New proof that the sum of natural numbers is -1/12 of the zeta function

2016/07/09
Published 2014/3/30
K. Sugiyama[1]

We prove that the sum of natural numbers is -1/12 of the value of the zeta function by the new method.

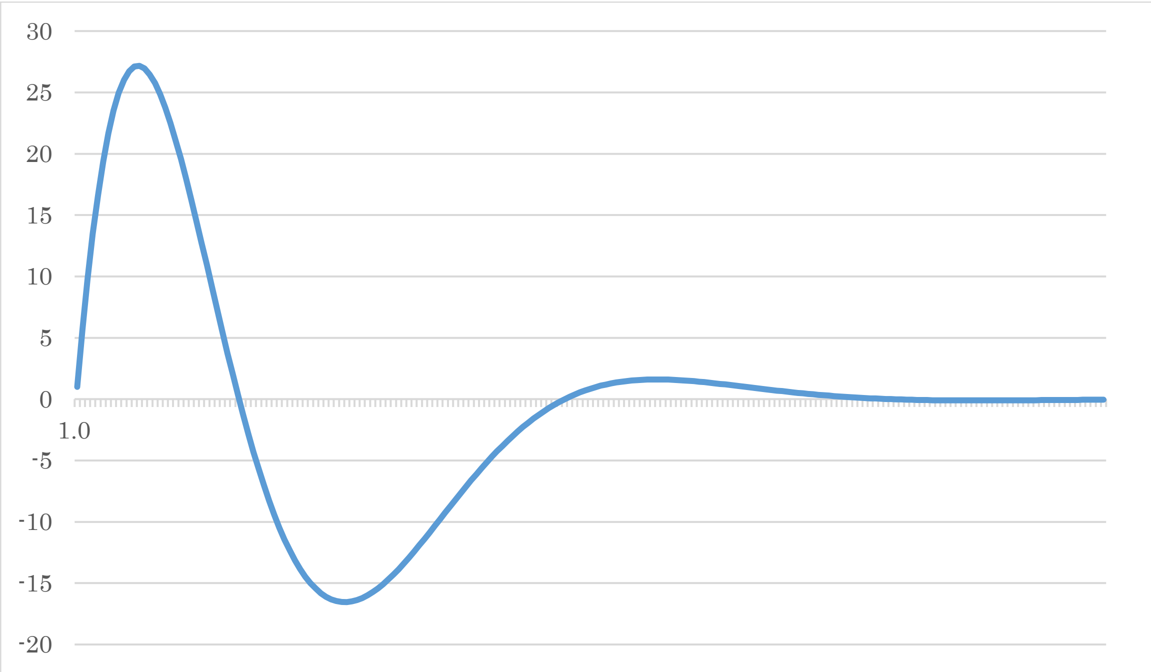

Figure 5.1: Damped oscillation of natural number

Abel calculated the sum of the divergent series by the Abel summation method. However, we cannot calculate the sum of natural numbers by the method. In this paper, we calculate the sum of the natural numbers by the extended Abel summation method.

2010 *Mathematics Subject Classification*: Primary 11M06; Secondary 11B65.

CONTENTS

1 Introduction 2
1.1 Issue 2
1.2 Importance of the issue 2
1.3 Research trends so far 2
1.4 New derivation method of this paper 3
1.5 Old method by the Abel summation method 3
2 New method 6
2.1 New proof for the summation formula of natural numbers 6
2.2 Proof of the proposition 1 7
2.3 Proof of the proposition 2 8

3 Conclusion 10
4 Future issues 10
5 Supplement 11
5.1 Numerical calculation of the oscillating Abel summation method 11
5.2 General formula of the oscillating Abel summation method 12
5.3 The mean Abel summation method by residue theorem 15
5.4 Definitions of the propositions by the (ε, δ)-definition of limit 19
6 Appendix 22
6.1 Proof of the summation formula of the powers of the natural numbers 22
6.1.1 Cauchy's integral formula and mean formula 22
6.1.2 Laurent series and the pole of the order zero 24
6.1.3 The proof of the summation formula of the natural numbers by harmonic derivative 25
6.1.4 The proof of the summation formula of the powers of the natural numbers by harmonic overtone derivative 27
6.1.5 The second proof 29
6.2 Riemann zeta function 32
6.3 Jacobi's theta function 34
6.4 Relation with the q-analog 35
6.4.1 The q-analog 35
7 Acknowledgment 36
8 Bibliography 36

1 Introduction

1.1 Issue

The integral representation of the zeta function converges by the analytic continuation. On the other hand, the zeta function also has a series representation. One of the representation is the sum of all natural numbers. The sum does not converge. It diverges. This paper converges the sum by the extended Abel summation method.

$$"1+2+3+\cdots" = -\frac{1}{12} \tag{1.1}$$

1.2 Importance of the issue

It is desirable that both the integral representation and the series representation have a same value for mathematical consistency. One method to converge the divergent series is the Abel summation method. The method converges the divergent series by multiplying convergence factor. However, the sum of all natural numbers does not converge by the method. Therefore, it is an important issue to find a new summation method.

1.3 Research trends so far

Leonhard Euler[2] suggested that the sum of all natural numbers is -1/12 in 1749. Bernhard Riemann[3] showed that the integral representation of the zeta function is -1/12 in 1859. Srinivasa Ramanujan[4] proposed that the sum of all natural numbers is -1/12 by Ramanujan summation method in 1913.

Niels Abel[5] introduced the Abel summation method in order to converge the divergent series in about 1829. J. Satoh[6] constructed q-analogue of zeta-function in 1989. M. Kaneko, N. Kurokawa, and M. Wakayama[7] derived the sum of the double quoted natural numbers by the q-analogue of zeta-function in 2002.

1.4 New derivation method of this paper

We define a new sum of all natural numbers by a new summation method, oscillating Abel summation method (damped oscillation summation method). The method converges the divergent series by multiplying convergence factor, which is damped and oscillating very slowly. The traditional sum diverges for the infinite terms. On the other hand, the new sum is equal to the traditional sum for the finite term. In addition, the sum converges on -1/12 for the infinite term.

(Summation formula of natural numbers)

$$\lim_{x \to 0+} \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx) = -\frac{1}{12} \tag{1.2}$$

1.5 Old method by the Abel summation method

Niels Abel introduced the Abel summation method in about 1829.

We consider the following sum of the series.

$$S = \sum_{k=1}^{\infty} a_k \tag{1.3}$$

Then we define the following function.
(The Abel summation method)

$$F(x) = \sum_{k=1}^{\infty} a_k \exp(-kx) \tag{1.4}$$

$$0 < x \tag{1.5}$$

We define Abel sum as follows.

$$S_A = \lim_{x \to 0+} F(x) \tag{1.6}$$

We consider the following divergent series.

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} k \tag{1.7}$$

Then we define the following function.

$$G(x) = \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-kx) \tag{1.8}$$

$$0 < x \tag{1.9}$$

Here, we will use the following formula.
(Formula of the geometric series that has coefficients of natural numbers)

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \tag{1.10}$$

We obtain the following equation by the above formula.

$$F(x) = \frac{-e^{-x}}{(1+e^{-x})^2} \tag{1.11}$$

We calculate the Abel sum as follows.

$$S_A = \lim_{x \to 0+} F(x) = \frac{-e^{-0}}{(1+e^{-0})^2} = -\frac{1}{4} \tag{1.12}$$

Abel calculated the sum of the divergent series by the Abel summation method as above stated. However, we cannot calculate the sum of natural numbers by the method. We explain the reason as follows.

We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{1.13}$$

Then we define the following function.

$$G(x) = \sum_{k=1}^{\infty} k \exp(-kx) \tag{1.14}$$

$$0 < x \tag{1.15}$$

Here, we will use the following formula.
(Formula of the geometric series that has coefficients of natural numbers)

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \tag{1.16}$$

We obtain the following equation by the above formula.

$$G(x) = \frac{e^{-x}}{(1-e^{-x})^2} \tag{1.17}$$

We calculate the Abel sum as follows.

$$S_A = \lim_{x \to 0+} G(x) = \frac{-e^{-0}}{(1-e^{-0})^2} = \infty \tag{1.18}$$

The Abel sum diverges as above stated. Therefore, we cannot calculate the sum of the natural numbers by the method.

In order to examine the mechanism of the divergence, we use the following definitional formula of Bernoulli numbers.

(Definitional formula of Bernoulli numbers)

$$\frac{ze^z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \tag{1.19}$$

In this paper, Bernoulli numbers B_n is Bernoulli polynomial B_n (1).

We will square the both sides of the above formula. In addition, we will divide the both sides by z^2. Then we obtain the following equation.

$$\frac{e^{2z}}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)^2 \tag{1.20}$$

We will multiply the left side of the above formula by e^{-z}. In addition, we will multiply the right side by Maclaurin series of e^{-z}. Then we obtain the following equation.

$$\frac{e^z}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)^2 \sum_{n=0}^{\infty} \frac{1}{n!}(-z)^n \tag{1.21}$$

Therefore, we express the function G (x) as follows.

$$G(x) = \frac{1}{x^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!}(-x)^n\right)^2 \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{1.22}$$

We obtain the following equation by calculating the above formula.

$$G(x) = \frac{1}{x^2}\left(1 - \frac{x}{2} + \frac{x^2}{12} + O(x^3)\right)^2 \left(1 + x + \frac{x^2}{2} + O(x^3)\right) \tag{1.23}$$

$$G(x) = \frac{1}{x^2}\left(1 - x + \frac{5}{12}x^2 + O(x^3)\right)\left(1 + x + \frac{x^2}{2} + O(x^3)\right) \tag{1.24}$$

$$G(x) = \frac{1}{x^2}\left(1 + (1-1)x + \left(\frac{1}{2} - 1 + \frac{5}{12}\right)x^2 + O(x^3)\right) \tag{1.25}$$

$$G(x) = \frac{1}{x^2}\left(1 - \frac{1}{12}x^2 + O(x^3)\right) \tag{1.26}$$

$$G(x) = \frac{1}{x^2} - \frac{1}{12} + O(x) \tag{1.27}$$

Here the symbol O (x) is Landau symbols. The symbol means that the error has the order of the variable x.
The first term diverges. The first term is called singular term.

