Generalizations of the Distance and Dependent Function in Extenics to $2D$, $3D$, and $n-D$

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Dr. Cai Wen defined in his 1983 paper: — the distance formula between a point x_0 and a one-dimensional (1D) interval $<a, b>$; — and the dependence function which gives the degree of dependence of a point with respect to a pair of included $1D$-intervals. His paper inspired us to generalize the Extension Set to two-dimensions, i.e. in plane of real numbers R^2 where one has a rectangle (instead of a segment of line), determined by two arbitrary points $A(a_1, a_2)$ and $B(b_1, b_2)$. And similarly in R^3, where one has a prism determined by two arbitrary points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$. We geometrically define the linear and non-linear distance between a point and the $2D$ and $3D$-extension set and the dependent function for a nest of two included $2D$ and $3D$-extension sets. Linearly and non-linearly attraction point principles towards the optimal point are presented as well. The same procedure can be then used considering, instead of a rectangle, any bounded $2D$-surface and similarly any bounded $3D$-solid, and any bounded $(n-D)$-body in R^n. These generalizations are very important since the Extension Set is generalized from one-dimension to 2, 3 and even n-dimensions, therefore more classes of applications will result in consequence.

1 Introduction

Extension Theory (or Extenics) was developed by Professor Cai Wen in 1983 by publishing a paper called Extension Set and Non-Compatible Problems. Its goal is to solve contradictory problems and also nonconventional, nontraditional ideas in many fields. Extenics is at the confluence of three disciplines: philosophy, mathematics, and engineering. A contradictory problem is converted by a transformation function into a non-contradictory one. The functions of transformation are: extension, decomposition, combination, etc. Extenics has many practical applications in Management, Decision-Making, Strategic Planning, Methodology, Data Mining, Artificial Intelligence, Information Systems, Control Theory, etc. Extenics is based on matter-element, affair-element, and relation-element.

2 Extension Distance in $1D$-space

Let's use the notation $<a, b>$ for any kind of closed, open, or half-closed interval $[a, b]$, (a, b), $(a, b]$, $[a, b)$. Prof. Cai Wen has defined the extension distance between a point x_0 and a real interval $X = <a, b>$, by

$$\rho\,(x_0, X) = \left| x_0 - \frac{a+b}{2} \right| - \frac{b-a}{2}, \tag{1}$$

where in general:

$$\rho : (R, R^2) \to (-\infty, +\infty). \tag{2}$$

Algebraically studying this extension distance, we find that actually the range of it is:

$$\rho\,(x_0, X) \in \left[-\frac{b-a}{2}, +\infty \right] \tag{3}$$

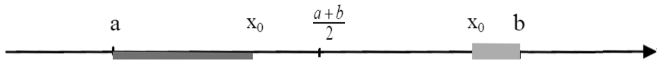

Fig. 1:

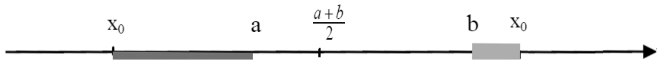

Fig. 2:

or its minimum range value $-\left(\frac{b-a}{2}\right)$ depends on the interval X extremities a and b, and it occurs when the point x_0 coincides with the midpoint of the interval X, i.e. $x_0 = \frac{a+b}{2}$. The closer is the *interior point* x_0 to the midpoint of the interval $<a, b>$, the negatively larger is $\rho\,(x_0, X)$.

In Fig. 1, for interior point x_0 between a and $\frac{a+b}{2}$, the extension distance $\rho\,(x_0, X) = a - x_0$ is the *negative length of the brown line segment* [left side]. Whereas for interior point x_0 between $\frac{a+b}{2}$ and b, the extension distance $\rho\,(x_0, X) = x_0 - b$ is the *negative length of the blue line segment* [right side]. Similarly, the further is *exterior point* x_0 with respect to the closest extremity of the interval $<a, b>$ to it (i.e. to either a or b), the positively larger is $\rho\,(x_0, X)$.

