

# Derivation of the reflection integral equation of the zeta function by the complex analysis

[Home](#) > [Quantum mechanics](#) > [Zeta function and Bernoulli numbers](#)

2016/07/09  
Published 2013/9/15  
K. Sugiyama<sup>1</sup>

In this paper, we derive the reflection integral equation of the zeta function by the complex analysis.

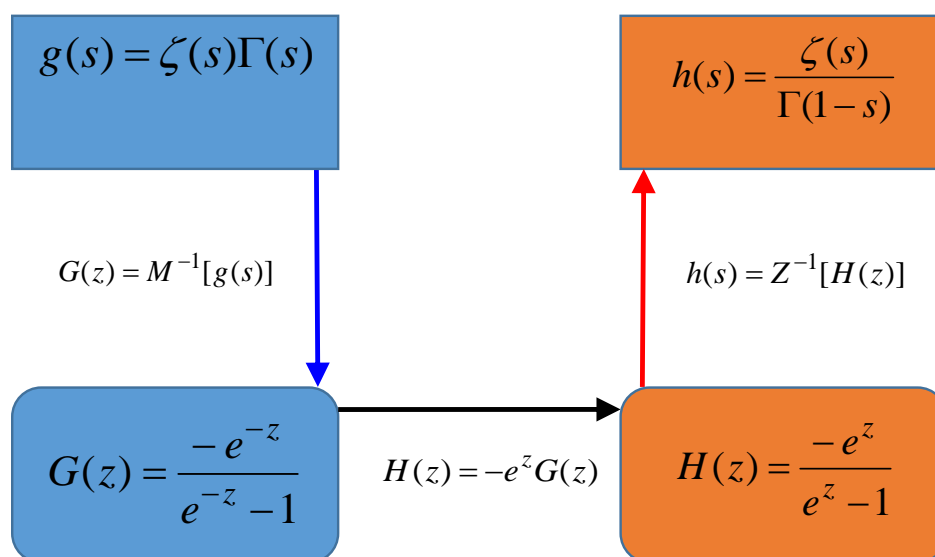


Figure 3-1: The framework of the method of derivation

Many researchers have attempted the proof of the Riemann hypothesis, but have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation by the complex analysis as preparation proving the Riemann hypothesis.

We obtain a generating function of the inverse Mellin-transform. We obtain a new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from the inverse Z-transform of the generating function.

We derive the Faulhaber’s formula, and Nörlund-Rice integral from the reflection integral equation.

## CONTENTS

1	Introduction.....	2
1.1	Issue .....	2
1.2	Importance of the issue .....	2
1.3	Research trends so far .....	3

1.4	New derivation method of this paper .....	3
2	Confirmations of known results .....	3
2.1	The Mellin transform .....	3
2.2	Hurewicz's Z-transform .....	5
2.3	Cauchy's residue theorem .....	7
2.4	Euler's gamma function.....	7
2.5	Euler's beta function.....	8
2.6	The Riemann zeta function .....	9
2.7	The Bernoulli polynomials.....	12
2.8	The Bernoulli numbers.....	13
3	Derivation of the reflection integral equation .....	14
3.1	The framework of the method of derivation .....	14
3.2	Derivation of the reflection integral equation from the inverse Mellin transform.....	16
4	Conclusion .....	20
5	Future issues.....	20
6	Supplement.....	20
6.1	Confirmations of known results (Part 2).....	20
6.1.1	Hurwitz zeta function.....	21
6.1.2	Euler-Maclaurin formula.....	22
6.1.3	Asymptotic expansion.....	22
6.1.4	Faulhaber's formula.....	23
6.1.5	The Ramanujan master theorem .....	24
6.1.6	Woon's introduction of the continuous Bernoulli numbers .....	24
6.1.7	Nörlund-Rice integral .....	25
6.2	Derivation of Faulhaber's formula .....	26
6.2.1	Derivation of the summation equation of the Riemann zeta function .....	26
6.2.2	Derivation of the summation equation of the Hurwitz zeta function.....	28
6.2.3	Derivation of asymptotic expansion .....	30
6.2.4	Analytic continuation of asymptotic expansion.....	30
6.2.5	Derivation of Faulhaber's formula .....	32
6.3	Derivation of Nörlund-Rice integral .....	34
6.3.1	Derivation of the reflection integral formula .....	34
6.3.2	Derivation of Nörlund-Rice integral .....	36
7	Appendix.....	38
7.1	The table of the Z-transform .....	38
7.2	Derivation of the reflection integral equation from the reflection integral formula .....	41
8	Bibliography.....	41

## 1 Introduction

### 1.1 Issue

Many researchers have attempted the proof of the Riemann hypothesis, but have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation by the complex analysis as preparation proving the Riemann hypothesis.

### 1.2 Importance of the issue

The proof of the Riemann hypothesis is one of the most important unsolved problems in mathematics.

For this reason, many researchers have attempted the proof of the Riemann hypothesis, but have not been successful. One of the methods proving Riemann hypothesis is interpreting the zeros of the zeta function as the eigenvalues of a certain operator. However, no one found the operator until now. We are able to consider the reflection integral equation as one of the operators. For this reason, the derivation of the reflection integral equation is an important issue.

### 1.3 Research trends so far

Leonhard Euler introduced the infinite series of the zeta function in 1737. Bernhard Riemann introduced the analytic continuation of the zeta function in 1859.

David Hilbert and George Polya<sup>2</sup> suggested that the zeros of the zeta function were probably eigenvalues of a certain operator around 1914. This conjecture is called Hilbert-Polya conjecture. Zeev Rudnick and Peter Sarnak<sup>3</sup> are studying the distribution of zeros by random matrix theory in 1996. Shigenobu Kurokawa is studying the field with one element<sup>4</sup> around 1996. Alain Connes<sup>5</sup> showed the relation between noncommutative geometry and the Riemann hypothesis in 1998. Christopher Deninger<sup>6</sup> is studying the eigenvalue interpretation of the zeros in 1998.

### 1.4 New derivation method of this paper

We obtain a generating function of the inverse Mellin-transform. We obtain a new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from the inverse Z-transform of the generating function.

(The reflection integral equation)

$$\zeta(1-s) = \oint_C \frac{-dt}{2\pi i} B(s, t) \zeta(t) \quad (1.1)$$

We derive the Faulhaber's formula, and the Nörlund-Rice integral from the reflection integral equation.

(The Faulhaber's formula)

$$\sum_{k=0}^{q-1} k^n = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (1.2)$$

## 2 Confirmations of known results

In this chapter, we confirm the known results.

### 2.1 The Mellin transform

Hjalmar Mellin<sup>7</sup> published the Mellin transform in 1904.

(The Mellin transform)

$$f(s) = M[F(x)] \quad (2.1)$$

$$f(s) = \int_0^{\infty} x^{s-1} F(x) dx \quad (2.2)$$

If a function  $f(s)$  is analytic in the strip  $S = \{a < \text{Re}(s) < b\}$ , and if it tends to zero uniformly as  $\text{Im}(s) \rightarrow \pm\infty$  for any real value  $c$  between  $a$  and  $b$  the following line integral converges absolutely. (The inverse Mellin transform)

$$F(z) = M^{-1}[f(s)] \quad (2.3)$$

$$F(z) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{f(s)}{z^s} \quad (2.4)$$

$$S = \{a < \text{Re}(s) < b\} \quad (2.5)$$

The real part of the strip  $S$  needs to be greater than the real part of all poles of the integrand. We show the strip  $S$  in the following figure. The white circles mean poles.

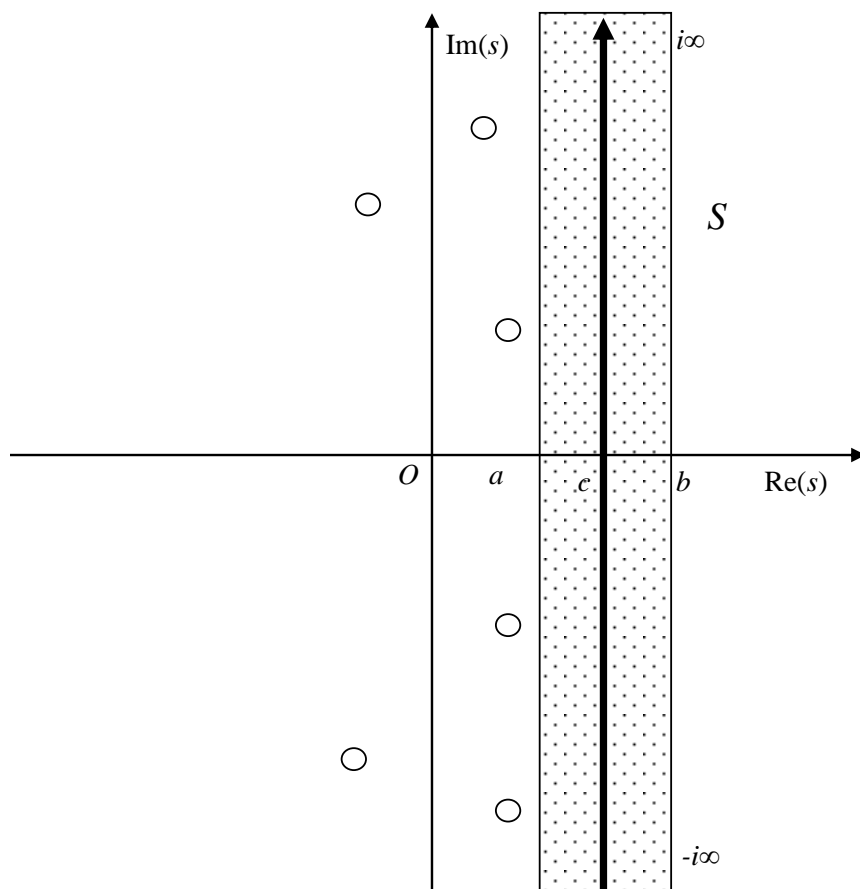


Figure 2-1: The strip  $S$  of the inverse Mellin transform

We define the inverse Mellin transform by the following contour integration. (The inverse Mellin transform)

$$F(z) = M^{-1}[f(s)] \quad (2.6)$$

$$F(z) = \oint_C \frac{ds}{2\pi i} \frac{f(s)}{z^s} \quad (2.7)$$

We suppose that the circuit of integration  $C$  circles around all poles of the integrand. For example, we suppose the circuit of integration  $C = C_I + C_R$  as follows. The white circles mean poles.