Therefore, the following the Abel sum diverges.

$$S_A = \lim_{x \to 0+} G(x) = \infty \tag{1.28}$$

It is the purpose of this paper to remove this divergence.

2 New method

2.1 New proof for the summation formula of natural numbers

We explain the new proof for the summation formula of natural numbers.

We have the following proposition. We will prove the proposition in the next sections.

Proposition 1.

$$\lim_{x \to 0+} \sum_{k=1}^{n} k \exp(-kx) \cos(kx) = \sum_{k=1}^{n} k \tag{2.1}$$

We have the following proposition. We will prove the proposition in the next sections, too.

Proposition 2.

$$\lim_{x \to 0+} \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx) = -\frac{1}{12} \tag{2.2}$$

We define the function S_n and H_n (x) as follows.

$$S_n = \sum_{k=1}^{n} k \tag{2.3}$$

$$H_n(x) \ = \sum_{k=1}^{n} k \exp(-kx) \cos(kx) \tag{2.4}$$

Then we define the following symbols.

$$1 + 2 + 3 + \cdots + n := S_n \tag{2.5}$$

$$1 + 2 + 3 + \cdots := \lim_{n\to\infty} S_n \tag{2.6}$$

$$"1 + 2 + 3 + \cdots + n" := \lim_{x\to 0+} H_n(x) \tag{2.7}$$

$$"1 + 2 + 3 + \cdots" := \lim_{x\to 0+} \left(\lim_{n\to\infty} H_n(x)\right) \tag{2.8}$$

Then we have the following propositions.

$$1 + 2 + 3 + \cdots = \infty \tag{2.9}$$

$$"1 + 2 + 3 + \cdots + n" = 1 + 2 + 3 + \cdots + n \tag{2.10}$$

$$"1 + 2 + 3 + \cdots" = -\frac{1}{12} \tag{2.11}$$

The double quotes mean the analytic continuation of the sum of natural numbers.

The traditional sum diverges for the infinite terms. On the other hand, the new “sum” is equal to the traditional sum for the finite term. In addition, the “sum” converges on -1/12 for the infinite term.

2.2 Proof of the proposition 1

Proposition 1.

$$\lim_{x\to 0+} \sum_{k=1}^{n} k \exp(-kx) \cos(kx) = \sum_{k=1}^{n} k \tag{2.12}$$

Proof. We define the function $H_n(x)$ as follows.

$$H_n(x) = \sum_{k=1}^{n} k \exp(-kx) \cos(kx) \tag{2.13}$$

On the other hand, we have the following equation.

$$\lim_{x \to 0+} H_n(x) = \sum_{k=1}^{n} k \exp(-k0) \cos(k0) = \sum_{k=1}^{n} k \tag{2.14}$$

Therefore, we have the following equation.

$$\lim_{x \to 0+} \sum_{k=1}^{n} k \exp(-kx) \cos(kx) = \sum_{k=1}^{n} k \tag{2.15}$$

This completes the proof.

2.3 Proof of the proposition 2

Proposition 2.

$$\lim_{x \to 0+} \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx) = -\frac{1}{12} \tag{2.16}$$

Proof. We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{2.17}$$

Then we define the following function.

$$\boxed{H(x) := \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx)} \tag{2.18}$$

$$0 < x \tag{2.19}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(i\theta) = \cos(\theta) + i \ \sin(\theta) \tag{2.20}$$

We obtain the following formula from the above formula.

$$\cos(\theta) = \frac{1}{2}(\exp(i\theta) + \exp(-i\theta)) \tag{2.21}$$

We obtain the following formula from the above formula by putting $\theta = k\,x$.

$$\cos(kx) = \frac{1}{2}(\exp(ikx) + \exp(-ikx)) \tag{2.22}$$

Hence, we express the function $H\,(x)$ as follows.

$$H(x) = \sum_{k=1}^{\infty} k \exp(-kx) \frac{1}{2} (\exp(ikx) + \exp(-ikx)) \tag{2.23}$$

$$H(x) = \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-kx) \exp(ikx) + \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-kx) \exp(-ikx) \tag{2.24}$$

$$H(x) = \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-kx + ikx) + \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-kx - ikx) \tag{2.25}$$

$$H(x) = \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-k(x - ix)) + \frac{1}{2} \sum_{k=1}^{\infty} k \exp(-k(x + ix)) \tag{2.26}$$

In addition, we define the following function.

$$G(\mathrm{z}) = \sum_{k=1}^{\infty} k \exp(-kz) \tag{2.27}$$

$$z \in \mathbb{C} \tag{2.28}$$

We express the function *G* (*z*) from the formula (1.27).

$$G(z) = \frac{1}{z^2} - \frac{1}{12} + O(z) \tag{2.29}$$

Hence, we express the function *G* (*x* - *ix*) as follows.

$$G(x - ix) = \sum_{k=1}^{\infty} k \exp(-k(x - ix)) \tag{2.30}$$

$$G(x - ix) = \frac{1}{(x - ix)^2} - \frac{1}{12} + O(x) \tag{2.31}$$

Then, we express the function *H* (*x*).

$$H(x) = \frac{1}{2} G(x - ix) + \frac{1}{2} G(x + ix) \tag{2.32}$$

$$H(x) = \frac{1}{2} \left(\frac{1}{(x - ix)^2} - \frac{1}{12} + O(x) \right) + \frac{1}{2} \left(\frac{1}{(x + ix)^2} - \frac{1}{12} + O(x) \right) \tag{2.33}$$

$$H(x) = \frac{1}{2(x - ix)^2} + \frac{1}{2(x + ix)^2} - \frac{1}{12} + O(x) \tag{2.34}$$

The first term is shown below.

$$\frac{1}{2(x-ix)^2}=\frac{1}{2(x^2-2ix^2-x^2)}=\frac{1}{-4ix^2} \tag{2.35}$$

On the other hand, the second term is shown below.

$$\frac{1}{2(x+ix)^2}=\frac{1}{2(x^2+2ix^2-x^2)}=\frac{1}{4ix^2} \tag{2.36}$$

Therefore, the sum of the first term and the second term vanishes.

$$\frac{1}{2(x-ix)^2}+\frac{1}{2(x+ix)^2}=0 \tag{2.37}$$

Since the singular term vanished, we have the following equation.

$$H(x)=-\frac{1}{12}+O(x) \tag{2.38}$$

We express the function $H_n(x)$ as follows.

$$H_n(x)=\sum_{k=1}^{n} k\exp(-kx)\cos(kx) \tag{2.39}$$

On the other hand, from the formula (2.38) we have the following equation.

$$\lim_{n\to\infty} H_n(x)=\sum_{k=1}^{\infty} k\exp(-kx)\cos(kx)=H(x)=-\frac{1}{12}+O(x) \tag{2.40}$$

Therefore, we have the following equation.

$$\lim_{x\to 0+}\left(\lim_{n\to\infty} H_n(x)\right)=\lim_{x\to 0+}\left(-\frac{1}{12}+O(x)\right)=-\frac{1}{12} \tag{2.41}$$

This completes the proof.

3 Conclusion

We obtained the following results in this paper.
- We proved that the sum of natural numbers is -1/12 of the value of the zeta function by the new method.

4 Future issues

The future issues are shown below.
- To study the relation between the oscillating Abel summation method and the q-analog.

5 Supplement

5.1 Numerical calculation of the oscillating Abel summation method

We will calculate the following value numerically.

$$H = \sum_{k=1}^{3000} k \exp(-k0.01) \cos(k0.01) \tag{5.1}$$

The result is shown below.

$$H = -0.0833333498 \cdots \tag{5.2}$$

This value is very close to the following -1/12.

$$-\frac{1}{12} = -0.0833333333 \cdots \tag{5.3}$$

The graph is shown in the Figure 5.1.