In Fig. 2, for exterior point $x_0 < a$, the extension distance $\rho\,(x_0, X) = a - x_0$ is the positive length of the brown line segment [left side]. Whereas for exterior point $x_0 > b$, the extension distance $\rho\,(x_0, X) = x_0 - b$ is the *positive length of the blue line segment* [right side].

3 Principle of the Extension $1D$-Distance

Geometrically studying this extension distance, we find the following principle that Prof. Cai Wen has used in 1983 defining it:

> $\rho(x_0, X)$ is *the geometric distance between the point* x_0 *and the closest extremity point of the interval* $<a, b>$ *to it (going in the direction that connects* x_0 *with the optimal point), distance taken as negative if* $x_0 \in <a, b>$, *and as positive if* $x_0 \subset <a, b>$.

This principle is very important in order to generalize the extension distance from $1D$ to $2D$ (two-dimensional real space), $3D$ (three-dimensional real space), and $n-D$ (n-dimensional real space).

The extremity points of interval $<a, b>$ are the point a and b, which are also the boundary (frontier) of the interval $<a, b>$.

4 Dependent Function in $1D$-Space

Prof. Cai Wen defined in 1983 in $1D$ the Dependent Function $K(y)$. If one considers two intervals X_0 and X, that have no common end point, and $X_0 \subset X$, then:

$$K(y) = \frac{\rho(y, X)}{\rho(y, X) - \rho(y, X_0)}. \tag{4}$$

Since $K(y)$ was constructed in $1D$ in terms of the extension distance $\rho(.,.)$, we simply generalize it to higher dimensions by replacing $\rho(.,.)$ with the generalized in a higher dimension.

5 Extension Distance in $2D$-Space

Instead of considering a segment of line AB representing the interval $<a, b>$ in $1R$, we consider a rectangle $AMBN$ representing all points of its surface in $2D$. Similarly as for $1D$-space, the rectangle in $2D$-space may be closed (i.e. all points lying on its frontier belong to it), open (i.e. no point lying on its frontier belong to it), or partially closed (i.e. some points lying on its frontier belong to it, while other points lying on its frontier do not belong to it).

Let's consider two arbitrary points $A(a_1, a_2)$ and $B(b_1, b_2)$. Through the points A and B one draws parallels to the axes of the Cartesian system XY and one thus one forms a rectangle $AMBN$ whose one of the diagonals is just AB.

Let's note by O the midpoint of the diagonal AB, but O is also the center of symmetry (intersection of the diagonals) of the rectangle $AMBN$. Then one computes the distance between a point $P(x_0, y_0)$ and the rectangle $AMBN$. One can do that following the same principle as Dr. Cai Wen did:

— compute the distance in $2D$ (two dimensions) between the point P and the center O of the rectangle (intersection of rectangle's diagonals);

— next compute the distance between the point P and the closest point (let's note it by P') to it on the frontier (the rectangle's four edges) of the rectangle $AMBN$.

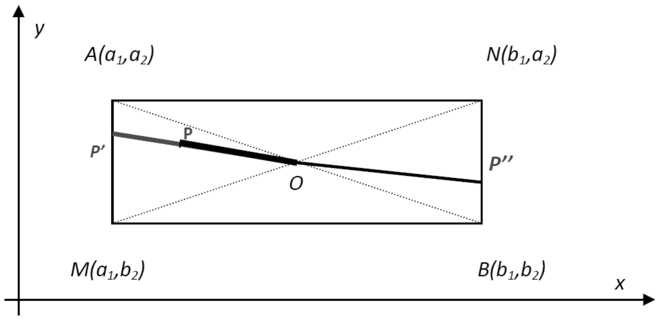

Fig. 3: P is an interior point to the rectangle $AMBN$ and the optimal point O is in the center of symmetry of the rectangle.

