

$$\frac{B\Omega}{\sin\omega} = \frac{c}{\sin(B\Omega A)} = \frac{A\Omega}{\sin(B-\omega)}$$

and

$$\frac{A\Omega}{\sin\omega} = \frac{b}{\sin(A\Omega C)}.$$

Because

$$m(\angle B\Omega A) = 180^O - m(\angle B);$$
$$m(\angle A\Omega C) = 180^O - m(\angle A),$$

it results that:

$$\frac{A\Omega}{B\Omega} = \frac{b}{c}\frac{\sin B}{\sin A} = \frac{\sin(B-\omega)}{\sin\omega}$$

Expanding $\sin(B-\omega)$ and taking into account that $\frac{b}{c} = \frac{\sin B}{\sin C}$ and $\sin(A+C) = \sin B$ we obtain:

$$ctg\omega = ctgA + ctgB + ctgC.$$

If we note $m(\angle\Omega' AC) = \omega'$, and making a similar rational, it results:

$$ctg\omega' = ctgA + ctgB + ctgC.$$

The precedent relations lead to $\omega = \omega'$, which shows that Ω and Ω' are isogonal points.

Observation

The angle ω is called Brocard's angle and it appears in many relations and formulae of the triangle geometry (see [1]).

Theorem 3 The adjoin circles $C\overline{A}$ and $B\overline{A}$ intersect on the symmedian from A on the triangle ABC.

Proof

Let S the second point of intersection of the circles $C\overline{A}$ and $B\overline{A}$ (see Fig. 2) and $\{P\} = AS \cap BC$.

We observe that

$$\angle SCA \equiv \angle SAB \text{ and } \angle SBA \equiv \angle SAC.$$

Therefore $\Delta SBA \sim \Delta SAC$, from which we obtain:

$$\frac{SB}{SC} = \frac{AB^2}{AC^2} \qquad (1)$$

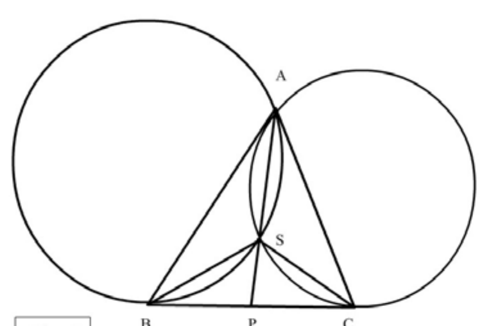


Fig. 2

On the other side, from the above angular congruencies we obtain: $\angle BSP \equiv \angle CSP$, and taking into account the bisectrix in the triangle BSC, it results that;

$$\frac{SB}{SC} = \frac{PB}{PC} \qquad (2)$$

From (1) and (2) it results

$$\frac{PB}{PC} = \frac{AB^2}{AC^2}$$

which shows that AP is the symmedian from A of the triangle ABC.

Observation

Theorem 3 expresses the fact that two adjoin circles, which are tangent to two sides of a triangle, having a common vertex of these sides; have as radical axes the symmedian of the triangle constructed from that vertex.

Theorem 4

The adjoin circles $A\overline{B}$ and $A\overline{C}$ intersect on the median from A of the triangle ABC.

Proof

Let D the second point of intersection of the circles $A\overline{B}$ and $A\overline{C}$ and $\{M\} = AD \cap BC$. The line AD is the radical axes of the circles $A\overline{B}$ and $A\overline{C}$, we have:

$$MB^2 = MD \cdot MA = MC^2,$$

it results that M is the middle point of (BC).

Observations

a). Theorem 4 expresses the fact that two adjoin circles of a triangle, tangent to the same side, have as radical axes the opposed median to the side.

b). From the results proved, it results that the radical axes of two adjoin circle of a triangle can be: the Brocard's cevian $A\Omega$, $A\Omega'$, etc., triangle's symmedian, triangle's medians or the triangle's sides. Indeed, if we consider in a given triangle the adjoin circles $B\overline{C}$ and $C\overline{B}$ their radical axes is the side BC

c) Relative to the radical centers of the adjoin circles we proved that Ω and Ω' have this property. Because, in general, a triangle has 6 adjoin circles, it means that there exists $C_6^3 = 20$ radical centers corresponding to the different triplets of adjoin circles.

d). The vertexes of the triangle ABC are the radical centers of certain triplets of adjoin circles Indeed, for example, the vertex C of the triangle ABC is the radical center of the adjoin circles $B\overline{C}, C\overline{B}$ and $A\overline{C}$ because these pass through the same vertex C. Also, C is the radical center of the circles $B\overline{C}$, $C\overline{B}$ and $C\overline{A}$ as well as of the circles $B\overline{C}$

Theorem 5 (L. Carnot-1803)