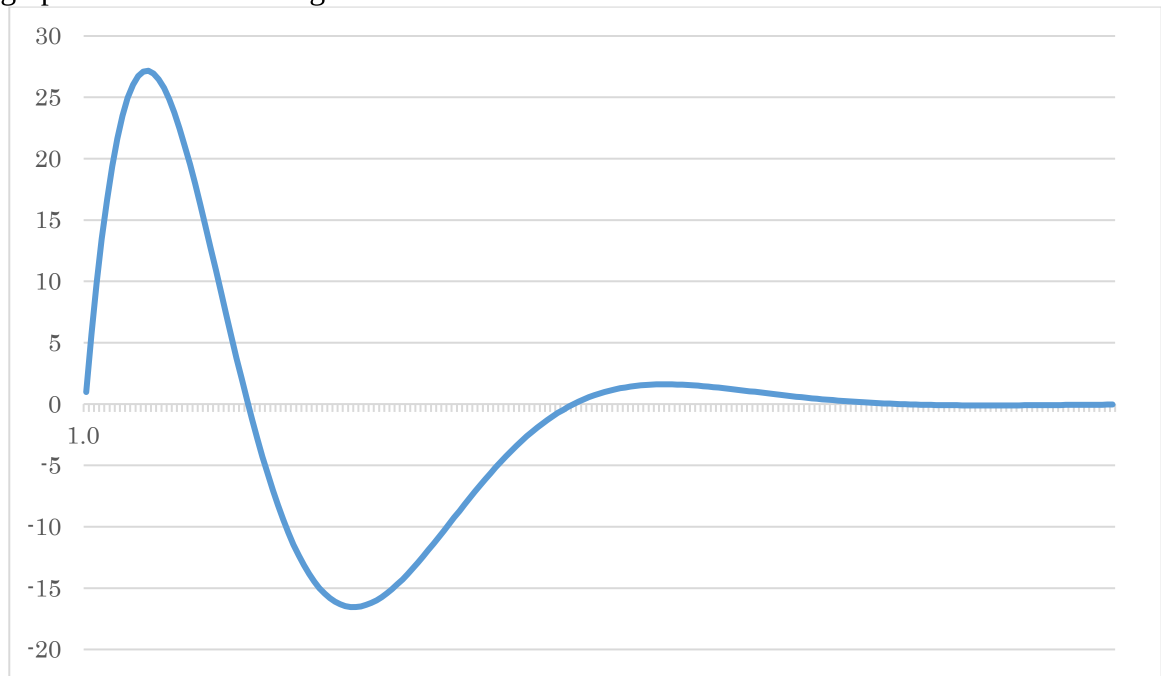

Figure 5.1: Damped oscillation of natural number

Here, we will double the attenuation factor as follows. We do not change the vibration period.

$$H = \sum_{k=1}^{3000} k \exp(-k0.02) \cos(k0.01) \tag{5.4}$$

The result is shown below.

$$H = 1199.9166679 \cdots \tag{5.5}$$

Therefore, the series does not converge if the attenuation factor and the vibration period do not have a special relationship. We will consider the relation in the next section.

5.2 General formula of the oscillating Abel summation method

We consider the following series.

$$S = \sum_{k=1}^{\infty} a_k \tag{5.6}$$

Then we define the following function.

$$H(x) = \sum_{k=1}^{\infty} a_k \exp(-k\phi(x)) \cos(kx) \tag{5.7}$$

$$0 < x \tag{5.8}$$

$$0 < \phi(x) \tag{5.9}$$

In addition, we define the following function.

$$G(\mathrm{z}) = \sum_{k=1}^{\infty} a_k \exp(-kz) \tag{5.10}$$

$$z \in \mathbb{C} \tag{5.11}$$

We express the function *G* (*z*) by using the singular term *A* (*z*), and the constant *C*.

$$G(z) = A(z) + C + O(z) \tag{5.12}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(i\theta) = \cos(\theta) + i \sin(\theta) \tag{5.13}$$

We express the function *H* (*x*) as follows by using the above formula.

$$H(x) = \frac{1}{2}(G(z) + G(\bar{z})) \tag{5.14}$$

$$z = \phi(x) + ix \tag{5.15}$$

$$\bar{z} = \phi(x) - ix \tag{5.16}$$

We obtain the following equation by using the formula (5.12) of the function *G* (*z*).

$$H(x) = \frac{1}{2}\left(A(z) + A(\overline{z})\right) + C + \ O(z) \tag{5.17}$$

We determine the function $\phi(x)$ by the following singular equation in order to remove the singular term of the above equation.

(Singular equation)

$$\boxed{\frac{1}{2}\left(A(z) + A(\bar{z})\right) = \ O(z)} \tag{5.18}$$

The series a_k and the function G (z) and $\phi(x)$ are shown below.

Series a_k	Function $G(z)$	Function $\phi(x)$
1	$G(z) = -\frac{1}{z} - \frac{1}{2} + O(z)$	$\phi(x) = \cot\left(\frac{\pi}{2}\right) x + O(x^3)$
k	$G(z) = \frac{1}{z^2} - \frac{1}{12} + O(z)$	$\phi(x) = \cot\left(\frac{\pi}{4}\right) x + O(x^4)$
k^2	$G(z) = -\frac{2}{z^3} + O(z)$	$\phi(x) = \cot\left(\frac{\pi}{6}\right) x + O(x^5)$
k^3	$G(z) = \frac{6}{z^4} + \frac{1}{120} + O(z)$	$\phi(x) = \cot\left(\frac{\pi}{8}\right) x + O(x^6)$
k^n	$G(z) = \frac{A}{z^{n+1}} + \zeta(-n) + O(z)$	$\phi(x) = \cot\left(\frac{\pi}{2n+2}\right) x + O(x^{n+3})$

The series and the example of the damped oscillation summation method are shown below.

Series a_k	Example of the damped oscillation summation method
1	$H(x) = \sum_{k=1}^{\infty} 1 \exp(-kx^3) \cos(kx)$
k	$H(x) = \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx)$
k^2	$H(x) = \sum_{k=1}^{\infty} k^2 \exp(-kx\sqrt{3}) \cos(kx)$
k^3	$H(x) = \sum_{k=1}^{\infty} k^3 \exp\left(-kx(1+\sqrt{2})\right) \cos(kx)$
k^n	$H(x) = \sum_{k=1}^{\infty} k^n \exp\left(-kx \cot\left(\frac{\pi}{2n+2}\right)\right) \cos(kx)$

Therefore, we have the following equations.

$$"1+1+1+\cdots" = -\frac{1}{2} \qquad (5.19)$$

$$"1+2+3+\cdots" = -\frac{1}{12} \qquad (5.20)$$

$$"1^2+2^2+3^2+\cdots" = 0 \qquad (5.21)$$

$$"1^3+2^3+3^3+\cdots" = \frac{1}{120} \qquad (5.22)$$

We calculate the above equations as follows.

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} 1\exp(-kx^3)\cos(kx) = -\frac{1}{2} \tag{5.23}$$

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k\exp(-kx)\cos(kx) = -\frac{1}{12} \tag{5.24}$$

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k^2\exp\left(-kx\sqrt{3}\right)\cos(kx) = 0 \tag{5.25}$$

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k^3\exp\left(-kx\left(1+\sqrt{2}\right)\right)\cos(kx) = \frac{1}{120} \tag{5.26}$$

General formula of the oscillating Abel summation method is shown below.

$$"1^n + 2^n + 3^n + \cdots" = \zeta(-n) \tag{5.27}$$

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k^n\exp\left(-kx\cot\left(\frac{\pi}{2n+2}\right)\right)\cos(kx) = \zeta(-n) \tag{5.28}$$

5.3 The mean Abel summation method by residue theorem

We consider the following sum of all natural numbers.

$$S := \sum_{k=1}^{\infty} k \tag{5.29}$$

Then we define the following function.

$$H(x) = \sum_{k=1}^{\infty} k\exp(-kx)\cos(kx) \tag{5.30}$$

$$G(z) = \sum_{k=1}^{\infty} k\exp(-kz) \tag{5.31}$$

We express the function $H(x)$ as follows.

$$H(x) = \frac{1}{2}(G(x-ix) + G(x+ix)) \tag{5.32}$$

We interpret the above function as a sum of the function $G(z_1)$ and $G(z_2)$ like the following Figure 5.2.

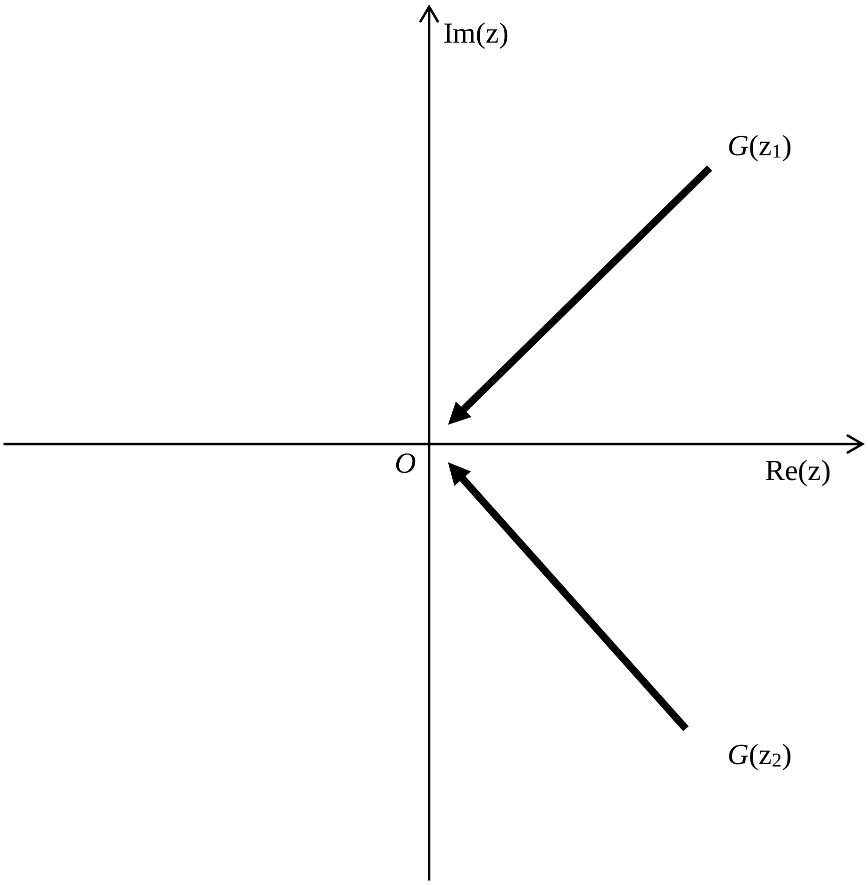

Figure 5.2: The oscillating Abel summation method

In the above Figure 5.2, the two functions $G\ (z_1)$ and $G\ (z_2)$ approach to the origin O at the two angles. This is a special condition. Is it possible to make the condition more general?