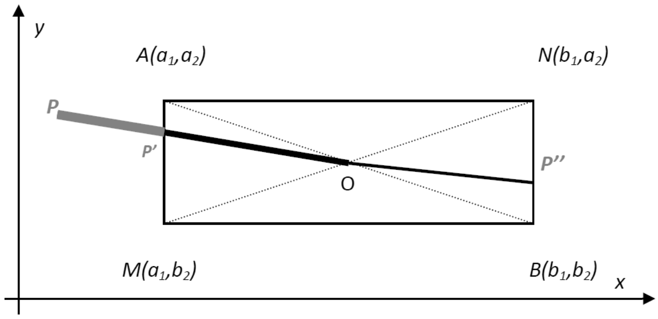

Fig. 4: P is an exterior point to the rectangle $AMBN$ and the optimal point O is in the center of symmetry of the rectangle.

This step can be done in the following way: considering P' as the intersection point between the line PO and the frontier of the rectangle, and taken among the intersection points that point P' which is the closest to P; this case is entirely consistent with Dr. Cai's approach in the sense that when reducing from a $2D$-space problem to two $1D$-space problems, one exactly gets his result.

The Extension $2D$-Distance, for $P \neq O$, will be:

$$\rho((x_0, y_0), AMBN) = d(\text{point } P, \text{ rectangle } AMBN) =$$
$$= |PO| - |P'O| = \pm|PP'|, \tag{5}$$

i) which is equal to the negative length of the red segment $|PP'|$ in Fig. 3, when P is interior to the rectangle $AMBN$;

ii) or equal to zero, when P lies on the frontier of the rectangle $AMBN$ (i.e. on edges AM, MB, BN, or NA) since P coincides with P';

iii) or equal to the positive length of the blue segment $|PP'|$ in Fig. 4, when P is exterior to the rectangle $AMBN$, where $|PO|$ means the classical $2D$-distance between the point P and O, and similarly for $|P'O|$ and $|PP'|$.

The Extension $2D$-Distance, for the optimal point, i.e. $P = O$, will be

$$\rho(O, AMBN) = d(\text{point } O, \text{ rectangle } AMBN) =$$
$$= -\max d(\text{point } O, \text{ point } M \text{ on the frontier of } AMBN). \tag{6}$$

The last step is to devise the Dependent Function in $2D$-space similarly as Dr. Cai's defined the dependent function in $1D$. The midpoint (or center of symmetry) O has the coordinates

$$O\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right). \tag{7}$$

Let's compute the

$$|PO| - |P'O|. \tag{8}$$

In this case, we extend the line OP to intersect the frontier of the rectangle $AMBN$. P' is closer to P than P'', therefore we consider P'. The equation of the line PO, that of course passes through the points $P(x_0, y_0)$ and $O\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$, is:

$$y - y_0 = \frac{\frac{a_2+b_2}{2} - y_0}{\frac{a_1+b_1}{2} - x_0}(x - x_0). \tag{9}$$

Since the x-coordinate of point P' is a_1 because P' lies on the rectangle's edge AM, one gets the y-coordinate of point P' by a simple substitution of $x_{P'} = a_1$ into the above equality:

$$y_{P'} = y_0 + \frac{a_2 + b_2 - 2y_0}{a_1 + b_1 - 2x_0}(a_1 - x_0). \tag{10}$$

Therefore P' has the coordinates

$$P'\left[x_{P'} = a_1,\ y_{P'} = y_0 + \frac{a_2 + b_2 - 2y_0}{a_1 + b_1 - 2x_0}(a_1 - x_0)\right]. \tag{11}$$

The distance

$$d(PQ) = |PQ| = \sqrt{\left(x_0 - \frac{a_1+b_1}{2}\right)^2 + \left(y_0 - \frac{a_2+b_2}{2}\right)^2}, \tag{12}$$

while the distance

$$d(P', Q) = |P'Q| =$$
$$= \sqrt{\left(a_1 - \frac{a_1+b_1}{2}\right)^2 + \left(y_{P'} - \frac{a_2+b_2}{2}\right)^2} =$$
$$= \sqrt{\left(\frac{a_1-b_1}{2}\right)^2 + \left(y_{P'} - \frac{a_2+b_2}{2}\right)^2}. \tag{13}$$