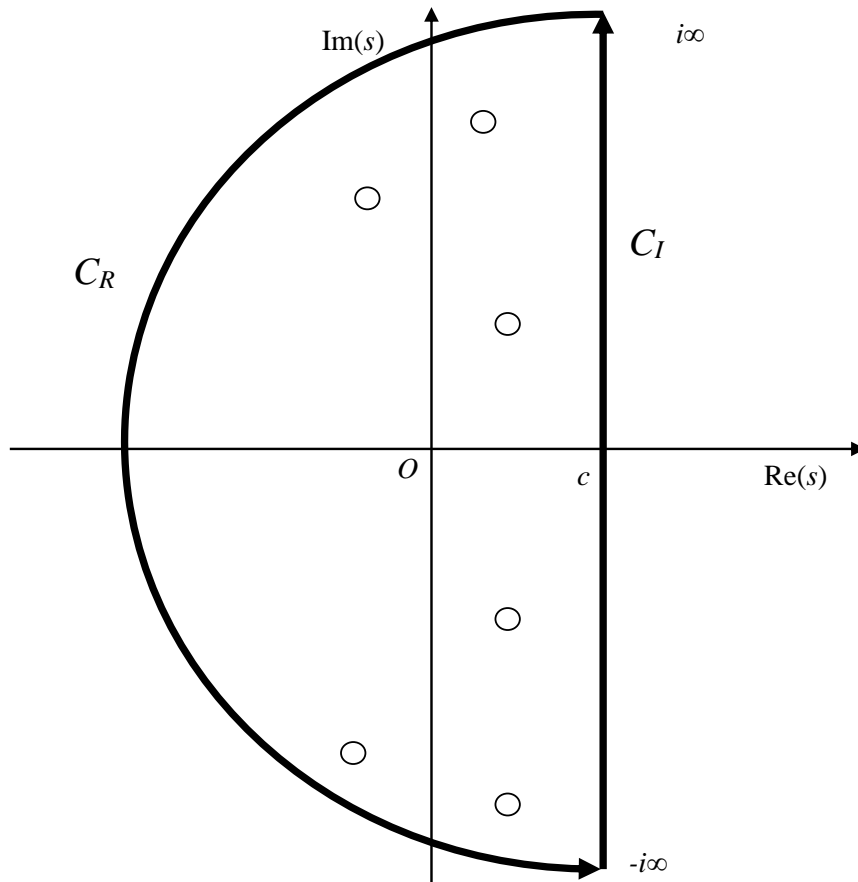


Figure 2-2: The inverse Mellin transform

If the line integration of the path  $C_R$  is zero, the contour integration of the circuit of integration  $C$  equals the line integration of the path  $C_I$ . Then we have the following formula.

$$\oint_C \frac{ds f(s)}{2\pi i z^s} = \int_{c-i\infty}^{c+i\infty} \frac{ds f(s)}{2\pi i z^s} \quad (2.8)$$

## 2.2 Hurewicz's Z-transform

Witold Hurewicz<sup>8</sup> published the Z-transform in 1947. When a function  $F(z)$  is holomorphic over the domain  $D = \{0 < |z| < R\}$ , we are able to transform the function to the series which converges uniformly in a wider sense over the domain.

(Z-transform)

$$F(z) = Z[f(n)] \quad (2.9)$$

$$F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n} \quad (2.10)$$

$$D = \{0 < |z| < R\} \quad (2.11)$$

Therefore, when the minimum distance between an origin and poles is  $R$ , we have the domain  $D$  of the Z-transform in the following figure.

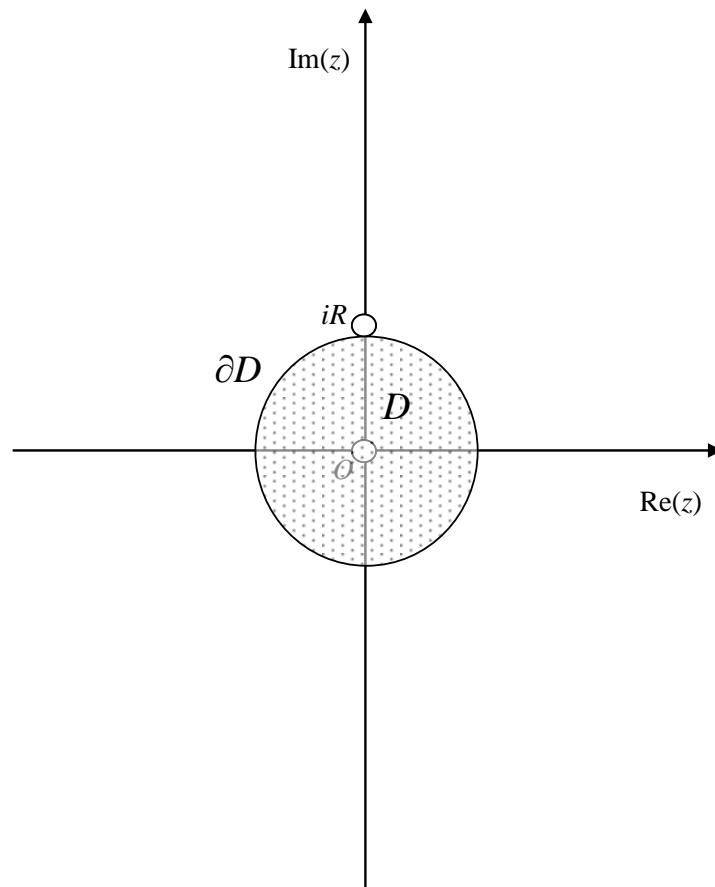


Figure 2-3: The domain of the Z-transform

The inverse Z-transform is the contour integration along the circuit of integration  $\partial D$  circles the domain  $D$ .

(Inverse Z-transform)

$$f(n) = Z^{-1}[F(z)] \quad (2.12)$$

$$f(n) = \oint_{\partial D} \frac{dz}{2\pi iz} z^n F(z) \quad (2.13)$$

### 2.3 Cauchy's residue theorem

Augustin-Louis Cauchy published the residue theorem<sup>9</sup> in 1831.

We suppose that a function  $F(z)$  has an isolated singularity  $c$  on a domain  $D$  inside of the simple closed curve  $\partial D$  and is holomorphic on both the domain  $D$  and the closed curve  $\partial D$  except for the isolated singularity. Then, we have the following formula.

(The residue theorem)

$$\oint_{\partial D} \frac{dz}{2\pi i} F(z) = \operatorname{Res}_{z \rightarrow c} F(z) dz \quad (2.14)$$

If a function  $F(z)$  has isolated singularities  $c_k$ , we have the following formula.

(The residue theorem)

$$\oint_{\partial D} \frac{dz}{2\pi i} F(z) = \sum_{k=1}^n \operatorname{Res}_{z \rightarrow c_k} F(z) dz \quad (2.15)$$

### 2.4 Euler's gamma function

Leonhard Euler<sup>10</sup> introduced the gamma function as a generalization of the factorial in 1729.

(The gamma function)

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \quad (2.16)$$

We replace the variable  $x$  with  $(qx)$ , and we replace the variable  $(dx)$  with  $(qdx)$ .

$$\frac{\Gamma(s)}{q^s} = \int_0^{\infty} x^{s-1} e^{-qx} dx \quad (2.17)$$

Hermann Hankel published the following integral representation<sup>11</sup> in 1863.

(The contour integration of the gamma function)

$$\frac{1}{\Gamma(1-s)} = \oint_{\gamma} \frac{dz}{2\pi iz} z^s e^z \quad (2.18)$$

An integral path of the gamma function is the path  $\gamma$  in the following figure. The white circles mean poles.

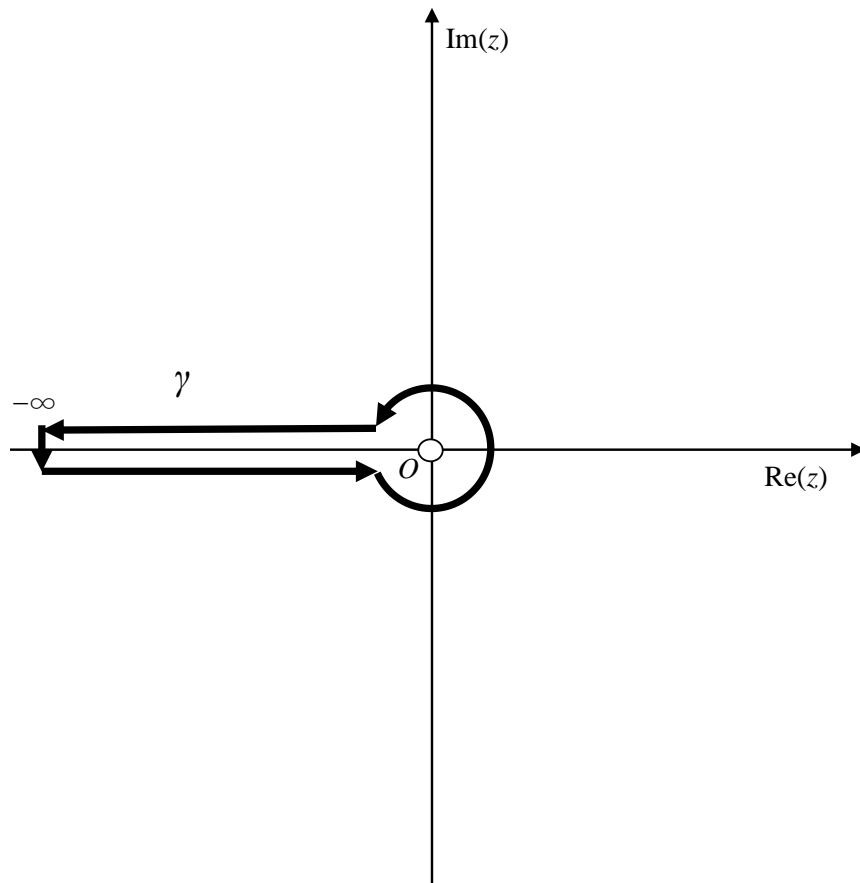


Figure 2-4: An integral path of the gamma function

We have the following formula for the gamma function.  
(Euler's reflection formula)

$$\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)} \frac{1}{\Gamma(s)} \quad (2.19)$$

## 2.5 Euler's beta function

Leonhard Euler introduced the beta function in 1768 in his book<sup>12</sup>.  
(The beta function)

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s + t)} \quad (2.20)$$

We obtain the following reflection formula of the beta function from the reflection formula of the gamma function.

(The reflection formula of the Beta function)

$$B(1 - s, 1 - t) = \frac{\pi \sin(\pi(s + t + 1))}{\sin(\pi s) \sin(\pi t)} \frac{1}{B(s, t - 1)(t - 1)} \quad (2.21)$$



## 2.6 The Riemann zeta function

Leonhard Euler introduced the infinite series of the zeta function in 1737.  
(The zeta function)

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (2.22)$$

Bernhard Riemann<sup>13</sup> expressed the zeta function by the gamma function in 1859.  
(The zeta function)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-x}}{1 - e^{-x}} dx \quad (2.23)$$

We interpret the above formula as the following Mellin transform.  
(The Mellin transform)

$$g(s) = M[G(x)] \quad (2.24)$$

$$g(s) = \int_0^{\infty} x^{s-1} G(x) dx \quad (2.25)$$

$$G(x) = \frac{e^{-x}}{1 - e^{-x}} \quad (2.26)$$

$$g(s) = \zeta(s)\Gamma(s) \quad (2.27)$$

We have the inverse Mellin transform as follows.  
(The inverse Mellin transform)

$$G(x) = M^{-1}[g(s)] \quad (2.28)$$

$$G(x) = \oint_C \frac{ds}{2\pi i} \frac{g(s)}{x^s} \quad (2.29)$$

$$G(x) = \frac{e^{-x}}{1 - e^{-x}} \quad (2.30)$$

$$g(s) = \zeta(s)\Gamma(s) \quad (2.31)$$

Bernhard Riemann introduced the analytic continuation of the zeta function in 1859.  
(The analytic continuation of the zeta function)

$$\frac{\zeta(s)}{\Gamma(1-s)} = \oint_{\gamma} \frac{dz}{2\pi i z} z^s \frac{e^z}{1 - e^z} \quad (2.32)$$

We interpret the above formula as the following the inverse Z-transform.  
(The inverse Z-transform)

$$h(s) = Z^{-1}[H(z)] \tag{2.33}$$

$$h(s) = \oint_{\gamma} \frac{dz}{2\pi iz} z^s H(z) \tag{2.34}$$

$$H(z) = \frac{e^z}{1 - e^z} \tag{2.35}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \tag{2.36}$$

We show the integral path  $\gamma$  in the following figure. The white circles mean poles.