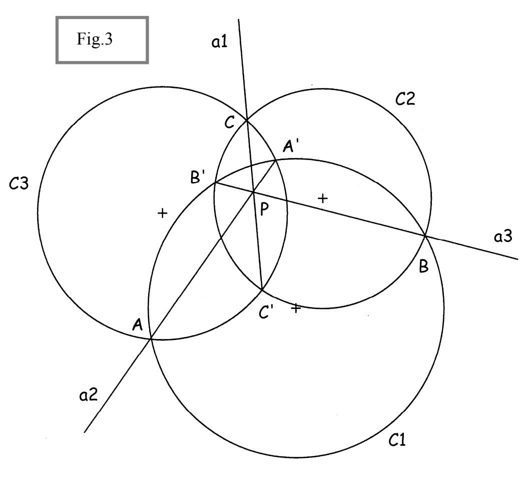
The common cords of three secant circles two by two are concurrent.

Proof:

Let C_1, C_2, C_3 three secant circles and a_1, a_2, a_3 the radical axes of the circles (C_2, C_3), (C_1, C_3) respectively (C_1, C_2) (see fig. 3.)

We note P the intersection point between a_1 and a_2, it results that the point P will have equal

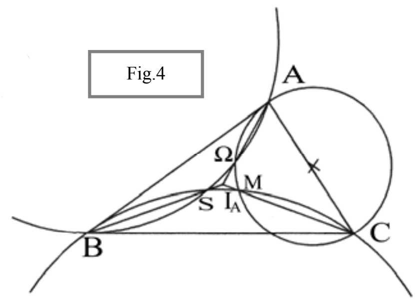
Fig.3



powers in rapport to all the circles, therefore P will be situated on the radical axes a_3 of the circles (C_1, C_2).

Theorem 6 (R. A. Johnson, 1929)
In a triangle ABC the cevian $A\Omega$, symmedian from B and the median from C are concurrent.

Proof:



We'll construct the circles $C\overline{A}$, $A\overline{B}$ and $C\overline{B}$.
In conformity to theorem 1, the second intersection point of the circles $C\overline{A}$ and $A\overline{B}$ is Ω, from the theorem 2 it results that the second point common to the circles $A\overline{B}$ and $C\overline{B}$ is S, situated on the symmedian from B (see fig. 4), and in conformity to theorem 3, the circles $C\overline{A}$ and $C\overline{B}$ intersect the second time in M, which belongs to the median from C of the triangle ABC. The Carnot's theorem affirms that the cevian $A\Omega$, BS and CM are concurrent. We noted the intersection point I_A.

Observations

a) The point I_A is the radical center of the adjoin circles $C\overline{A}$, $A\overline{B}$, $C\overline{B}$.
b) The theorem 6 can be proved using the reciprocal theorem of Ceva; for this it has to be computed $\frac{BA_1}{A_1C}$, where $\{A_1\} = A\Omega \cap BC$, we find that $\frac{BA_1}{A_1C} = \frac{AC^2}{BC^2}$
c) In the same manner we can state and prove the following theorems:

Theorem 7
The cevian BΩ, symmedian from C and the median from A are concurrent in a point I_B, which is the radical center of the adjoin circles $A\overline{C}, C\overline{B}, A\overline{B}$ of the triangle ABC.

Theorem 8
The cevian CΩ, symmedian from A and the median from B of the triangle ABC are concurrent in a point I_C , which is the radical center of the adjoin circles $C\overline{A}, B\overline{C}, B\overline{A}$.

Theorem 9
The cevian AΩ', symmedian from C and the median from B of the triangle ABC are concurrent in a point I_A' which is the radical center of the adjoin circles $A\overline{C}, B\overline{C}, B\overline{A}$.

Theorem 10
The cevian BΩ', symmedian from A and the median from C of the triangle ABC are concurrent in a point I_B' which is the radical center of the circles $B\overline{A}, C\overline{B}, C\overline{A}$.

Theorem 11
The cevian CΩ', symmedian from B and the median from A of the triangle ABC are concurrent in a point I_C' which is the radical center of the adjoin circles $A\overline{C}, C\overline{B}, A\overline{B}$.

Observation:

If the triangle ABC is isosceles, AB=AC, then the adjoin circle $B\overline{C}$ coincides with the adjoin circle $C\overline{B}$ and we obtain the theorem:

In the isosceles triangle ABC, AB=AC, the symmedian from B and the median from C intersect in the Brocard's point **Ω**. See [3].

References:

[1] Mihăileanu, N.N. - Lecţii complementare de geometrie, Editura didactică şi pedagogică, Bucureşti 1976; pp. 56 -57.

[2] Johnson, R. A. - Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle. Boston, MA: Houghton Mifflin, pp. 263/286, 1929.

[3] Pătraşcu, I. - O teoremă relativă la punctul lui Brocard, Gazeta Matematică anul LXXXIX, nr. 9/1984, pp. 328 -329.