Also, the distance

$$d(PP') = |PP'| = \sqrt{(a_1 - x_0)^2 + (y_{P'} - y_0)^2}. \tag{14}$$

Whence the Extension $2D$-distance formula

$$\rho\,[(x_0, y_0),\ AMBN] =$$
$$= d[P(x_0, y_0),\ A(a_1, a_2)\,MB(b_1, b_2)\,N] =$$
$$= |PQ| - |P'Q| \tag{15}$$
$$= \sqrt{\left(x_0 - \frac{a_1+b_1}{2}\right)^2 + \left(y_0 - \frac{a_2+b_2}{2}\right)^2} - \sqrt{\left(\frac{a_1-b_1}{2}\right)^2 + \left(y_{P'} - \frac{a_2+b_2}{2}\right)^2} \tag{16}$$
$$= \pm|PP'| \tag{17}$$
$$= \pm\sqrt{(a_1 - x_0)^2 + (y_{P'} - y_0)^2}, \tag{18}$$

where

$$y_{P'} = y_0 + \frac{a_2 + b_2 - 2y_0}{a_1 + b_1 - 2x_0}(a_1 - x_0). \tag{19}$$

6 Properties

As for $1D$-distance, the following properties hold in $2D$:

6.1 Property 1

a) $(x, y) \in \mathrm{Int}\,(AMBN)$ if $\rho\,[(x, y), AMBN] < 0$, where $\mathrm{Int}\,(AMBN)$ means interior of $AMBN$;

b) $(x, y) \in \mathrm{Fr}\,(AMBN)$ if $\rho\,[(x, y), AMBN] = 0$, where $\mathrm{Fr}\,(AMBN)$ means frontier of $AMBN$;

c) $(x, y) \notin AMBN$ if $\rho\,[(x, y), AMBN] > 0$.

6.2 Property 2

Let $A_0M_0B_0N_0$ and $AMBN$ be two rectangles whose sides are parallel to the axes of the Cartesian system of coordinates, such that they have no common end points, and $A_0M_0B_0N_0 \subset AMBN$. We assume they have the same optimal points $O_1 \equiv O_2 \equiv O$ located in the center of symmetry of the two rectangles. Then for any point $(x, y) \subset R^2$ one has $\rho\,[(x, y), A_0M_0B_0N_0] \geqslant \rho\,[(x, y), AMBN]$. See Fig. 5.

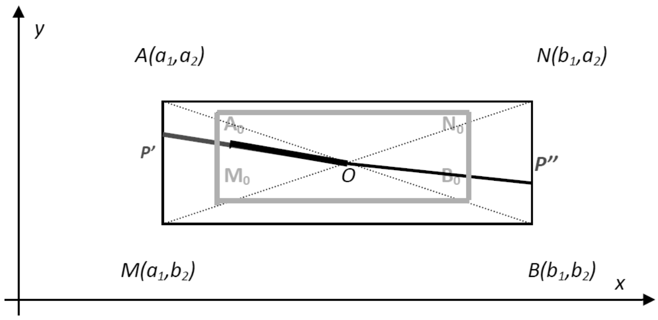

Fig. 5: Two included rectangles with the same optimal points $O_1 \equiv O_2 \equiv O$ located in their common center of symmetry.

7 Dependent $2D$-Function

Let $\mathrm{A}_0M_0B_0N_0$ and $AMBN$ be two rectangles whose sides are parallel to the axes of the Cartesian system of coordinates, such that they have no common end points, and $A_0M_0B_0N_0 \subset AMBN$.

The Dependent $2D$-Function formula is:

$$K_{2D(x,y)} = \frac{\rho\,[(x, y), AMBN]}{\rho\,[(x, y), AMBN,] - \rho\,[(x, y), A_0M_0B_0N_0]}. \tag{20}$$

7.1 Property 3

Again, similarly to the Dependent Function in $1D$-space, one has:

a) If $(x, y) \in \mathrm{Int}\,(A_0M_0B_0N_0)$, then $K_{2D(x,y)} > 1$;

b) If $(x, y) \in \mathrm{Fr}\,(A0M0B0N0)$, then $K_{2D(x,y)} = 1$;

c) If $(x, y) \in \text{Int}\,(AMBN - A_0M_0B_0N_0)$, then $0 < K_{2D(x,y)} < 1$;
d) If $(x, y) \in \text{Fr}\,(AMBN)$, then $K_{2D(x,y)} = 0$;
e) If $(x, y) \notin AMBN$, then $K2D(x, y) < 0$.