Then we consider the case that the many functions $G\ (z)$ approach to the origin from the all points on the circle of the radius x like the following Figure 5.3

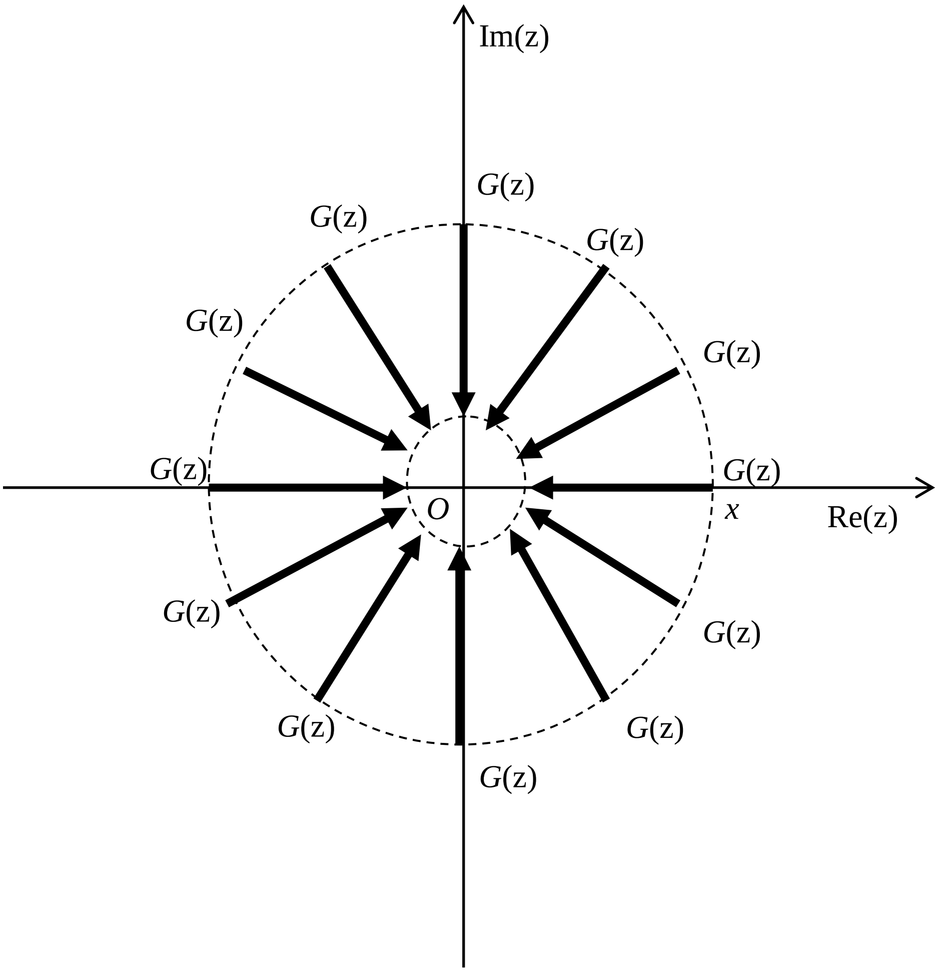

Figure 5.3: The mean Abel summation method

We express the above consideration in the following equations.

$$H(x) = \frac{1}{2}(G(z_1) + G(z_2)) \tag{5.33}$$

$$H(x) = \frac{1}{3}(G(z_1) + G(z_2) + G(z_3)) \tag{5.34}$$

$$H(x) = \frac{1}{4}(G(z_1) + G(z_2) + G(z_3) + G(z_4)) \tag{5.35}$$

$$H(x) = \frac{1}{n}(G(z_1) + G(z_2) + G(z_3) + \cdots + G(z_n)) \tag{5.36}$$

$$H(x) = \sum_{k=1}^{n} \frac{1}{n} G(z_k) \tag{5.37}$$

The function *H* (*x*) is the mean value of the function *G* (*z*).

We suppose that the circumference of the circle of the radius *x* is *L*. We express the circumference *L* of the circle as follows.

$$L = 2\pi x \tag{5.38}$$

On the other hand, we approximately express the circumference L as follows for sufficiently large n.

$$L = n|z_{k+1} - z_k| \tag{5.39}$$

Therefore, we have the following equation.

$$2\pi x = n|z_{k+1} - z_k| \tag{5.40}$$

Then, we express the normalized constant $1/n$ as follows.

$$\frac{1}{n} = \frac{|z_{k+1} - z_k|}{2\pi|z_k|} \tag{5.41}$$

$$|z_k| = x \tag{5.42}$$

On the other hand, the direction of -i (z_{k+1} - z_k) is same as the direction of z_k as shown in the Figure 5.4

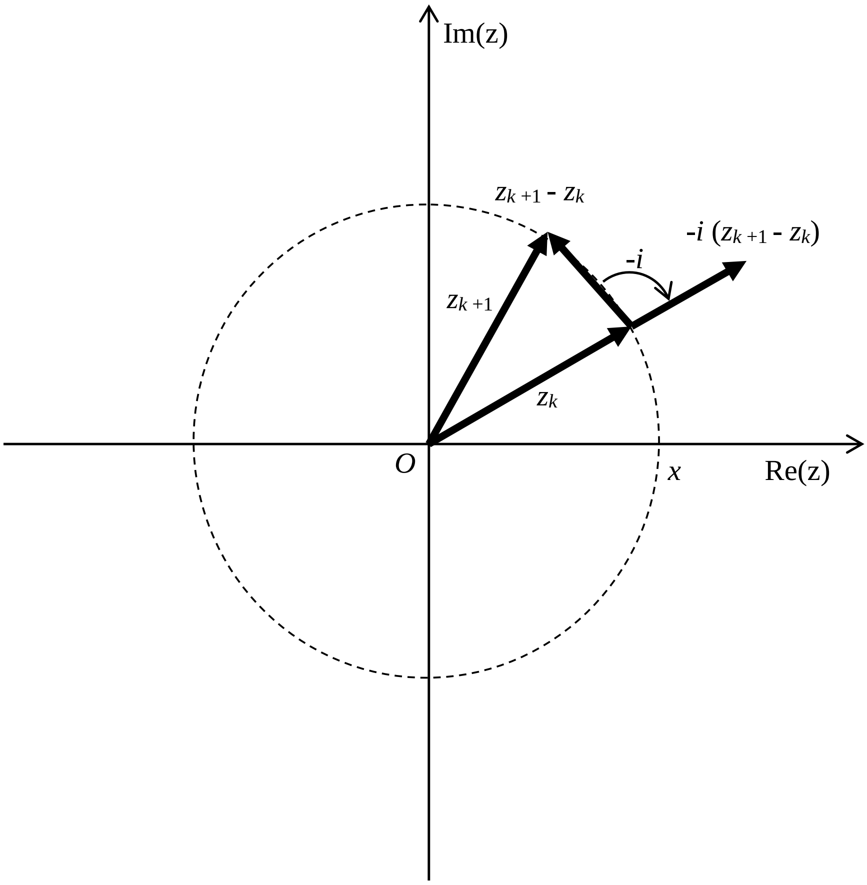

Figure 5.4: General damped oscillation summation method

Therefore, we express $1/n$ as follows.

$$\frac{1}{n}=\frac{|z_{k+1}-z_k|}{2\pi|z_k|}=\frac{-i(z_{k+1}-z_k\ )}{2\pi z_k}=\frac{-i\delta z_k}{2\pi z_k} \tag{5.43}$$

$$\delta z_k := z_{k+1}-z_k \tag{5.44}$$

Therefore, we express the function $H\ (x)$ as follows.

$$H(x)=\sum_{k=1}^{n}\frac{-i\delta z_k}{2\pi z_k}G(z_k) \tag{5.45}$$

We express the above sum as the following integration.

$$H(x)=\oint_{|z|=x}\frac{-idz}{2\pi z}G(z) \tag{5.46}$$

$$H(x)=\oint_{|z|=x}\frac{dz}{2\pi iz}G(z) \tag{5.47}$$

The above result is equal to the residue theorem. We can interpret the residue theorem the mean value theorem of contour integration.

We obtain the following result by the residue theorem.

$$H(x)=\oint_{|z|=x}\frac{dz}{2\pi iz}\left(\frac{1}{z^2}-\frac{1}{12}+O(z)\right)=-\frac{1}{12} \tag{5.48}$$

Therefore, we obtain the following new general summation method.
(The mean Abel summation method)

$$S\ =\sum_{k=1}^{\infty}a_k \tag{5.49}$$

$$G(z)\ =\sum_{k=1}^{\infty}a_k\exp(-kz) \tag{5.50}$$

$$H(x)\ =\oint_{|z|=x}\frac{dz}{2\pi iz}G(z) \tag{5.51}$$

$$S_H=\lim_{x\to 0+}H(x) \tag{5.52}$$

This summation method can sum any series by the residue theorem.

5.4 Definitions of the propositions by the (*ε*, *δ*)-definition of limit

The value of the limit depends on the order of the limit as follows.

$$\lim_{x\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n} k\exp(-\mathrm{kx})\cos(kx)\right) = -\frac{1}{12} \tag{5.53}$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0}\sum_{k=1}^{n} k\exp(-\mathrm{kx})\cos(kx)\right) = \infty \tag{5.54}$$

In this paper, we define the order of the limit as follows.

$$\lim_{x\to 0}\sum_{k=1}^{\infty} f_k(x) := \lim_{x\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)\right) \tag{5.55}$$

$$f_k(x) = k\exp(-\mathrm{kx})\cos(kx) \tag{5.56}$$

This order of the limit means the following relation.

$$0 \ll \frac{1}{x} \ll n \tag{5.57}$$

We confirm the above relation by the (ε, δ)-definition of limit.