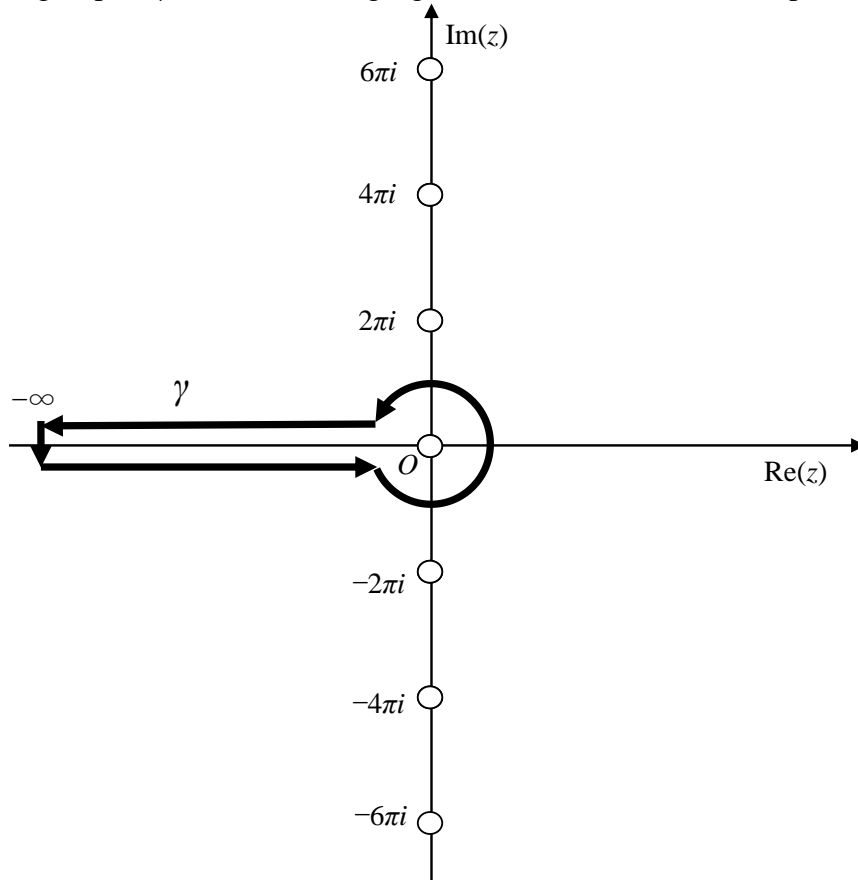


Figure 2-5: The integral path of the zeta function

We have the Z-transform of the zeta function as follows.  
(Z-transform)

$$H(z) = Z[h(s)] \quad (2.37)$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \quad (2.38)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (2.39)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \quad (2.40)$$

The generating functions of the Mellin transform and the Z-transform have the following relationship.

$$H(z) = -e^z G(z) \quad (2.41)$$

$$H(z) = G(-z) \quad (2.42)$$

Riemann showed the following formula.  
(Riemann's reflection formula)

$$\zeta(1 - s) = \frac{2}{(2\pi)^s} \Gamma(s) \cos\left(\frac{\pi}{2}s\right) \zeta(s) \quad (2.43)$$

Riemann proposed the following conjecture in 1859.  
(Riemann hypothesis)

Nontrivial zeros all have real part 1/2.

We express the examples of nontrivial zeros  $\rho_1$  and  $\rho_2$  in the following figure. The black circles are zeros, and the white circle means a pole.

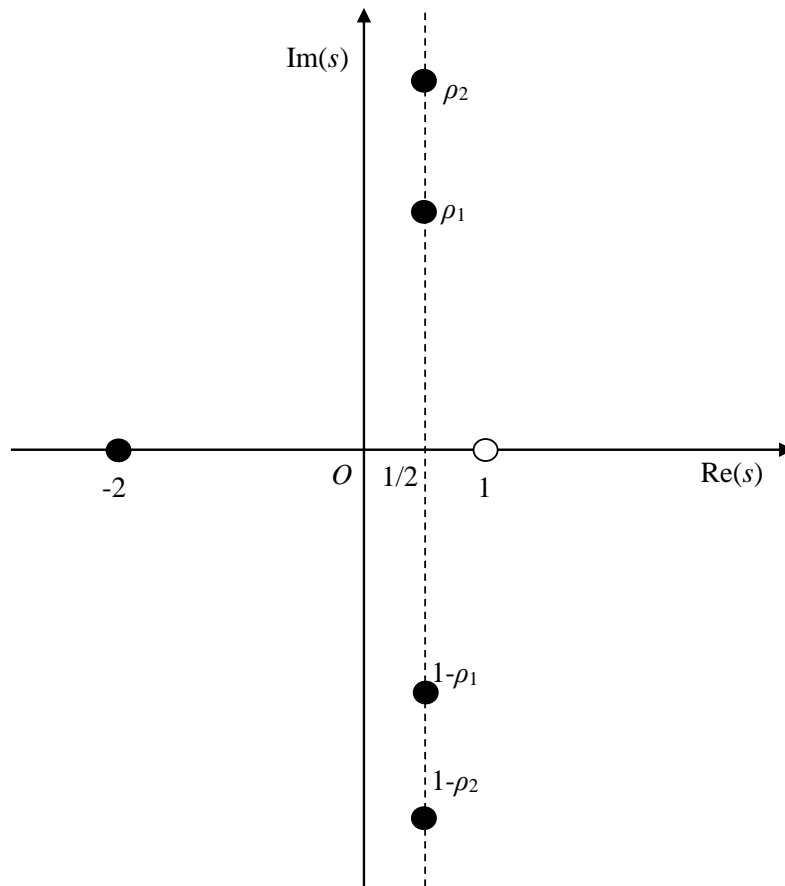


Figure 2-6: Nontrivial zeros of the zeta function

$$\rho_1 = \frac{1}{2} + i(14.13 \dots) \quad (2.44)$$

$$\rho_2 = \frac{1}{2} + i(21.02 \dots) \quad (2.45)$$

Since the proof of the Riemann hypothesis has not been successful, it has been an important mathematical issue.

## 2.7 The Bernoulli polynomials

We define the Bernoulli polynomials as follows.  
(The Bernoulli polynomials)

$$\frac{x e^{qx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(q)}{n!} x^n \quad (2.46)$$

The above series does not converge over the whole domain. The radius of convergence is  $2\pi$  because the minimum distance between the origin and poles of the generating function is  $2\pi$ .

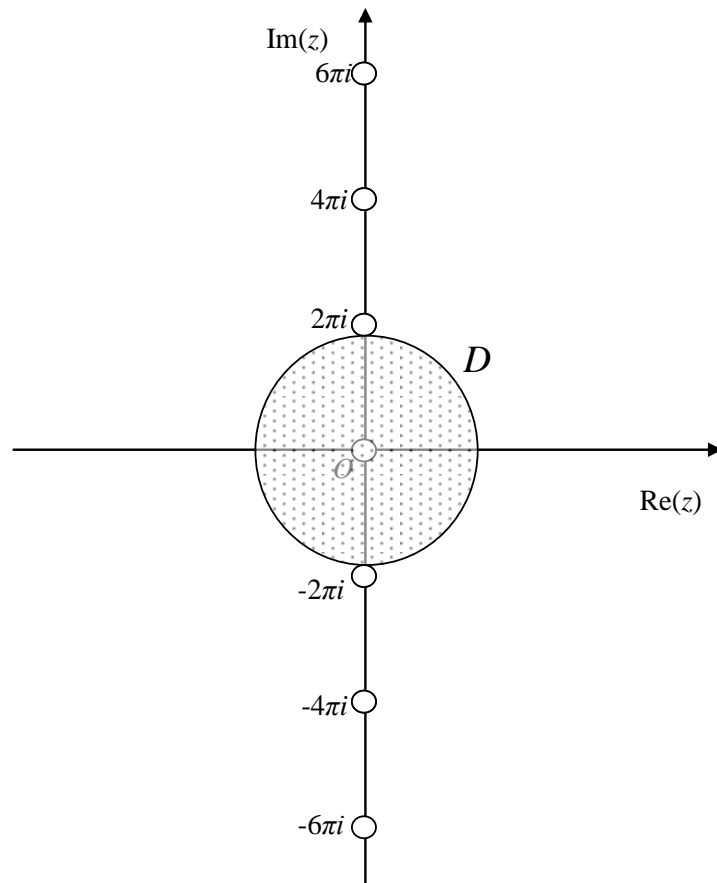


Figure 2-7: The radius of convergence of the Bernoulli polynomials

## 2.8 The Bernoulli numbers

Jakob Bernoulli introduced the Bernoulli numbers in 1713 in his book<sup>14</sup>.  
(The Bernoulli numbers)

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \quad (2.47)$$

We have the following formula for the even positive integer  $n$ .  
(The reflection formula of the Bernoulli numbers)

$$\zeta(n) = \frac{(2\pi)^n}{2} \frac{1}{n!} (-1)^{\frac{n}{2}+1} B_n \quad (2.48)$$

According to Vich's book<sup>15</sup>, we have the following Z-transform.  
(The Z-transform of the Bernoulli numbers)

$$\frac{\left(\frac{1}{z}\right) \exp\left(\frac{1}{z}\right)}{\exp\left(\frac{1}{z}\right) - 1} = Z \left[ \frac{B_n}{n!} \right] \quad (2.49)$$

In this paper, we use the following Z-transform.

(The Z-transform of the Bernoulli numbers)

$$H(z) = Z[h(s)] \quad (2.50)$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \quad (2.51)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (2.52)$$

$$h(s) = \frac{-B_{1-s}}{1-s} \frac{1}{\Gamma(1-s)} \quad (2.53)$$

On the other hand, we have the Z-transform of the zeta function as follows.

(Z-transform of the zeta function)

$$H(z) = Z[h(s)] \quad (2.54)$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \quad (2.55)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (2.56)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1-s)} \quad (2.57)$$

Therefore, we have the following equation.

$$\zeta(s) = -\frac{B_{1-s}}{1-s} \quad (2.58)$$

We obtain the following formula by replacing the variable  $s$  with the negative integer  $-n$ .

(The formula of the Bernoulli numbers)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (2.59)$$

Traditionally, we consider that the Bernoulli numbers are discrete. I consider that the Bernoulli numbers are continuous. We call the continual Bernoulli numbers the **Bernoulli function**. I interpret that the Bernoulli function is the following different representation of the zeta function.

$$B_s = -s\zeta(1-s) \quad (2.60)$$

### 3 Derivation of the reflection integral equation

#### 3.1 The framework of the method of derivation

We have the following inverse Mellin transform of the zeta function.

$$G(z) = M^{-1}[g(s)] \quad (3.1)$$

$$G(z) = \oint_C \frac{ds}{2\pi i} \frac{g(s)}{z^s} \quad (3.2)$$

$$G(z) = \frac{e^{-z}}{1 - e^{-z}} \quad (3.3)$$

$$g(s) = \zeta(s)\Gamma(s) \quad (3.4)$$

We have the following inverse Z-transform of the zeta function.

$$h(s) = Z^{-1}[H(z)] \quad (3.5)$$

$$h(s) = \oint_\gamma \frac{dz}{2\pi i z} z^s H(z) \quad (3.6)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (3.7)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \quad (3.8)$$

The generating functions of the Mellin transform and Z-transform have the following relations.

$$H(z) = -e^z G(z) \quad (3.9)$$

$$H(z) = G(-z) \quad (3.10)$$

We show the framework of the method of derivation in the following figure.