8 General Case in 2*D*-Space

One can replace the rectangles by any finite surfaces, bounded by closed curves in 2*D*-space, and one can consider any optimal point *O* (not necessarily the symmetry center). Again, we assume the optimal points are the same for this nest of two surfaces. See Fig. 6.

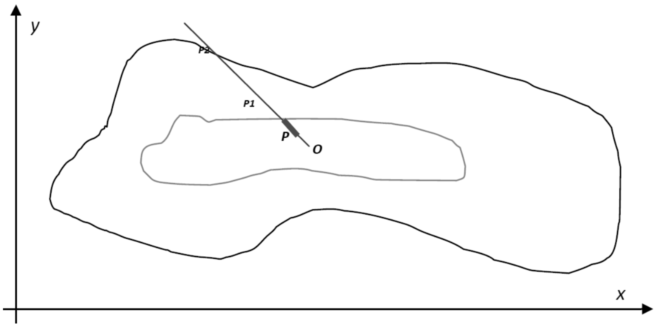

Fig. 6: Two included arbitrary bounded surfaces with the same optimal points situated in their common center of symmetry.

9 Linear Attraction Point Principle

We introduce the Attraction Point Principle, which is the following:

Let S be a given set in the universe of discourse U, and the optimal point $O \subset S$. Then each point $P\,(x_1, x_2, \ldots, x_n)$ from the universe of discourse tends towards, or is attracted by, the optimal point *O*, because the optimal point *O* is an ideal of each point. That's why one computes the extension $(n-D)$-distance between the point *P* and the set S as $\rho\,[(x_1, x_2, \ldots, x_n), S\,]$ on the direction determined by the point *P* and the optimal point *O*, or on the line *PO*, i.e.:

a) $\rho\,[(x_1, x_2, \ldots, x_n), S\,]$ is the negative distance between *P* and the set frontier, if *P* is inside the set S;
b) $\rho\,[(x_1, x_2, \ldots, x_n), S\,] = 0$, if *P* lies on the frontier of the set S;
c) $\rho\,[(x_1, x_2, \ldots, x_n), S\,]$ is the positive distance between *P* and the set frontier, if *P* is outside the set.

It is a king of convergence/attraction of each point towards the optimal point. There are classes of examples where such attraction point principle works. If this principle is good in all cases, then there is no need to take into consideration the center of symmetry of the set S, since for example if we have a 2*D* piece which has heterogeneous material density, then its center of weight (barycenter) is different from the center of symmetry. Let's see below such example in the 2*D*-space: Fig. 7.

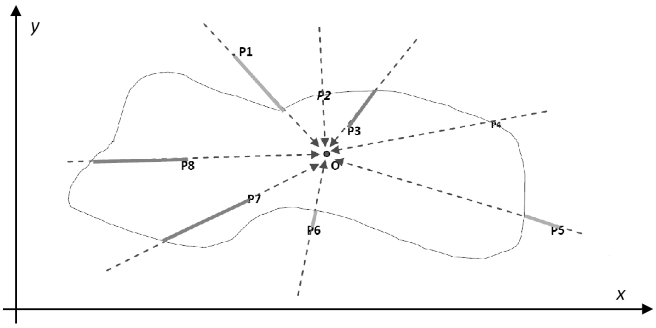

Fig. 7: The optimal point O as an attraction point for all other points $P_1, P_2, \ldots, P_8$ in the universe of discourse R^2.