In this section, we define the following proposition by the (ε, δ)-definition of limit.
We define the function S_n and H_n (x), and the constant α as follows.

$$S_n = \sum_{k=1}^{n} k \tag{5.58}$$

$$H_n(x) = \sum_{k=1}^{n} k\exp(-kx)\cos(kx) \tag{5.59}$$

$$\alpha = -\frac{1}{12} \tag{5.60}$$

We have the following proposition.

$$\lim_{n\to\infty} S_n = \infty \tag{5.61}$$

We can express the above proposition by (R, N)-definition of the limit as follows.

Given any number $R > 0$, there exists a natural number N such that for all n satisfying

$$N < n, \tag{5.62}$$

we have the following inequation.

$$R < S_n \tag{5.63}$$

We have the following proposition.

Proposition 1.

$$\lim_{x\to 0+} H_n(x) = S_n \tag{5.64}$$

We can express the above proposition by (ε, δ)-definition of the limit as follows.

Given any number $x > 0$ and any natural number n, there exists a number $\delta > 0$ such that for all $x > 0$ satisfying

$$x < \delta, \tag{5.65}$$

we have the following inequation.

$$|H_n(x) - S_n| < \varepsilon \tag{5.66}$$

We can summarize the above definition as follows.

$$\begin{gathered} \forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists \delta > 0 \text{ s.t.} \\ \forall x > 0, x < \delta \Rightarrow |H_n(x) - S_n| < \varepsilon \end{gathered} \tag{5.67}$$

We have the following proposition.

Proposition 2.

$$\lim_{x\to 0+} \left(\lim_{n\to\infty} H_n(x) \right) = \alpha \tag{5.68}$$

$$0 \ll \frac{1}{x} \ll n \tag{5.69}$$

We can express the above proposition by (ε, δ)-definition of the limit as follows.

Given any number $\varepsilon > 0$, there exist a natural number N and a number $\delta > 0$ such that for all n satisfying

$$N < n, \tag{5.70}$$

we have the following inequation.

$$|H_n(\delta) - \alpha| < \varepsilon \tag{5.71}$$

We can summarize the above definition as follows.

$$\begin{gathered} \forall \varepsilon > 0, \exists N \in \mathbb{N}, \exists \delta > 0 \text{ s.t.} \\ \forall n \in \mathbb{N}, N < n \Rightarrow |H_n(\delta) - \alpha| < \varepsilon \end{gathered} \tag{5.72}$$

The proposition 1 is the region of 1+2+3+…+n and the proposition 2 is the region -1/12 in the following figure.

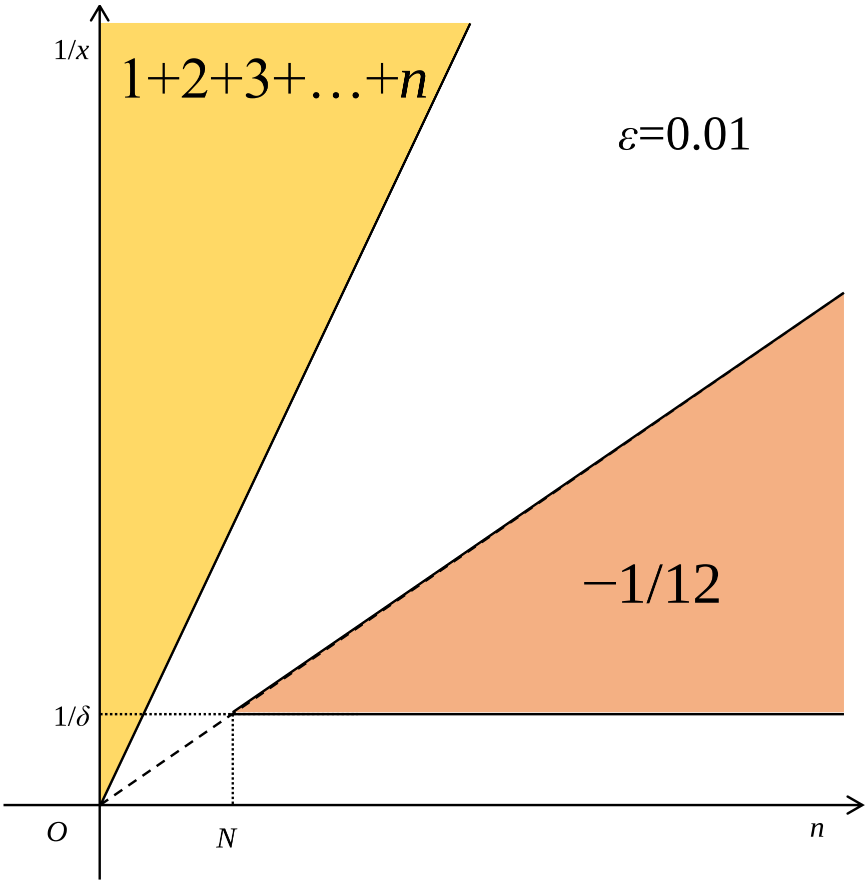

Figure 5.5: (*ε, δ*)-definition of limit

6 Appendix

6.1 Proof of the summation formula of the powers of the natural numbers

The author found the following formula and proved it in March 2014.
(Summation formula of the natural numbers)

$$\zeta(-1) = \lim_{x\to 0+} \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx) \tag{6.1}$$

N.S found the following formula by generalizing the above formula and published in March 2015.
(Summation formula of the powers of the natural numbers)

$$\zeta(1-2t) = \lim_{x\to 0+} \sum_{k=1}^{\infty} k^{2t-1} \exp(-k^t x) \cos(k^t x) \tag{6.2}$$

In this section, we prove the above formula in the following condition.

$$1 \le t \tag{6.3}$$

6.1.1 Cauchy's integral formula and mean formula

We express the function $F(z)$ which is differentiated s-times as follows.

$$F^{(s)}(z) = \frac{d^s}{dz^s} F(z) \tag{6.4}$$

We call the differentiated function $F^{(s)}(z)$ the **derivative**. We call the function we differentiate the **generating function**. On the other hand, Cauchy's integral formula is shown below.
(Cauchy's integral formula)

$$F(a) = \oint_C \frac{dz}{2\pi i(z-a)} F(z) \tag{6.5}$$

The function $F(z)$ is holomorphic in the Cauchy's integral formula. If the function $F(z)$ is not holomorphic, the integrated function is different from the original function by residue theorem. In this paper, we call the formula with the non-holomorphic function $F(z)$ Cauchy's mean formula in order to distinguish the Cauchy's integral formula.
(Cauchy's mean formula)

$$f(a) = \oint_C \frac{dz}{2\pi i(z-a)} F(z) \tag{6.6}$$

We interpret the function $f(a)$ as the mean value of the values on the contour path C of the function $F(z)$. Therefore, we interpret the Cauchy's mean formula as the mean value theorem. We interpret the function $f(a)$ is the analytic continuation of the function $f(z)$. We call the analytic continuation of the function the **mean function**. We use different function name for the function $f(a)$ since the mean function is different from the original function generally.

Cauchy's mean differentiation formula is shown below.
(Cauchy's mean differentiation formula)

$$f^{(s)}(a) = \Gamma(s+1) \oint_C \frac{dz}{2\pi i(z-a)} \frac{1}{(z-a)^s} F(z) \tag{6.7}$$

We call the differentiated function $f^{(s)}(z)$ the **mean derivative**. We interpret that the Cauchy's mean differentiation formula is the formula that averages and differentiates the function. We interpret that the mean derivative $f^{(s)}(z)$ is the function we average and differentiate s-times. We express the mean derivative by from derivative by Cauchy's mean formula.
(Cauchy's mean formula)

$$f^{(s)}(a) = \oint_C \frac{dz}{2\pi i(z-a)} F^{(s)}(z) \tag{6.8}$$

We express the generating function by from mean derivative by **Laurent series**.
(Laurent series)

$$F(z) = \sum_{s=-\infty}^{\infty} \frac{f^{(s)}(a)}{\Gamma(s+1)} (z-a)^s \tag{6.9}$$

The relation of the functions in this section is shown below.