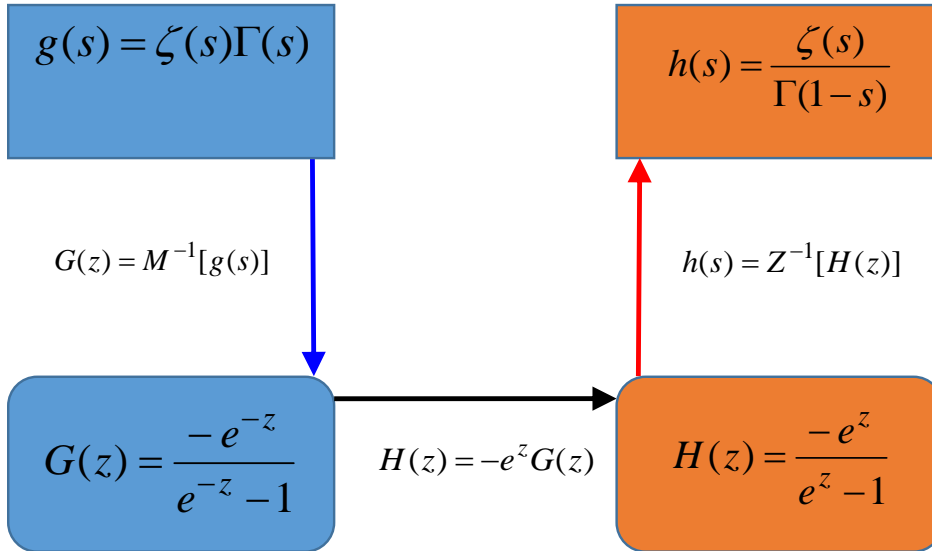


Figure 3-1: The framework of the method of derivation

In this paper, we obtain the following equation.  
 (The reflection integral equation)

$$\zeta(1-s) = \oint_C \frac{-dt}{2\pi i} B(s,t)\zeta(t) \quad (3.11)$$

This paper explains this derivation method.

### 3.2 Derivation of the reflection integral equation from the inverse Mellin transform

We have the following inverse Mellin transform of the zeta function.

$$G(z) = M^{-1}[g(t)] \quad (3.12)$$

$$G(z) = \oint_C \frac{dt}{2\pi i} \frac{g(t)}{z^t} \quad (3.13)$$

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \quad (3.14)$$

$$g(t) = \zeta(t)\Gamma(t) \quad (3.15)$$

In the inverse Mellin transform, the circuit of the integration  $C$  needs to circle around all poles of the integrand. Then, we adopt the following circuit of integration  $C$ . The white circles mean poles.



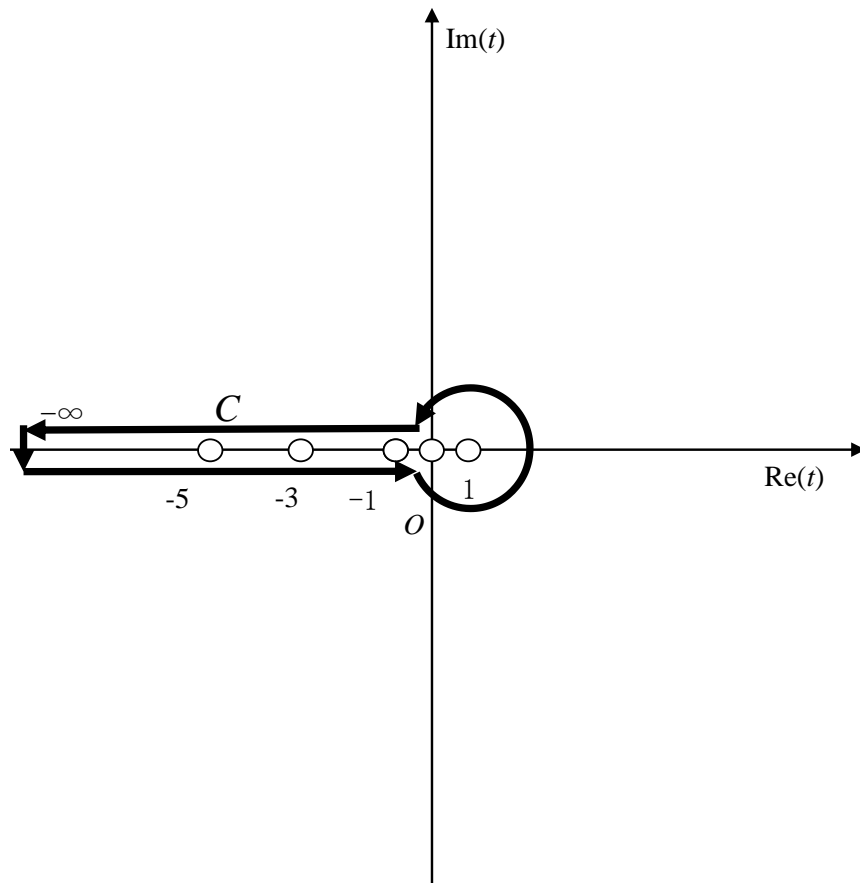


Figure 3-2: The integral path of the inverse Mellin transform

On the other hand, we have the following inverse Z-transform of the zeta function.  
(The inverse Z-transform)

$$h(s) = Z^{-1}[H(z)] \quad (3.16)$$

$$h(s) = \oint_{\gamma} \frac{dz}{2\pi iz} z^s H(z) \quad (3.17)$$

$$H(z) = \frac{e^z}{1 - e^z} \quad (3.18)$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1 - s)} \quad (3.19)$$

We show the integral path  $\gamma$  in the following figure. The white circles mean poles.

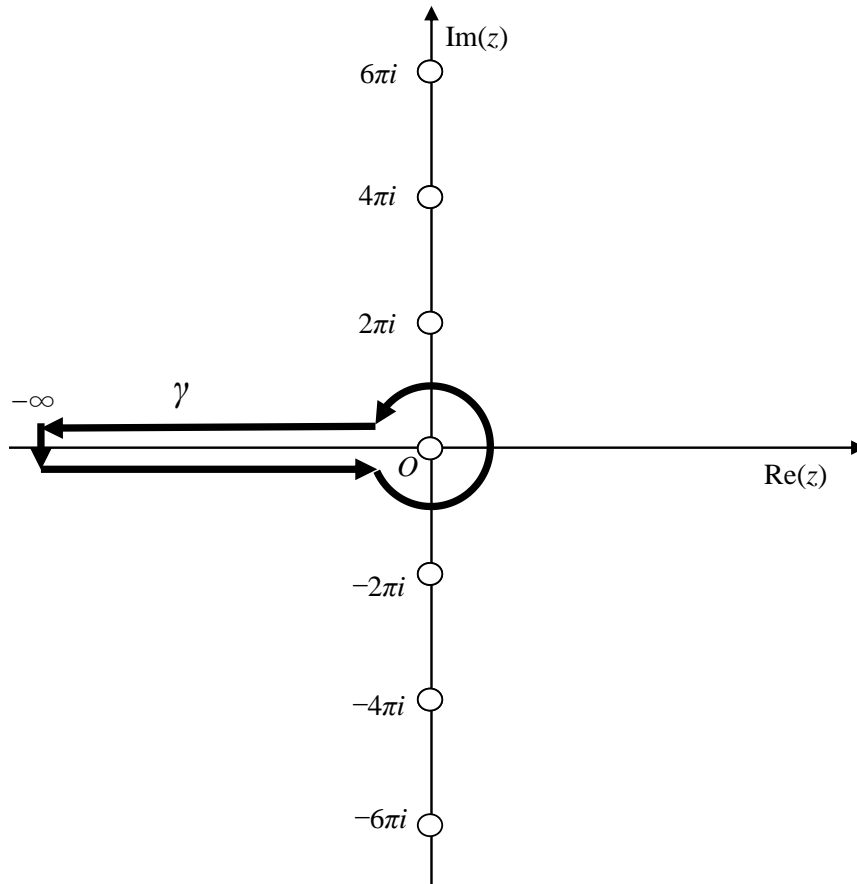


Figure 3-3: The integral path of the zeta function

By using the equation (2.41), we deform the inverse Z-transform as follows.

$$h(s) = \oint_{\gamma} \frac{dz}{2\pi iz} z^s \{-e^z G(z)\} \quad (3.20)$$

We replace the variable  $s$  with the variable  $(1-s)$  in the above formula.

$$h(1-s) = \oint_{\gamma} \frac{dz}{2\pi iz} \frac{1}{z^{s-1}} \{-e^z G(z)\} \quad (3.21)$$

We obtain the following equation by substituting the inverse Mellin transform for the function  $G(z)$ .

$$h(1-s) = \oint_{\gamma} \frac{dz}{2\pi iz} \frac{1}{z^{s-1}} \left\{ -e^z \oint_C \frac{dt}{2\pi i} \frac{g(t)}{z^t} \right\} \quad (3.22)$$

In order to take the integral of the above equation with the respect to the variable  $z$ , we deform the above equation.

$$h(1-s) = \oint_C \frac{-dt}{2\pi i} \left( \oint_\gamma \frac{dz}{2\pi i z} \frac{e^z}{z^{s+t-1}} \right) g(t) \quad (3.23)$$

We apply the following formula to the above equation.  
(Contour integration of gamma function)

$$\frac{1}{\Gamma(u)} = \oint_\gamma \frac{dz}{2\pi i z} \frac{e^z}{z^{u-1}} \quad (3.24)$$

Then, we get the following equation.

$$h(1-s) = \oint_C \frac{-dt}{2\pi i} \frac{g(t)}{\Gamma(s+t)} \quad (3.25)$$

$$\frac{\zeta(1-s)}{\Gamma(s)} = \oint_C \frac{-dt}{2\pi i} \frac{\Gamma(t)\zeta(t)}{\Gamma(s+t)} \quad (3.26)$$

$$\zeta(1-s) = \oint_C \frac{-dt}{2\pi i} \frac{\Gamma(s)\Gamma(t)\zeta(t)}{\Gamma(s+t)} \quad (3.27)$$

Here, we simplify the above equation by using the following beta function.

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad (3.28)$$

As the result, we obtain the following equation.  
(Reflection integral equation)

$$\zeta(1-s) = \oint_C \frac{-dt}{2\pi i} B(s, t)\zeta(t) \quad (3.29)$$

We show the integral path  $C$  in the following figure. The white circles mean poles.