10 Remark 1

Another possible way, for computing the distance between the point *P* and the closest point P' to it on the frontier (the rectangle's four edges) of the rectangle *AMBN*, would be by drawing a perpendicular (or a geodesic) from *P* onto the closest rectangle's edge, and denoting by P' the intersection between the perpendicular (geodesic) and the rectangle's edge. And similarly if one has an arbitrary set S in the 2*D*space, bounded by a closed urve. One computes

$$d(P, S\,) = \operatorname*{Inf}_{Q \in S} |PQ| \tag{21}$$

as in the classical mathematics.

11 Extension Distance in 3*D*-Space

We further generalize to 3*D*-space the Extension Set and the Dependent Function. Assume we have two points (a_1, a_2, a_3) and (b_1, b_2, b_3) in *D*. Drawing through *A* end *B* parallel planes to the planes' axes (XY, XZ, YZ) in the Cartesian system *XYZ* we get a prism $AM_1M_2M_3BN_1N_2N_3$ (with eight vertices) whose one of the transversal diagonals is just the line segment *AB*. Let's note by *O* the midpoint of the transverse diagonal *AB*, but *O* is also the center of symmetry of the prism.

Therefore, from the line segment *AB* in 1*D*-space, to a rectangle *AMBN* in 2*D*-space, and now to a prism $AM_1M_2M_3BN_1N_2N_3$ in 3*D*-space. Similarly to 1*D*- and 2*D*-space, the prism may be closed (i.e. all points lying on its frontier belong to it), open (i.e. no point lying on its frontier belong to it), or partially closed (i.e. some points lying on its frontier belong to it, while other points lying on its frontier do not belong to it).

Then one computes the distance between a point $P\,(x_0, y_0, z_0)$ and the prism $AM_1M_2M_3BN_1N_2N_3$. One can do that following the same principle as Dr. Cai's:

— compute the distance in 3*D* (two dimensions) between the point *P* and the center *O* of the prism (intersection of prism's transverse diagonals);
— next compute the distance between the point *P* and the closest point (let's note it by P') to it on the frontier of

the prism $AM_1M_2M_3BN_1N_2N_3$ (the prism's lateral surface); considering P' as the intersection point between the line OP and the frontier of the prism, and taken among the intersection points that point P' which is the closest to P; this case is entirely consistent with Dr. Cai's approach in the sense that when reducing from $3D$-space to $1D$-space one gets exactly Dr. Cai's result;

— the Extension $3D$-Distance $d(P, AM_1M_2M_3BN_1N_2N_3)$ is $d(P, AM_1M_2M_3BN_1N_2N_3) = |PO| - |P'O| = \pm|PP'|$, where $|PO|$ means the classical distance in $3D$-space between the point P and O, and similarly for $|P'O|$ and $|PP'|$. See Fig. 8.

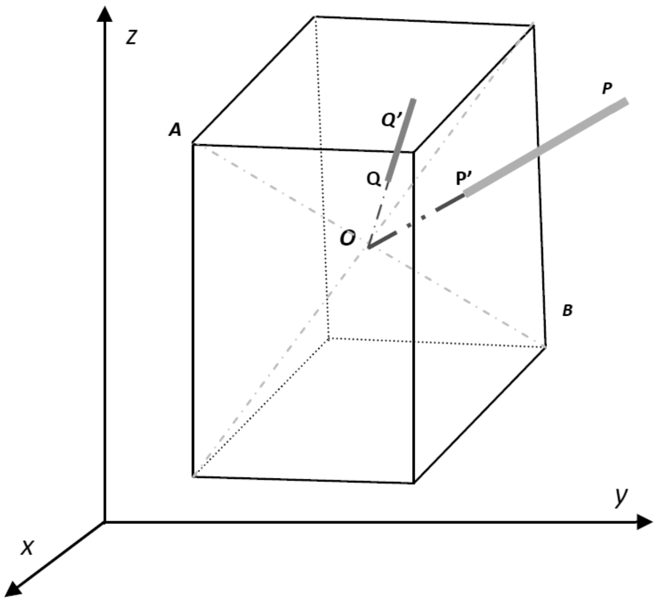

Fig. 8: Extension $3D$-Distance between a point and a prism, where O is the optimal point coinciding with the center of symmetry.