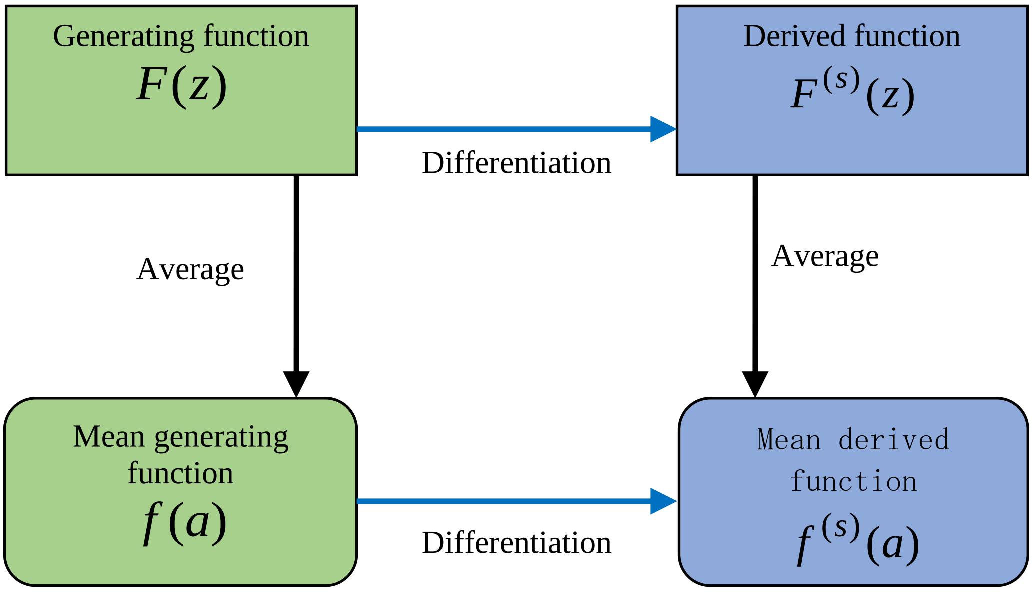

Figure 6.1: Mean generating function and mean derivative

6.1.2 Laurent series and the pole of the order zero

Generally, the order of the pole is a positive integer *n*.
(The pole of the order *n*)

$$\frac{1}{z^n} \tag{6.10}$$

$$1 \leq n \in \mathbb{Z} \tag{6.11}$$

Though the order of the pole is greater or equal to one generally, we introduce the following the **pole of the order zero** by using finite positive real number *h* in this paper.
(The pole of the order zero)

$$\frac{1}{hz^h} \tag{6.12}$$

$$0 < h \ll 1 \tag{6.13}$$

The pole of the order zero is divergent at *z*=0. The differentiation of the above pole is shown below.

$$\frac{d}{dz}\left(\frac{1}{hz^h}\right) = -\frac{1}{z} \tag{6.14}$$

On the other hand, the differentiation of the natural logarithm is shown below.

$$\frac{d}{dz}\left(\log\left(\frac{1}{z}\right)\right) = -\frac{1}{\mathrm{z}} \tag{6.15}$$

Therefore, we interpret the natural logarithm as the pole of the order zero.
We derive the constant term from the coefficient of the argument of the natural logarithm.

$$\log\left(\frac{C}{z}\right) = \log\left(\frac{1}{z}\right) + \log(C) = \log\left(\frac{1}{z}\right) + \gamma \tag{6.16}$$

Therefore, the new interpretation that we regard the term of the order zero as the pole of the order zero is consistent with the old interpretation that we regard the term of the order zero as the constant term.

We express any function by the new Laurent series with natural logarithm.
(Laurent series with natural logarithm)

$$F(z) = \sum_{k=-\infty}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} z^{k-h} \tag{6.17}$$

$$f^{(k)}(0) = \Gamma(k+1) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^k} F(z) \tag{6.18}$$

We express the function $F(z)$ as follows if the function has the pole of the order zero at most by using the constant A and γ.

$$F(z) = A \log\left(\frac{1}{z}\right) + \gamma + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} z^k \tag{6.19}$$

6.1.3 The proof of the summation formula of the natural numbers by harmonic derivative

Harmonic number is shown below.
(Harmonic number)

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{6.20}$$

We define the harmonic generating function as follows.
(Harmonic generating function)

$$H_n(z) := \sum_{k=1}^{n} \frac{1}{k} \exp(kz) \tag{6.21}$$

We obtain the **harmonic derivative** by differentiating the harmonic generating function s-times.
(Harmonic derivative)

$$H_n^{(s)}(z) = \sum_{k=1}^{n} \frac{k^s}{k} \exp(kz) \tag{6.22}$$

We introduce the following symbol.

$$H(z) = \lim_{n\to\infty} H_n^{(0)}(z) = \mathrm{Li}_1(e^z) \tag{6.23}$$

Then, we introduce the following symbol.

$$H^{(s)}(z) = \lim_{n\to\infty} H_n^{(s)}(z) = \mathrm{Li}_{(1-s)}(e^z) \tag{6.24}$$

We define the mean harmonic derivative by Cauchy's mean formula.
(Mean harmonic derivative)

$$\nu_n^{(s)}(0) := \oint_C \frac{dz}{2\pi i z}\, H_n^{(s)}(z) \tag{6.25}$$

We call the above function the natural derivative. We call the natural derivative the **natural function** shortly.

We introduce the following symbol.

$$\nu^{(s)}(0) = \lim_{n\to\infty} \nu_n^{(s)}(0) = \zeta(1-s) \tag{6.26}$$

Then, we introduce the following symbol.

$$\nu_n^{(s)} = \nu_n^{(s)}(0) \tag{6.27}$$

We obtain the natural function by Cauchy's mean differentiation formula.
(The natural function)

$$\nu^{(s)}(0) = \Gamma(s+1) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^s} H(z) \tag{6.28}$$

We choose the following condition.

$$\mathrm{Re}(z) < 0 \tag{6.29}$$

On the other hand, the harmonic generating function is shown below.

$$H(z) = -\log(1-e^z) \tag{6.30}$$

We calculate the order of the pole of the harmonic generating function at z=0 by using Landau's symbol as follows.

$$H(z) = -\log(1-e^z) = -\log(1-(1+O(z))) = O\left(\log\left(\frac{1}{z}\right)\right) \tag{6.31}$$

Since the harmonic generating function has the pole of the order zero at z=0, we express the function as the Laurent series by using the constant A and γ.
(Laurent series of the harmonic generating function)

$$H(z) = A\log\left(\frac{1}{z}\right) + \gamma + \sum_{k=1}^{\infty} \frac{\nu^{(k)}(0)}{\Gamma(k+1)} z^k \tag{6.32}$$

We differentiate the above function two-times.

$$H^{(2)}(z) = \frac{A}{z^2} + \nu^{(2)}(0) + O(z) \tag{6.33}$$

We prove the summation formula of the natural numbers as follows.

$$S = \lim_{x\to 0+} \sum_{k=1}^{\infty} k \exp(-kx)\cos(kx) \tag{6.34}$$

$$S = \lim_{x\to 0+} \frac{1}{2}\left(H^{(2)}(-x+ix) + H^{(2)}(-x-ix)\right) \tag{6.35}$$

$$S = \lim_{x\to 0+} \frac{1}{2}\left(\frac{A}{(-x+ix)^2} + \frac{A}{(-x-ix)^2}\right) + \nu^{(2)}(0) + O(x) \tag{6.36}$$

$$S = \nu^{(2)}(0) = \zeta(-1) = -\frac{1}{12} \tag{6.37}$$

(Q.E.D.)

6.1.4 The proof of the summation formula of the powers of the natural numbers by harmonic overtone derivative

We define the harmonic overtone generating function as follows.
(Harmonic overtone generating function)

$$H_n^t(z) := \sum_{k=1}^{n} \frac{1}{k} \exp(k^t z) \tag{6.38}$$

We obtain the **harmonic overtone derivative** by differentiating the harmonic overtone generating function s-times.
(Harmonic overtone derivative)

$$H_n^{t(s)}(z) = \sum_{k=1}^{n} \frac{k^{st}}{k} \exp(k^t z) \tag{6.39}$$

We introduce the following symbol.

$$H^t(z) = \lim_{n\to\infty} H_n^{t(0)}(z) \tag{6.40}$$

Then, we introduce the following symbol.

$$H^{t(s)}(z) = \lim_{n\to\infty} H_n^{t(s)}(z) \tag{6.41}$$

We define the mean harmonic overtone derivative by Cauchy's mean formula.
(Mean harmonic overtone derivative)

$$\nu^{t(s)}(0) := \oint_C \frac{dz}{2\pi i z}\, H^{t(s)}(z) \tag{6.42}$$

We call the above function the natural overtone derivative. We call the natural overtone derivative the **natural overtone function** shortly.

We introduce the following symbol.

$$\nu^{t(s)}(0) = \lim_{n\to\infty} \nu_n^{t(s)}(0) = \zeta(1-st) \tag{6.43}$$

Then, we introduce the following symbol.

$$\nu_n^{t(s)} = \nu_n^{t(s)}(0) \tag{6.44}$$

We obtain the natural overtone function by Cauchy's mean differentiation formula.
(The natural overtone function)

$$\nu^{t(s)}(0) = \Gamma(s+1) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^s} H^t(z) \tag{6.45}$$

We choose the following condition.

$$1 \le t \tag{6.46}$$

Then, we have the following equation for the positive *k*.

$$k \le k^t \tag{6.47}$$

In addition, we choose the following condition.

$$\mathrm{Re}(z) < 0 \tag{6.48}$$

Then, we have the following equation for the positive *k*.

$$|\exp(k^t z)| \le |\exp(kz)| \tag{6.49}$$

Therefore, we have the following equation by choosing the absolute value of the *z* is very small.

$$|H^t(z)| = \left|\sum_{k=1}^{\infty} \frac{\exp(k^t z)}{k}\right| \le \left|\sum_{k=1}^{\infty} \frac{\exp(kz)}{k}\right| = |H(z)| \tag{6.50}$$

From the above result, we found that the harmonic overtone generating function does not have the pole of higher order than the pole of the harmonic generating function.