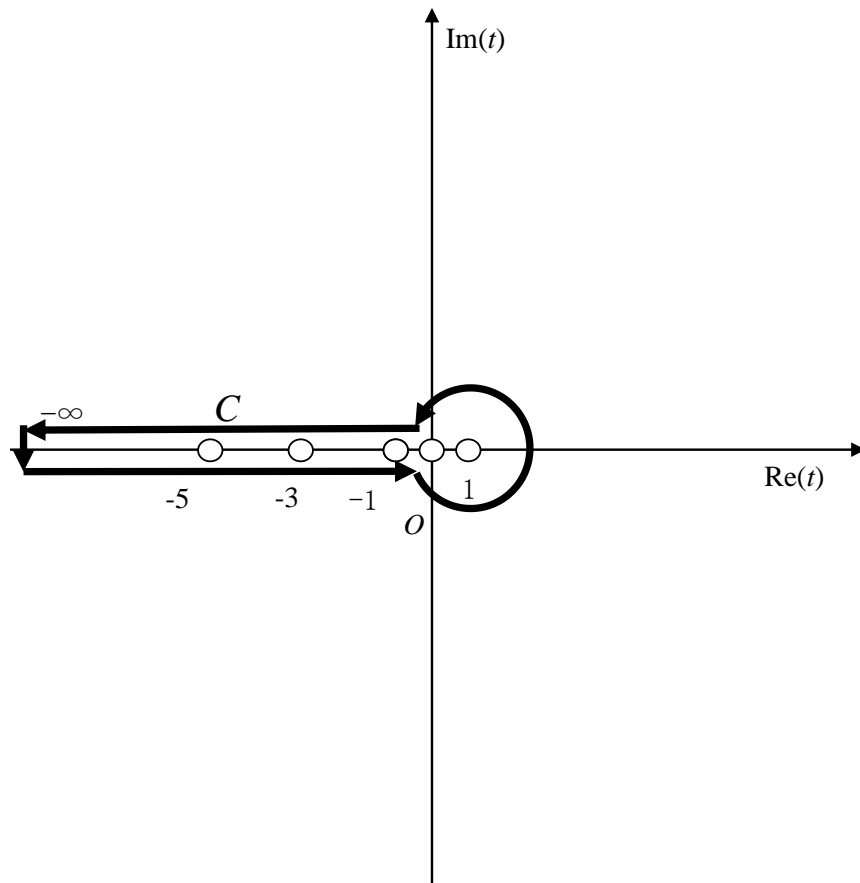


Figure 3-4: The integral path of the reflection integral equation of the zeta function

#### 4 Conclusion

We obtained the following result in this paper.

- We derived the reflection integral equation of the zeta function.

#### 5 Future issues

We have the following future issues.

- To derive Faulhaber's formula
- To derive Nörlund-Rice integral
- To study the eigenvalues of the integral equation
- To study the relation between the generating function of Z-transform and zeros

#### 6 Supplement

In the supplement, after we confirm the known results, we derive the following formulas.

- Faulhaber's formula
- Nörlund-Rice integral

##### 6.1 Confirmations of known results (Part 2)

In this chapter, we confirm the known results.

### 6.1.1 Hurwitz zeta function

Adolf Hurwitz<sup>16</sup> introduced the following zeta function in 1882.  
(The Hurwitz zeta function)

$$\zeta(s, q) = \frac{1}{(q+0)^s} + \frac{1}{(q+1)^s} + \frac{1}{(q+2)^s} + \dots = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s} \quad (6.1)$$

We have the following relationship between the Hurwitz zeta function and the Riemann zeta function.

$$\zeta(s) = \left\{ \frac{1}{1^s} + \frac{1}{2^s} + \dots + \frac{1}{(q-1)^s} \right\} + \left\{ \frac{1}{(q+0)^s} + \frac{1}{(q+1)^s} + \dots \right\} \quad (6.2)$$

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \zeta(s, q) \quad (6.3)$$

The Hurwitz zeta function becomes the Riemann zeta function when  $q$  is equal to one.

$$\zeta(s, 1) = \zeta(s) \quad (6.4)$$

We express the Hurwitz zeta function by the gamma function.  
(The Hurwitz zeta function)

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{e^{-x} - 1} dx \quad (6.5)$$

We interpret the above formula as the following Mellin transform.  
(The Mellin transform)

$$g(s, q) = M[G(x, q)] \quad (6.6)$$

$$g(s, q) = \int_0^{\infty} x^{s-1} G(x, q) dx \quad (6.7)$$

$$G(x, q) = \frac{e^{-qx}}{1 - e^{-x}} \quad (6.8)$$

$$g(s, q) = \zeta(s, q) \Gamma(s) \quad (6.9)$$

We have the following analytic continuation of the Hurwitz zeta function.  
(The analytic continuation of the Hurwitz zeta function)

$$\frac{\zeta(s, q)}{\Gamma(1-s)} = \oint_{\gamma} z^{s-1} \frac{e^{qz}}{1-e^z} \frac{dz}{2\pi i} \quad (6.10)$$

We interpret the above formula as the following inverse Z-transform.  
(The inverse Z-transform)

$$h(s, q) = Z^{-1}[H(z, q)] \quad (6.11)$$

$$h(s, q) = \oint_{\gamma} z^{s-1} H(z, q) \frac{dz}{2\pi i} \quad (6.12)$$

$$H(z, q) = \frac{e^{qz}}{1-e^z} \quad (6.13)$$

$$h(s, q) = \frac{\zeta(s, q)}{\Gamma(1-s)} \quad (6.14)$$

We have the following equation for the natural number  $n$ .  
(Formula of Bernoulli polynomials)

$$\zeta(-n, q) = -\frac{B_{n+1}(q)}{n+1} \quad (6.15)$$

### 6.1.2 Euler-Maclaurin formula

Euler<sup>17</sup> discovered the following formula in 1738. Maclaurin<sup>18</sup> have also discovered the same formula in 1742 independently.  
(The Euler-Maclaurin formula)

$$I = \int_p^q f(x) dx \quad (6.16)$$

$$S = \frac{1}{2} f(p) + f(p+1) + \dots + f(q-1) + \frac{1}{2} f(q) \quad (6.17)$$

$$S - I = \sum_{k=2}^r \frac{B_k}{k!} (f^{(k-1)}(q) - f^{(k-1)}(p)) + R \quad (6.18)$$

In the above formula, the variable  $R$  is an error term.

### 6.1.3 Asymptotic expansion

Euler<sup>19</sup> calculated the value of the zeta function by Euler-Maclaurin formula in 1755.

(The asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \frac{q^{1-s}}{s-1} + \sum_{k=1}^r \frac{B_k}{k!} \frac{(s+k-2)!}{(s-1)! q^{s+k-1}} + R \quad (6.19)$$

In the above formula, the variable  $R$  is an error term. The detail method to derive the above formula is shown in the book<sup>20</sup> by Edwards in 1974.

Here, we use the following equation.

$$\frac{q^{1-s}}{s-1} = \sum_{k=0}^0 \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s) q^{s+k-1}} \quad (6.20)$$

Then, we obtain the following formula.

(The asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s) q^{s+k-1}} + R \quad (6.21)$$

#### 6.1.4 Faulhaber's formula

Johann Faulhaber<sup>21</sup> published the formula of the sum of powers in 1631. We have the following formula for the natural number  $n$ .

(Faulhaber's formula)

$$\sum_{k=1}^q k^n = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k} B_k(0) q^{n+1-k} \quad (6.22)$$

We express the above formula by using the Bernoulli polynomial  $B_k(1)$ .

(Faulhaber's formula)

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k} B_k(1) q^{n+1-k} \quad (6.23)$$

In this paper, we express the above formula by the Bernoulli numbers  $B_k$  as follows.

(Faulhaber's formula)

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (6.24)$$

### 6.1.5 The Ramanujan master theorem

Srinivasa Ramanujan obtained the following theorem<sup>22</sup> about 1910.  
(The Ramanujan master theorem)

$$F(x) = \sum_{n=0}^{\infty} \frac{f(n)}{n!} (-x)^n \quad (6.25)$$

$$f(-s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} F(x) dx \quad (6.26)$$

We have the above equations for the following Bernoulli numbers and the zeta function.

$$F(x) = \frac{-x}{e^{-x} - 1} \quad (6.27)$$

$$f(n) = B_n \quad (6.28)$$

$$f(-s) = s\zeta(s+1) \quad (6.29)$$

This theorem suggests that the following relation.

$$\zeta(1-s) = -\frac{B_s}{s} \quad (6.30)$$

### 6.1.6 Woon's introduction of the continuous Bernoulli numbers

S. C. Woon<sup>23</sup> introduced the continuous Bernoulli numbers in 1997.

We have the following formula for the natural number  $n$ .  
(Formula of the Bernoulli numbers)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (6.31)$$

Woon proposed the following formula for the complex number  $s$ .  
(The formula of the Bernoulli function)

$$\zeta(1-s) = -\frac{B(s)}{s} \quad (6.32)$$

In this paper, we use the following notation for the Bernoulli function based on the notation for the Bernoulli numbers.

(The formula of the Bernoulli function)



$$\zeta(1-s) = -\frac{B_s}{s} \quad (6.33)$$

We obtain the following equation by substituting the above formula into Riemann's reflection formula.

$$-\frac{B_s}{s} = \frac{2}{(2\pi)^s} \Gamma(s) \cos\left(\frac{\pi}{2}s\right) \zeta(s) \quad (6.34)$$

We obtain the following equation by deforming the above formula.  
(The reflection formula of the Bernoulli function)

$$\zeta(s) = -\frac{(2\pi)^s}{2} \frac{1}{\Gamma(s+1)} \frac{1}{\cos(\pi s/2)} B_s \quad (6.35)$$

The above formula becomes the following formula for the even positive integer  $s$ .

$$\zeta(s) = \frac{(2\pi)^s}{2} \frac{1}{s!} (-1)^{\frac{s}{2}+1} B_s \quad (6.36)$$

The above formula is equal to the following formula for the even positive integer  $n$ .  
(The reflection formula of the Bernoulli numbers)

$$\zeta(n) = \frac{(2\pi)^n}{2} \frac{1}{n!} (-1)^{\frac{n}{2}+1} B_n \quad (6.37)$$

The above result suggests that the validity of the formula of the Bernoulli function.

### 6.1.7 Nörlund-Rice integral

Niels Erik Nörlund<sup>24</sup> published the Nörlund-Rice integral in 1924.  
(The Nörlund-Rice integral)

$$\sum_{k=c}^n \binom{n}{k} (-1)^k f(k) = \oint_C B(n+1, -t) f(t) \frac{dt}{2\pi i} \quad (6.38)$$

Here, the path  $C$  circles around poles  $c, \dots, n$  for positive integer  $c$ .  $B(x, y)$  is Euler's beta function.

Philippe Flajolet<sup>25</sup> published the Poisson-Mellin-Newton cycle in 1985 for the Nörlund-Rice integral.

(The Poisson-Mellin-Newton cycle)

$$F(x) = \frac{1}{e^x} \sum_{n=0}^{\infty} a_n x^n \quad (6.39)$$

$$f(s) = \int_0^{\infty} x^{s-1} F(x) dx \quad (6.40)$$

$$a_n = (-1)^n \oint_C \frac{n! \Gamma(t-n)}{\Gamma(t+1)} \frac{f(t)}{\Gamma(-t)} \frac{dt}{2\pi i} \quad (6.41)$$

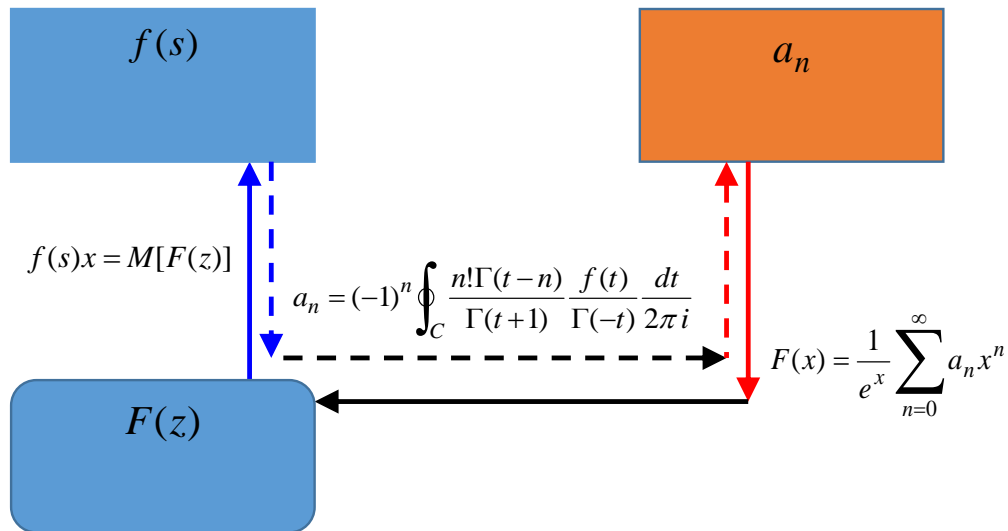


Figure 6-1: The Poisson-Mellin-Newton cycle

## 6.2 Derivation of Faulhaber's formula

In order to derive Faulhaber's formula, we derive the following formulas.

- The summation equation of the Riemann zeta function
- The summation equation of the Hurwitz zeta function
- The asymptotic expansion of the Riemann zeta function

### 6.2.1 Derivation of the summation equation of the Riemann zeta function

We derive the summation equation of the Riemann zeta function.

We have the following reflection integral equation.

(The reflection integral equation)

$$\zeta(1-s) = \oint_C -B(s,t)\zeta(t) \frac{dt}{2\pi i} \quad (6.42)$$

We replace the variable  $s$  with  $(1-s)$  and we replace the variable  $t$  with  $(1-t)$  in the above equation.  
(The reflection integral equation)

$$\zeta(s) = \oint_{C_0} B(1-s,1-t)\zeta(1-t) \frac{dt}{2\pi i} \quad (6.43)$$

We show the integral path  $C_0$  in the following figure. The white circles mean poles.

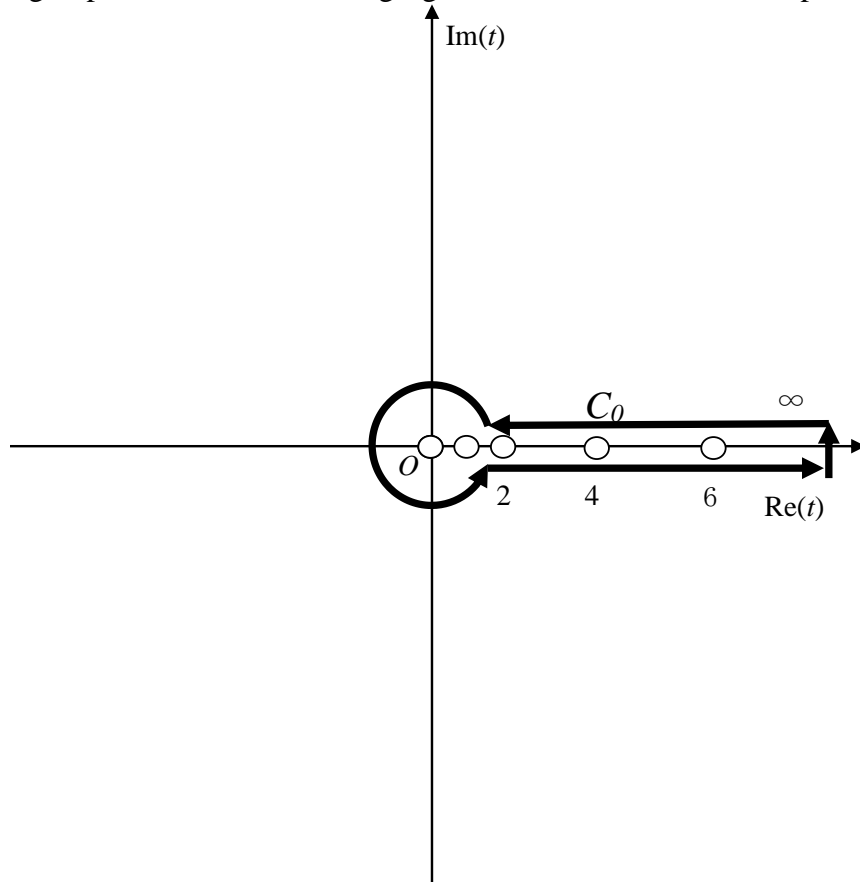


Figure 6-2: The integral path of the reflection integral equation

We substitute the following formula into the above equation.  
(Reflection formula of Beta function)

$$B(1-s,1-t) = \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s) \sin(\pi t)} \frac{1}{B(s,t-1)(t-1)} \quad (6.44)$$

Then, we obtain the following formula.

$$\zeta(s) = \oint_{C_0} \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s) \sin(\pi t)} \frac{\zeta(1-t)}{B(s, t-1)(t-1)} \frac{dt}{2\pi i} \quad (6.45)$$

We take the integral of the above integration by the residue theorem.

$$\zeta(s) = \sum_{k=1}^{\infty} \operatorname{Res}_{t=c_k} \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s) \sin(\pi t)} \frac{\zeta(1-t) dt}{B(s, t-1)(t-1)} \quad (6.46)$$

Here,  $c_k$  is  $k$ -th pole. The singularities are 0, 1, 2, 4, 6, ...

We have the following equation for integer  $n$ .

$$\operatorname{Res}_{t=n} \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s) \sin(\pi t)} dt = -1 \quad (6.47)$$

We express derive the following equation since the all singularities are integers and the value of the zeta function  $\zeta(1-t)$  is zero at  $t = 3, 5, 7, \dots$ .

(The summation equation)

$$\zeta(s) = \sum_{t=0}^{\infty} \frac{-1}{B(s, t-1)(t-1)} \zeta(1-t) \quad (6.48)$$

We are not able to calculate by the above equation because it diverges. In order to solve the problem, we derive the asymptotic expansion of the Hurwitz zeta function in the next section.

## 6.2.2 Derivation of the summation equation of the Hurwitz zeta function

In this section, we derive the summation equation of the Hurwitz zeta function.

We have the following Hurwitz zeta function.

(The Hurwitz zeta function)

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{e^{-x} - 1} dx \quad (6.49)$$

We express the above formula by the generating function of the Mellin transform.

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} G(x, q) dx \quad (6.50)$$

We obtain the following equation by deforming the above formula.

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \left\{ e^{-qx} H(x) \right\} dx \quad (6.51)$$

We obtain the following equation by substituting the  $Z$ -transform for the function  $H(x)$  of the above equation.

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \left\{ -e^{-qx} \sum_{t=0}^{\infty} \frac{\zeta(1-t)}{\Gamma(t)} x^{t-1} \right\} dx \quad (6.52)$$

$$D = \{0 < |x| < 2\pi\} \quad (6.53)$$

The Z-transform converges over the domain  $D$ . Therefore, we are able to change the order of the integration and the summation over the domain.

In order to take the integral of the above equation with respect to the variable  $x$ , we deform the above equation as follows.

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \sum_{t=0}^{\infty} \frac{-\zeta(1-t)}{\Gamma(t)} \int_0^{\infty} x^{s+t-2} e^{-qx} dx \quad (6.54)$$

We apply the following equation to the above equation.

$$\frac{\Gamma(s)}{q^s} = \int_0^{\infty} x^{s-1} e^{-qx} dx \quad (6.55)$$

As the result, we obtain the following equation.

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \sum_{t=0}^{\infty} \frac{-\zeta(1-t)}{\Gamma(t)} \frac{\Gamma(s+t-1)}{q^{s+t-1}} \quad (6.56)$$

Here, we simplify the above equation by using the following beta function.

$$\frac{1}{B(x, y)} = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \quad (6.57)$$

As the result, we obtain the following equation.

(The summation equation of the Hurwitz zeta function)

$$\zeta(s, q) = \sum_{t=0}^{\infty} \frac{-1}{B(s, t-1)(t-1)} \frac{\zeta(1-t)}{q^{s+t-1}} \quad (6.58)$$

The solution of the above equation reaches an infinite value because the convergent radius of a “definitional series of the Bernoulli function” is  $2\pi$ . Therefore, we change the upper limit of the summation to a variable  $r$  which depends on the variable  $q$ .

(The summation equation of the Hurwitz zeta function)

$$\zeta(s, q) = \sum_{t=0}^r \frac{-1}{B(s, t-1)(t-1)} \frac{\zeta(1-t)}{q^{s+t-1}} \quad (6.59)$$

### 6.2.3 Derivation of asymptotic expansion

In this section, we derive the asymptotic expansion of the zeta function.

We have the following relation between the Riemann and Hurwitz zeta function.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \zeta(s, q) \quad (6.60)$$

Therefore, we are able to express the summation equation of the Riemann zeta function as follows.  
(The summation equation)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{t=0}^r \frac{-1}{B(s, t-1)(t-1)} \frac{\zeta(1-t)}{q^{s+t-1}} \quad (6.61)$$

We obtain the following equation by replacing the beta function to the gamma function.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{-\Gamma(s+k-1)}{\Gamma(s)\Gamma(k-1)(k-1)} \frac{\zeta(1-k)}{q^{s+k-1}} \quad (6.62)$$

We deform the above equation by the formula of the Bernoulli polynomials as follows.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{-\Gamma(s+k-1)}{\Gamma(s)\Gamma(k-1)(k-1)} \frac{-kB_k}{q^{s+k-1}} \quad (6.63)$$

We obtain the following asymptotic expansion of the zeta function by deforming the above equation.

(The asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s)} \frac{1}{q^{s+k-1}} \quad (6.64)$$

### 6.2.4 Analytic continuation of asymptotic expansion

In this section, we derive the analytic continuation of asymptotic expansion of the zeta function.

(The analytic continuation of asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(1-s)}{\Gamma(2-s-k)} \frac{(-1)^{k-1}}{q^{s+k-1}} \quad (6.65)$$

The asymptotic expansion of the zeta function is shown below.

(The asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s)q^{s+k-1}} \quad (6.66)$$

We focus the following part of the above formula.

$$\frac{\Gamma(s+k-1)}{\Gamma(s)} \quad (6.67)$$

Here, we use the reflection formula of the gamma function.

$$\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \frac{1}{\Gamma(s)} \quad (6.68)$$

Then, we have the following equations.

$$\Gamma(s) = \frac{\pi}{\sin(\pi(1-s))} \frac{1}{\Gamma(1-s)} \quad (6.69)$$

$$\Gamma(s+k-1) = \frac{\pi}{\sin(\pi(2-s-k))} \frac{1}{\Gamma(2-s-k)} \quad (6.70)$$

Therefore, we obtain the following formula.

$$\frac{\Gamma(s+k-1)}{\Gamma(s)} = \frac{\sin(\pi(1-s))}{\sin(\pi(2-s-k))} \frac{\Gamma(1-s)}{\Gamma(2-s-k)} \quad (6.71)$$

Here we have the following equation for the integer  $k$ .

$$\frac{\sin(\pi(1-s))}{\sin(\pi(2-s-k))} = (-1)^{k-1} \quad (6.72)$$

Then, we obtain the following formula.

$$\frac{\Gamma(s+k-1)}{\Gamma(s)} = (-1)^{k-1} \frac{\Gamma(1-s)}{\Gamma(2-s-k)} \quad (6.73)$$

Therefore, we obtain the analytic continuation of asymptotic expansion of the zeta function.  
(The analytic continuation of asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(1-s)}{\Gamma(2-s-k)} \frac{(-1)^{k-1}}{q^{s+k-1}} \quad (6.74)$$

We expand the above formula by  $r = 3$ .