We calculate the order of the pole of the harmonic generating function at z=0 by using Landau's symbol as follows.

$$H(z) = -\log(1-e^z) = -\log(1-(1+O(z))) = O\left(\log\left(\frac{1}{z}\right)\right) \tag{6.51}$$

Since the harmonic overtone generating function has the pole of the order zero at z=0, we express the function as the Laurent series by using the function A(t) and γ(t).
(Laurent series of the harmonic overtone generating function)

$$H^t(z) = A(t)\log\left(\frac{1}{z}\right) + \gamma(t) + \sum_{k=1}^{\infty} \frac{\nu^{t(k)}(0)}{\Gamma(k+1)} z^k \tag{6.52}$$

We differentiate the above function two-times.

$$H^{t(2)}(z) = \frac{A(t)}{z^2} + \nu^{t(2)}(0) + O(z) \tag{6.53}$$

We prove the summation formula of the powers of the natural numbers as follows.

$$S = \lim_{x\to 0+} \sum_{k=1}^{\infty} k^{2t-1} \exp(-k^t x) \cos(k^t x) \tag{6.54}$$

$$S = \lim_{x\to 0+} \frac{1}{2}\left(H^{t(2)}(-x+ix) + H^{t(2)}(-x-ix)\right) \tag{6.55}$$

$$S = \lim_{x\to 0+} \frac{1}{2}\left(\frac{A(t)}{(-x+ix)^2} + \frac{A(t)}{(-x-ix)^2}\right) + \nu^{t(2)}(0) + O(x) \tag{6.56}$$

$$S = \nu^{t(2)}(0) = \zeta(1-2t) \tag{6.57}$$

(Q.E.D.)

6.1.5 The second proof

In the previous section, we prove the summation formula of the powers of the natural numbers on the following condition.

$$1 \le t \tag{6.58}$$

In this section, we prove the formula on the following condition.

$$0 < t \tag{6.59}$$

On the above condition, we have the following formula for the natural number *k*.

$$k^t < (k+1)^t \tag{6.60}$$

Therefore, we have the following formula.

$$\exp(k^t) < \exp((k+1)^t) \tag{6.61}$$

In addition, we choose the following condition.

$$\mathrm{Re}(z) < 0 \tag{6.62}$$

Then, we have the following formula for the natural number *k* and very small *z*.

$$|\exp(k^t z)| > |\exp((k+1)^t z)| \tag{6.63}$$

In the same way, we have the following formula for the natural number *k* and very small *z*.

$$\left|\frac{\exp(k^t z)}{k}\right| > \left|\frac{\exp((k+1)^t z)}{\mathrm{k}+1}\right| \qquad (6.64)$$

Therefore, the absolute value of the following function monotonically decreases.

$$f(x) = \frac{\exp(x^t z)}{x} \qquad (6.65)$$

As shown in the following figure, the first term and the integration of the function *f*(*x*) from one to infinity is greater than the harmonic overtone generating function.

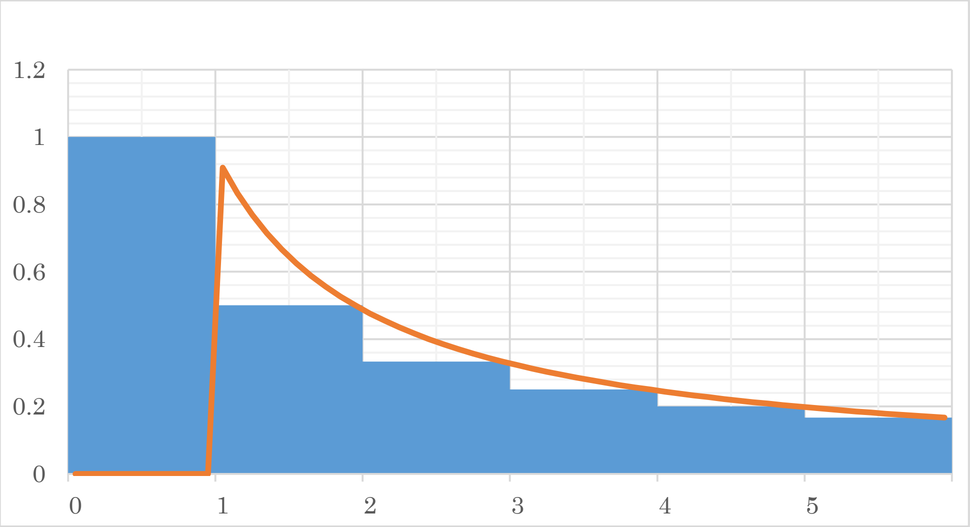

Figure 6.2: Harmonic overtone generating function

Therefore, we have the following formula for very small *z*.

$$|H^t(z)| = \left|\sum_{k=1}^{\infty}\frac{\exp(k^t z)}{k}\right| < \left|e^z + \int_1^{\infty}\frac{\exp(x^t z)}{x}dx\right| \qquad (6.66)$$

Here, we transform the variable as follows.

$$y = x^t \qquad (6.67)$$

We differentiate the both sides.

$$dy = tx^{t-1}dx \qquad (6.68)$$

$$dy = tx^t\frac{dx}{x} \qquad (6.69)$$

$$dy = ty\frac{dx}{x} \qquad (6.70)$$

$$\frac{1}{t}\frac{dy}{y} = \frac{dx}{x} \qquad (6.71)$$

Therefore, we transform the variable of the integration as follows.

$$\left|e^{z}+\int_{1}^{\infty}\frac{\exp(x^{t}z)}{x}dx\right| = \left|e^{z} + \frac{1}{t}\int_{1}^{\infty}\frac{\exp(yz)}{y}dy\right| \tag{6.72}$$

Here, we consider the following function.

$$g(y)=\frac{\exp(yz)}{y} \tag{6.73}$$

As shown in the following figure, the harmonic generating function is greater than the first term and the integration of the function $g(y)$ from one to infinity.

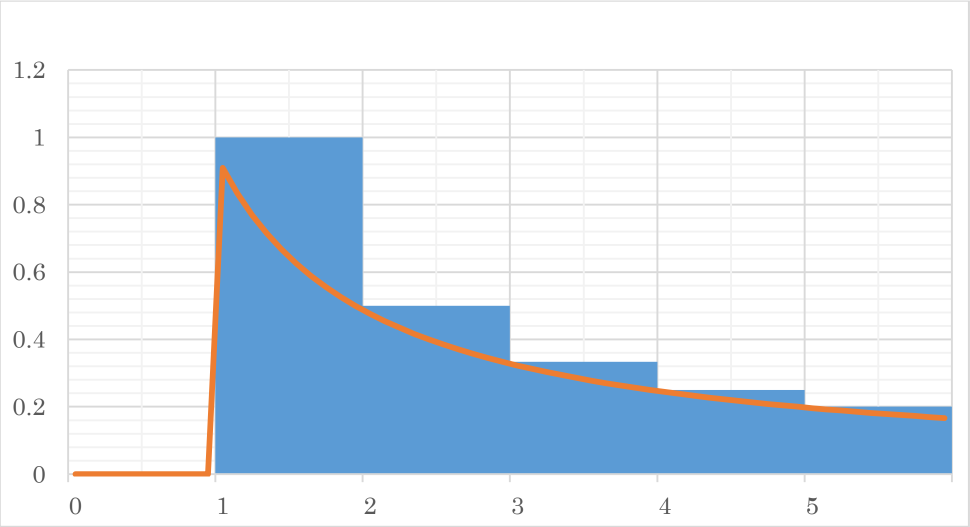

Figure 6.3: Harmonic generating function

Therefore, we have the following formula for very small z.

$$\left|e^{z}+\frac{1}{t}\int_{1}^{\infty}\frac{\exp(yz)}{y}dy\right| < \left|e^{z}+\frac{1}{t}\sum_{k=1}^{\infty}\frac{\exp(kz)}{k}\right| \tag{6.74}$$

As shown in the previous section, the right side of the following formula has the pole of zero-th at z=0.

$$|H^{t}(z)| = \left|\sum_{k=1}^{\infty}\frac{\exp(k^{t}z)}{k}\right| < \left|e^{z}+\frac{1}{t}\sum_{k=1}^{\infty}\frac{\exp(kz)}{k}\right| \tag{6.75}$$

Therefore, as shown in the previous section, we have the following formula.

$$\zeta(1-2t) = \lim_{x\to 0+}\sum_{k=1}^{\infty}k^{2t-1}\exp(-k^{t}x)\cos(k^{t}x) \tag{6.76}$$

6.2 Riemann zeta function

Harmonic derivative is shown below.
(Harmonic derivative)

$$H_n^{(s)}(z) = \sum_{k=1}^{n} \frac{k^s}{k} \exp(kz) \tag{6.77}$$

We express the harmonic number by the limit of the harmonic derivative.

$$H_n = \lim_{z\to 0}\left(\lim_{s\to 0} H_n^{(s)}(z)\right) \tag{6.78}$$

We define the **partition function** as follows.
(Partition function)

$$Z(z) = \sum_{k=1}^{\infty} \exp(kz) \tag{6.79}$$

We express the partition function by the limit of the harmonic derivative.

$$Z(z) = \lim_{s\to 1}\left(\lim_{n\to\infty} H_n^{(s)}(z)\right) \tag{6.80}$$

Riemann defined the zeta function.
(Zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6.81}$$

$$1 < \mathrm{Re}(s) \tag{6.82}$$

We express the zeta function by the limit of the harmonic derivative.

$$\zeta(s) = \lim_{z\to 0}\left(\lim_{n\to\infty} H_n^{(1-s)}(z)\right) \tag{6.83}$$

We summarize the above results as follows.
- The harmonic number is the limit of the harmonic derivative.
- The partition function is the limit of the harmonic derivative.
- The zeta function is the limit of the harmonic derivative.

We express these relations in the following figure.