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} - \frac{1}{(1-s)} \frac{B_0}{q^{s-1}} + \frac{B_1}{q^s} + \frac{s}{2} \frac{B_2}{q^{s+1}} \quad (6.75)$$

Here we substitute the values into the Bernoulli numbers in the above formula.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} - \frac{1}{(1-s)q^{s-1}} + \frac{1}{2} \frac{1}{q^s} + \frac{s}{12} \frac{1}{q^{s+1}} \quad (6.76)$$

On the other hand, we are able to separate the first term as follows.

$$\sum_{k=1}^{q-1} \frac{1}{k^s} = \sum_{k=1}^q \frac{1}{k^s} - \frac{1}{q^s} \quad (6.77)$$

Therefore, we obtain the following formula.

$$\zeta(s) = \sum_{k=1}^q \frac{1}{k^s} - \frac{1}{(1-s)q^{s-1}} - \frac{1}{2} \frac{1}{q^s} + \frac{s}{12} \frac{1}{q^{s+1}} \quad (6.78)$$

We are able to calculate the value of the zeta function of  $s=0, -1, -2$  by the above formula.

$$\zeta(0) = \sum_{k=1}^q \frac{1}{k^0} - \frac{1}{q^{0-1}} - \frac{1}{2} \frac{1}{q^0} + \frac{0}{12} \frac{1}{q^1} = -\frac{1}{2} \quad (6.79)$$

$$\zeta(-1) = \sum_{k=1}^q \frac{1}{k^{-1}} - \frac{1}{2} \frac{1}{q^{-2}} - \frac{1}{2} \frac{1}{q^{-1}} + \frac{-1}{12} \frac{1}{q^0} = -\frac{1}{12} \quad (6.80)$$

$$\zeta(-2) = \sum_{k=1}^q \frac{1}{k^{-2}} - \frac{1}{3} \frac{1}{q^{-3}} - \frac{1}{2} \frac{1}{q^{-2}} + \frac{-2}{12} \frac{1}{q^{-1}} = 0 \quad (6.81)$$

The variable  $q$  comes from the following formula.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \zeta(s, q) \quad (6.82)$$

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=q}^{\infty} \frac{1}{k^s} \quad (6.83)$$

Therefore, the variable  $q$  of the following formula is not infinite but finite.  
(The analytic continuation of asymptotic expansion of the zeta function)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(1-s)}{\Gamma(2-s-k)} \frac{(-1)^{k-1}}{q^{s+k-1}} \quad (6.84)$$

### 6.2.5 Derivation of Faulhaber's formula

In this section, we derive Faulhaber's formula.



We have the following formula.  
(The asymptotic expansion of the zeta function)

$$\zeta(n) = \sum_{k=1}^{q-1} \frac{1}{k^n} + \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(n+k-1)}{\Gamma(n)} q^{n+k-1} \quad (6.85)$$

We replace the variable  $n$  with  $(-n)$  in the above equation.

$$\sum_{k=1}^{q-1} k^n = \zeta(-n) - \sum_{k=0}^r \frac{B_k}{k!} \frac{\Gamma(-n+k-1)}{\Gamma(-n)} q^{n+1-k} \quad (6.86)$$

We deform the above equation by Euler's reflection formula as follows.

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) - \sum_{k=0}^r \frac{B_k}{k!} \frac{\sin(\pi(n+1))}{\sin(\pi(n-k+2))} \frac{\Gamma(n+1)q^{n+1-k}}{\Gamma(n-k+2)} \quad (6.87)$$

We have the following equation for the integer  $k$ .

$$\frac{\sin(\pi(n+1))}{\sin(\pi(n-k+2))} = -(-1)^k \quad (6.88)$$

Therefore, we obtain the following equation.

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^r \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1)q^{n+1-k}}{\Gamma(n-k+2)} \quad (6.89)$$

We have the following equation for the natural number  $n$  and the integer  $k > n+1$ .

$$\frac{1}{\Gamma(n-k+2)} = 0 \quad (6.90)$$

Therefore, we obtain the following equation.

$$\sum_{k=n+2}^r \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1)q^{n+1-k}}{\Gamma(n-k+2)} = 0 \quad (6.91)$$

We change the upper limit of summation to  $n+1$  of the equation (6.89) because the values of the all terms at  $k > n+1$  are zero.

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^{n+1} \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1)q^{n+1-k}}{\Gamma(n-k+2)} \quad (6.92)$$

We express the following equation by using the factorial.

$$\sum_{k=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^{n+1} (-1)^k \frac{n! B_k q^{n+1-k}}{k!(n-k+1)!} \quad (6.93)$$

We express the following equation by using the binomial coefficient.

$$\sum_{k=1}^{q-1} k^n = \zeta(-n) + \frac{1}{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (6.94)$$

We deform the above equation by the formula of the Bernoulli polynomials as follows.

$$\sum_{k=1}^{q-1} k^n = -\frac{B_{n+1}}{n+1} + \frac{1}{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (6.95)$$

We have the following equation for the natural number  $n$  and the integer  $k = n+1$ .

$$\frac{B_{n+1}}{n+1} = \frac{1}{n+1} (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (6.96)$$

According to the above result, we obtain the following equation.

$$-\frac{B_{n+1}}{n+1} + \frac{1}{n+1} \sum_{k=n+1}^{n+1} (-1)^k \binom{n+1}{k} B_k q^{n+1-k} = 0 \quad (6.97)$$

Therefore, we obtain the following Faulhaber's formula by deforming the formula (6.95).  
(Faulhaber's formula)

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \quad (6.98)$$

## 6.3 Derivation of Nörlund-Rice integral

### 6.3.1 Derivation of the reflection integral formula

We define a new function  $H(z)$ .

$$H(z) = -e^z G(z) \quad (6.99)$$

We obtain a new function  $g(s)$  from the Mellin transform of the function  $G(z)$ .

$$g(s) = M[G(z)] \quad (6.100)$$

We have the following inverse Mellin transform.

$$G(z) = M^{-1}[g(s)] \quad (6.101)$$

We obtain a new function  $h(s)$  from the inverse Z-transform of the function  $H(z)$ .

$$h(s) = Z^{-1}[H(z)] \quad (6.102)$$

We show the relation of the above functions in the following figure.

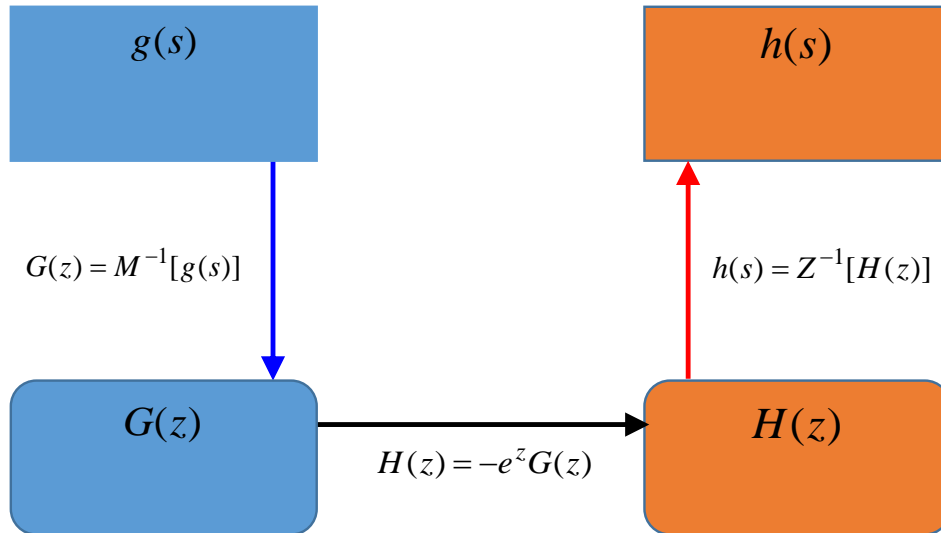


Figure 6-3: The inverse Mellin transform and the inverse Z-transform

We have the following formula of the inverse Z-transform.

$$h(s) = \oint_{\gamma} z^{s-1} H(z) \frac{dz}{2\pi i} \quad (6.103)$$

We deform the formula of the inverse Z-transform as follows.

$$h(s) = \oint_{\gamma} z^{s-1} \left\{ -e^z G(z) \right\} \frac{dz}{2\pi i} \quad (6.104)$$

We substitute the inverse Mellin transform for the function  $G(x)$  of the above formula.

$$h(s) = \oint_{\gamma} z^{s-1} \left\{ -e^z \oint_C \frac{g(t)}{z^t} \frac{dt}{2\pi i} \right\} \frac{dz}{2\pi i} \quad (6.105)$$

In order to take the integral of the above formula with respect to the variable  $z$ , we deform the above formula.

$$h(s) = \oint_C - \left\{ \oint_{\gamma} z^{s-t-1} e^z \frac{dz}{2\pi i} \right\} g(t) \frac{dt}{2\pi i} \quad (6.106)$$

We apply the following contour integration of the gamma function to the above equation.

$$\frac{1}{\Gamma(1-s)} = \oint_{\gamma} z^{s-1} e^z \frac{dz}{2\pi i} \quad (6.107)$$

As the result, we obtain the following equation.

$$h(s) = \oint_C \frac{-1}{\Gamma(1-s+t)} g(t) \frac{dt}{2\pi i} \quad (6.108)$$

We define the new functions  $\phi(s)$  and  $\chi(s)$  as follows.

$$\phi(s) = \frac{g(s)}{\Gamma(s)} \quad (6.109)$$

$$\chi(s) = h(s)\Gamma(1-s) \quad (6.110)$$

Then, we are able to deform the formula (6.108) as follows.

$$\chi(s) = \oint_C \frac{-\Gamma(1-s)\Gamma(t)}{\Gamma(1-s+t)} \phi(t) \frac{dt}{2\pi i} \quad (6.111)$$

We express the above formula by Euler's Beta function as follows.  
(The reflection integral formula)

$$\chi(s) = \oint_C -B(1-s, t) \phi(t) \frac{dt}{2\pi i} \quad (6.112)$$

### 6.3.2 Derivation of Nörlund-Rice integral

In this section, we derive Nörlund-Rice integral.

We obtain the summation formula by adopting residue theorem to the reflection integral formula.  
(The summation formula)

$$\chi(s) = \sum_{t=c+1}^{m+1} \frac{-1}{B(s, t-1)(t-1)} \phi(1-t) \quad (6.113)$$

Here, the function  $\phi(s)$  has the poles  $c+1, \dots, m+1$ .