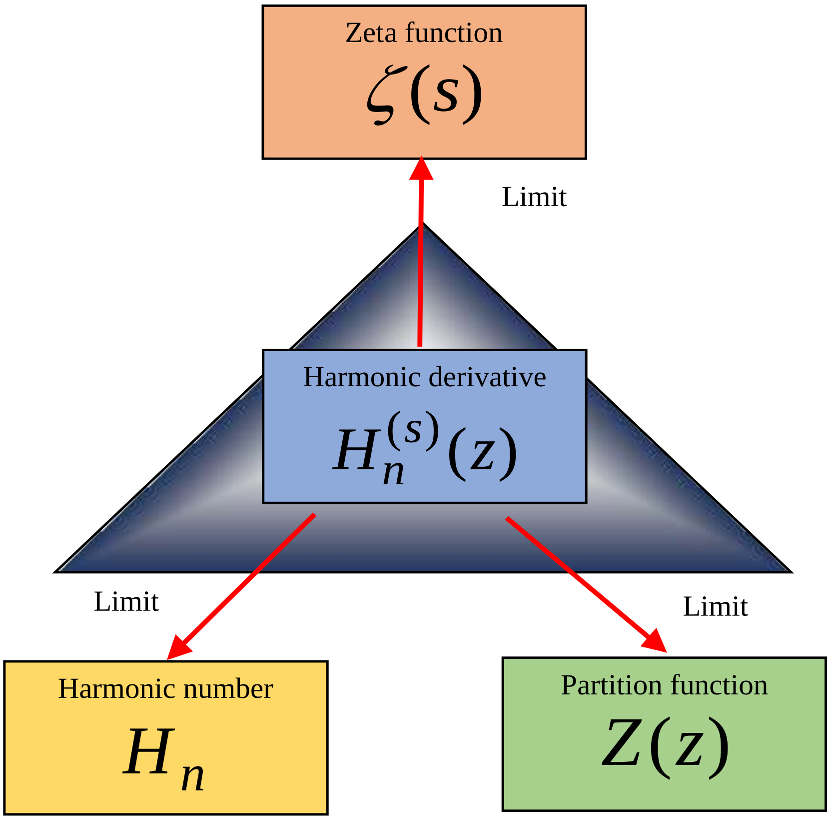

Figure 6.4: Zeta function

The natural function is shown below.
(The natural function)

$$v_n^{(s)}(0) = \oint_C \frac{dz}{2\pi i z} H_n^{(s)}(z) \qquad (6.84)$$

We obtain the following formula by integrating the above formula by parts (*s*-1)-times.

$$v_n^{(s)}(0) = \Gamma(s) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^{s-1}} H_n^{(1)}(z) \qquad (6.85)$$

We replace the variable s by 1-s.

$$v_n^{(1-s)}(0) = \Gamma(1-s) \oint_C \frac{dz}{2\pi i z} z^s H_n^{(1)}(z) \qquad (6.86)$$

On the other hand, Riemann defined the **analytic zeta function**.
(The analytic zeta function)

$$\zeta(s) = \Gamma(1-s) \oint_C \frac{dz}{2\pi i z} z^s \frac{e^z}{1-e^z} \qquad (6.87)$$

We express the analytic zeta function by the limit of the natural function.

$$\zeta(s) = \lim_{n\to\infty} v_n^{(1-s)}(0) \qquad (6.88)$$

We express these relations in the following figure.

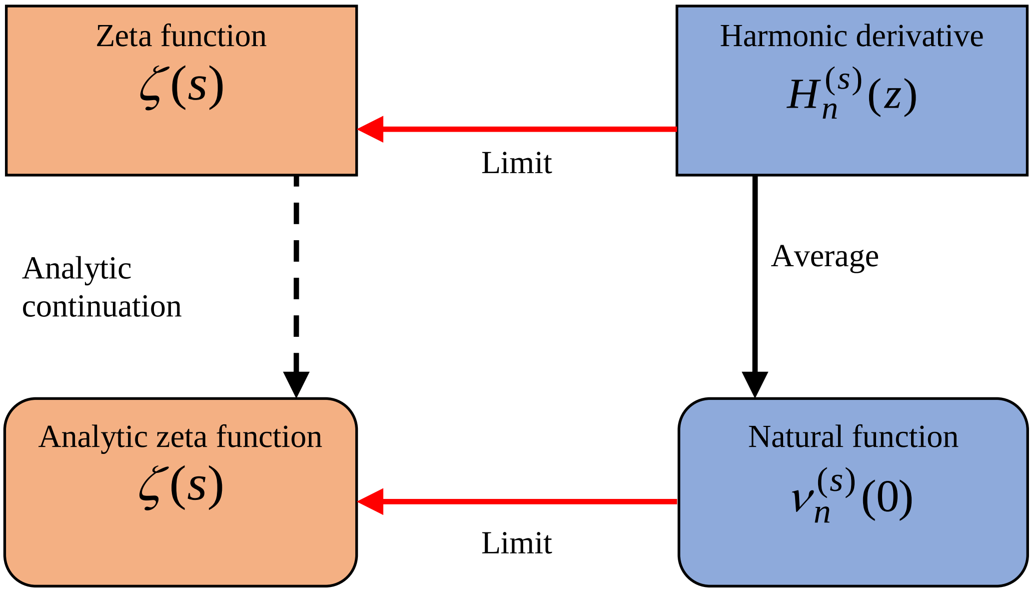

Figure 6.5: Analytic zeta function

6.3 Jacobi's theta function

Riemann expressed the zeta function by the Jacobi's theta function.
(The express of the zeta function by the Jacobi's theta function)

$$\zeta(-2s) = \frac{1}{\Gamma(-s)}\int_0^\infty \frac{dx}{x}\frac{1}{x^s}\frac{1}{\pi^s}\psi(x) \tag{6.89}$$

$$\psi(x) = \sum_{k=1}^{\infty} \exp(-k^2\pi x) \tag{6.90}$$

(Jacobi's theta function)

$$\theta(z,\tau) = \sum_{k=-\infty}^{\infty} \exp(\pi i k^2\tau + 2\pi i k z) \tag{6.91}$$

$$\theta(0, ix) = 1 + 2\psi(x) \tag{6.92}$$

On the other hand, we have the mean harmonic overtone derivative as follows.
(Mean harmonic overtone derivative)

$$\nu^{t(s)}(0) = \oint_C \frac{dz}{2\pi i z} H^{t(s)}(z) \tag{6.93}$$

We obtain the following formula by integrating the above formula by parts s-times.

$$\nu^{t(s)}(0) = \Gamma(s+1) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^s} H^{t(0)}(z) \qquad (6.94)$$

We express the zeta function by the harmonic overtone generating function.
(The express of the zeta function by the harmonic overtone generating function)

$$\zeta(1-st) = \Gamma(s+1) \oint_C \frac{dz}{2\pi i z} \frac{1}{z^s} H^{t(0)}(z) \qquad (6.95)$$

(Harmonic overtone generating function)

$$H^{t(0)}(z) = \sum_{k=1}^{\infty} \frac{1}{k} \exp(k^t z) \qquad (6.96)$$

6.4 Relation with the *q*-analog

6.4.1 The *q*-analog

F. H. Jackson[8] defined the following quantized new natural number, the ***q*-number** in 1904.
(The q-number)

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} \qquad (6.97)$$

$$[n]_q = \sum_{k=1}^{n} q^{k-1} \qquad (6.98)$$

$$[n]_q = \frac{1-q^n}{1-q} \qquad (6.99)$$

We call the quantized mathematical object the ***q*-analog** generally. The q-analog of the natural number n is the q-number $[n]_q$. We express the natural number by the classical limit of the q-number.

$$n = \lim_{q \to 1-} [n]_q \qquad (6.100)$$

The **zeta function** is shown below.
(Zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (6.101)$$

The q-analog of the zeta function is the ***q*-zeta function**. The function is shown below.[7]
(The q-zeta function)

$$\zeta_q(s) := \sum_{n=1}^{\infty} \frac{1}{[n]_q^s} q^{n(s-1)} \qquad (6.102)$$

We express the zeta function by the classical limit of the q-zeta function.

$$\zeta(s) = \lim_{q\to 1-} \zeta_q(s) \tag{6.103}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{[n]_1^s} 1^{n(s-1)} \tag{6.104}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6.105}$$

$$1 < \mathrm{Re}(s) \tag{6.106}$$

The relation with the harmonic derivative is shown below.

$$[n]_q = \frac{1}{\exp(z)} H_n^{(1)}(z) \tag{6.107}$$

$$q = \exp(z) \tag{6.108}$$

7 Acknowledgment

In writing this paper, I thank from my heart to NS who gave valuable advice to me.

8 Bibliography

---

[1] Mail: sugiyama_xs@yahoo.co.jp

[2] Leonhard Euler (1768), Remarques sur un beau rapport entre les series des puissances tant directes que reciproques (Remarks on a beautiful relation between direct as well as reciprocal power series), *Memoires de l'academie des sciences de Berlin*, **17**, 83-106, *Opera Omnia*, **1** (15), 70-90.

[3] Bernhard Riemann (1859), Uber die Anzahl der Primzahlen unter einer gegebenen Grosse (On the Number of Primes Less Than a Given Magnitude), *Monatsberichte der Berliner Akademie*, 671-680.

[4] Bruce C. Berndt (1939), Ramanujan's Notebooks, Ramanujan's Theory of Divergent Series, *Springer-Verlag (ed.)*, Chapter 6, 133-149.

[5] G. H. Hardy (1949), Divergent Series, Oxford: Clarendon Press, 71-77.

[6] Junya Satoh (1989), q - analogue of Riemann's ζ - function and q - Euler numbers, *J. Number Theory*, **31**, 346-362, 0022-314X(89)90078-4.

[7] M. Kaneko, N. Kurokawa, and M. Wakayama (2003), A variation of Euler's approach to values of the Riemann zeta function, *Kyushu J. Math.* **57** (1), 175-192, math/0206171.

[8] F. H. Jackson, (1905), The Basic Gamma-Function and the Elliptic Functions, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character (The Royal Society), **76** (508), 127-144, http://www.jstor.org/stable/92601.