We replace the variable  $s$  with  $(-n)$ , and we replace the variable  $t$  with  $(k+1)$  in the above equation.

$$\chi(-n) = \sum_{k=c}^m \frac{-1}{B(-n, k)(k)} \phi(-k) \quad (6.114)$$

Here, we define a new function  $f(k)$ .

$$f(k) = \phi(-k) \quad (6.115)$$

We express the formula by the function  $f(k)$ .

$$\chi(-n) = \sum_{k=c}^m \frac{-1}{B(-n, k)(k)} f(k) \quad (6.116)$$

We replace the gamma function for the beta function.

$$\chi(-n) = \sum_{k=c}^m \frac{-\Gamma(-n+k)}{\Gamma(-n)\Gamma(k+1)} f(k) \quad (6.117)$$

We obtain the following formula by using Euler's reflection formula.

$$\chi(-n) = \sum_{k=c}^m \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \frac{\sin(\pi(n+1))}{\sin(\pi(n-k+1))} f(k) \quad (6.118)$$

We have the following equation for the integer  $k$ .

$$\frac{\sin(\pi(n+1))}{\sin(\pi(n-k+1))} = (-1)^k \quad (6.119)$$

Therefore, we obtain the following formula.

$$\chi(-n) = \sum_{k=c}^m \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} (-1)^k f(k) \quad (6.120)$$

We have the following equation for the natural number  $n$  and the integer  $k > n$ .

$$\frac{1}{\Gamma(n-k+1)} = 0 \quad (6.121)$$

We change the variable  $m$  to  $n$ , because the values of the all terms at  $k > n$  are zero.

$$\chi(-n) = \sum_{k=c}^n \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} (-1)^k f(k) \quad (6.122)$$

We express the following equation by using the factorial.

$$\chi(-n) = \sum_{k=c}^n (-1)^k \frac{-n!}{(n-k)!k!} f(k) \quad (6.123)$$

We express the following equation by using the binomial coefficient.

$$-\chi(-n) = \sum_{k=c}^n \binom{n}{k} (-1)^k f(k) \quad (6.124)$$

On the other hand, we have the following reflection integral formula.  
(Reflection integral formula)

$$\chi(s) = \oint_C -B(1-s, t) \phi(t) \frac{dt}{2\pi i} \quad (6.125)$$

We replace the variable  $s$  with  $(-n)$ , and we replace the variable  $t$  to  $(-t)$  in the above equation.

$$-\chi(n) = \oint_C -B(1+n, -t) \phi(-t) \frac{dt}{2\pi i} \quad (6.126)$$

Here, we define a new function  $f(t)$ .

$$f(t) = \phi(-t) \quad (6.127)$$

Then, we obtain the following equation.

$$-\chi(-n) = \oint_C B(n+1, -t) f(t) \frac{dt}{2\pi i} \quad (6.128)$$

Therefore, we obtain the following equation.  
(The Nörlund-Rice integral)

$$\sum_{k=c}^n \binom{n}{k} (-1)^k f(k) = \oint_C B(n+1, -t) f(t) \frac{dt}{2\pi i} \quad (6.129)$$

## 7 Appendix

### 7.1 The table of the Z-transform

We show the table of the Z-transform as follows.

$f(s)$	$F(z) = Z[f(s)]$	Num.
$f(s) = \frac{\zeta(s)}{\Gamma(1-s)}$	$F(z) = \frac{e^z}{1-e^z}$	(7.1)
$f(s) = \frac{\eta(s)}{\Gamma(1-s)}$	$F(z) = \frac{e^z}{1+e^z}$	(7.2)
$f(s, q) = \frac{\zeta(s, q)}{\Gamma(1-s)}$	$F(z, q) = \frac{e^{qz}}{1-e^z}$	(7.3)
$f(s, q) = \frac{\eta(s, q)}{\Gamma(1-s)}$	$F(z, q) = \frac{e^{qz}}{1+e^z}$	(7.4)
$f(s, q, w) = \frac{\Phi(w, s, q)}{\Gamma(1-s)}$	$F(z, q, w) = \frac{e^{qz}}{1-we^z}$	(7.5)
$f(s, q, \lambda) = \frac{L(\lambda, q, s)}{\Gamma(1-s)}$	$F(z, q, \lambda) = \frac{e^{qz}}{1-\exp(2\pi i\lambda)e^z}$	(7.6)

We show the above functions as follows.

(The Riemann zeta function)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-x}}{1-e^{-x}} dx \quad (7.7)$$

(The Dirichlet<sup>26</sup> eta function)

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-x}}{1+e^{-x}} dx \quad (7.8)$$

(The Hurwitz zeta function)

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{1-e^{-x}} dx \quad (7.9)$$

(The Hurwitz eta function)

$$\eta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{1+e^{-x}} dx \quad (7.10)$$

(The Lerch transcendent<sup>27</sup>)

$$\Phi(w, s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{1 - we^{-x}} dx \quad (7.11)$$

(The Lerch zeta function)

$$L(\lambda, q, s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{1 - \exp(2\pi i\lambda)e^{-x}} dx \quad (7.12)$$

The formulas of the polynomials are shown below.

$$\zeta(1-n) = -\frac{B_n(1)}{n} \quad (7.13)$$

$$\eta(-n) = \frac{E_n(1)}{2} \quad (7.14)$$

$$\zeta(1-n, q) = -\frac{B_n(q)}{n} \quad (7.15)$$

$$\eta(-n, q) = \frac{E_n(q)}{2} \quad (7.16)$$

$$\Phi(w, 1-n, q) = -\frac{\beta_n(q, w)}{n} \quad (7.17)$$

We show the definition of the polynomials as follows.

(The Bernoulli polynomials)

$$\frac{xe^{qx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(q)}{n!} x^n \quad (7.18)$$

(The Euler polynomials)

$$\frac{2e^{qx}}{e^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(q)}{n!} x^n \quad (7.19)$$

(The Apostol-Bernoulli polynomials <sup>28</sup>)

$$\frac{xe^{qx}}{we^x - 1} = \sum_{n=0}^{\infty} \frac{\beta_n(q, w)}{n!} x^n \quad (7.20)$$



## 7.2 Derivation of the reflection integral equation from the reflection integral formula

We show the reflection integral formula as follows.  
(The reflection integral formula)

$$\chi(s) = \oint_C -B(1-s, t)\phi(t) \frac{dt}{2\pi i} \quad (7.21)$$

In the above equation, we suppose the following condition.

$$H(-z) = G(z) \quad (7.22)$$

Then, we obtain the following relation.

$$\phi(s) = \chi(s) \quad (7.23)$$

Therefore, the following two equations are equivalent.

$$G(-z) = -e^z G(z) \quad (7.24)$$

$$\phi(1-s) = \oint_C -B(s, t)\phi(t) \frac{dt}{2\pi i} \quad (7.25)$$

## 8 Bibliography

<sup>1</sup> Mail: [sugiyama\\_xs@yahoo.co.jp](mailto:sugiyama_xs@yahoo.co.jp)

<sup>2</sup> Andres Odlyzko, “[Correspondence about the origins of the Hilbert-Polya Conjecture](#)”, (1981).

<sup>3</sup> Zeev Rudnick; Peter Sarnak, “[Zeros of Principal L-functions and Random Matrix Theory](#)”, Duke Journal of Mathematics 81 (1996): 269-322.

<sup>4</sup> Yu. I. Manin, “Lectures on zeta functions and motives (according to Deninger and Kurokawa)”, Astérisque No. 228 (1995), 4, 121-163.

<sup>5</sup> Alain Connes, “Trace formula in noncommutative geometry and the zeros of the Riemann zeta function” (1998), <http://arxiv.org/abs/math/9811068>.

<sup>6</sup> C. Deninger, “Some analogies between number theory and dynamical systems on foliated spaces”, Doc. Math. J. DMV. Extra Volume ICMI (1998), 23-46.

<sup>7</sup> Hjalmar Mellin, “Die Dirichlet'schen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichem Geschlecht”, Acta Math. 28 (1904), 37-64.

<sup>8</sup> Witold Hurewicz, “Filters and Servo Systems with Pulsed Data”, in Theory of Servomechanics. McGraw-Hill (1947).

<sup>9</sup> Augustin-Louis Cauchy, “Mémoire sur les rapports qui existent entre le calcul des Résidus et le calcul des Limites, et sur les avantages qu'offrent ces deux calculs dans la résolution des équations algébriques ou transcendentes (Memorandum on the connections that exist between the residue calculus and the limit calculus, and on the advantages that these two calculi offer in solving algebraic and transcendental equations)”, presented to the Academy of Sciences of Turin, November 27, (1831).

<sup>10</sup> Leonhard Euler, Euler's letter to Goldbach 15 October (1729) (OO715).

<sup>11</sup> Hermann Hankel, “Die Euler'schen integrale bei unbeschränkter Variabilität der Arguments”, Zeitschrift für Math. und Physik 9 (1863) 1-21.

- 
- <sup>12</sup> Leonhard Euler, E342 - "Institutionum calculi integralis volumen primum (Foundations of Integral Calculus, volume 1)", First Section, De integratione formularum differentialum, Chapter 9, De evolutione integralium per producta infinita. (1768).
- <sup>13</sup> Bernhard Riemann, "Über die Anzahl der Primzahlen unter einer gegebenen Grösse (On the Number of Primes Less Than a Given Magnitude)", Monatsberichte der Berliner Akademie, 671-680 (1859).
- <sup>14</sup> Jakob Bernoulli, "Ars Conjectandi (The Art of Conjecturing)" (1713).
- <sup>15</sup> R. Vich, "Z-transform Theory and Applications", D. Reidel Publishing Company, (1987).
- <sup>16</sup> Adolf Hurwitz, Zeitschrift für Mathematik und Physik vol. 27 (1882) p. 95.
- <sup>17</sup> Leonhard Euler, Comment. Acad. Sci. Imp. Petrop. , 6 (1738) pp. 68-97.
- <sup>18</sup> Colin Maclaurin, "A treatise of fluxions", 1-2, Edinburgh (1742).
- <sup>19</sup> Leonhard Euler, E212 - "Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum" (Foundations of Differential Calculus, with Applications to Finite Analysis and Series), Part II, Chapter 6: De summatione progressionum per series infinitas. (1755).
- <sup>20</sup> H. M. Edwards, "Riemann's Zeta Function", Academic Press, (1974).
- <sup>21</sup> Johann Faulhaber, "Academia Algebrae - Darinnen die miraculosische Inventiones zu den höchsten Cossen waiters continuity und profiteer warden" (1631).
- <sup>22</sup> B. C. Berndt, "Ramanujan's Notebooks: Part I", New York: Springer-Verilog, p. 298, (1985).
- <sup>23</sup> S. C. Woon, "Analytic Continuation of Bernoulli Numbers, a New Formula for the Riemann Zeta Function, and the Phenomenon of Scattering of Zeros" (1997), <http://arxiv.org/abs/physics/9705021>
- <sup>24</sup> Niels Erik Nörlund, "Vorlesungen über Differenzenrechnung", Teubner, Leipzig and Berlin, (1924).
- <sup>25</sup> Philippe Flajolet, Mireille Regnier, and Robert Sedgewick, "Some uses of the Mellin integral transform in the analysis of algorithms", Combinatorics on Words, NATO ASI Series F, Vol. 12 (Springer, Berlin, 1985).
- <sup>26</sup> Dirichlet, P. G. L., "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält", Abhand. Ak. Wiss. Berlin 48 (1837).
- <sup>27</sup> Lerch, Mathias, "Note sur la fonction  $K(w, x, s) = \sum_{k=0}^{\infty} e^{2k\pi ix} (w+k)^{-s}$ ", Acta Mathematica (in French) 11 (1887) (1-4): 19-24.
- <sup>28</sup> Tom M. Apostol, "On the Lerch zeta function", Pacific J. Math., 1, 161-167 (1951).