12 Property 4

a) $(x, y, z) \in \mathrm{Int}\,(AM_1M_2M_3BN_1N_2N_3)$
if $\rho\,[(x, y, z), AM_1M_2M_3BN_1N_2N_3] < 0$,
where $\mathrm{Int}\,(AM_1M_2M_3BN_1N_2N_3)$ means interior of $AM_1M_2M_3BN_1N_2N_3$;

b) $(x, y, z) \in \mathrm{Fr}\,(AM_1M_2M_3BN_1N_2N_3)$
if $\rho\,[(x, y, z), AM_1M_2M_3BN_1N_2N_3] = 0$
means frontier of $AM_1M_2M_3BN_1N_2N_3$;

c) $(x, y, z) \notin AM_1M_2M_3BN_1N_2N_3$
if $\rho\,[(x, y, z), AM_1M_2M_3BN_1N_2N_3] > 0$.

13 Property 5

Let $A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03}$ and $AM_1M_2M_3BN_1N_2N_3$ be two prisms whose sides are parallel to the axes of the Cartesian system of coordinates, such that they have no common end points, and $A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03} \subset AM_1M_2M_3BN_1N_2N_3$. We assume they have the same optimal points $O_1 \equiv O_2 \equiv O$ located in the center of symmetry of the two prisms.

Then for any point $(x, y, z) \in R^3$ one has

$$\rho\,[(x, y, z), A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N]_{03} \geqslant \rho\,[(x, y, z)AM_1M_2M_3BN_1N_2N_3]\,.$$

14 The Dependent 3*D*-Function

The last step is to devise the Dependent Function in $3D$-space similarly to Dr. Cai's definition of the dependent function in $1D$-space. Let the prisms $A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03}$ and $AM_1M_2M_3BN_1N_2N_3$ be two prisms whose faces are parallel to the axes of the Cartesian system of coordinates XYZ, such that they have no common end points in such a way that $A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03} \subset AM_1M_2M3BN_1N_2N_3$. We assume they have the same optimal points $O_1 \equiv O_2 \equiv O$ located in the center of symmetry of these two prisms.

The Dependent 3D-Function formula is:

$$K_{3D(x,y,z)} = \Big(\rho\,[(x, y, z), AM_1M_2M_3BN_1N_2N_3]\Big) \times \times \Big(\rho\,[(x, y, z), AM_1M_2M_3BN_1N_2N_3,] - - \rho\,[(x, y, z), A_0M_{01}M_{02}M_{03}BN_{01}N_{02}N_{03}]\Big)^{-1}. \tag{22}$$

15 Property 6

Again, similarly to the Dependent Function in $1D$- and $2D$-spaces, one has:

a) If $(x, y, z) \in \mathrm{Int}\,(A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03})$,
then $K_{3D}(x, y, z) > 1$;

b) If $(x, y, z) \in \mathrm{Fr}\,(A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03})$,
then $K_{3D}(x, y, z) = 1$;

c) If $(x, y, z) \in \mathrm{Int}\,(AM_1M_2M_3BN_1N_2N_3-$
$-A_0M_{01}M_{02}M_{03}B_0N_{01}N_{02}N_{03})$,
then $0 < K_{3D}(x, y, z) < 1$;

d) If $(x, y, z) \in \mathrm{Fr}\,(AM_1M_2M_3BN_1N_2N_3)$,
then $K_{3D}(x, y, z) = 0$;

e) If $(x, y, z) \notin AM_1M_2M_3BN_1N_2N_3$,
then $K_{3D}(x, y, z) < 0$.

16 General Case in 3*D*-Space

One can replace the prisms by any finite $3D$-bodies, bounded by closed surfaces, and one considers any optimal point O (not necessarily the centers of surfaces' symmetry). Again, we assume the optimal points are the same for this nest of two $3D$-bodies.

17 Remark 2

Another possible way, for computing the distance between the point P and the closest point P' to it on the frontier (lateral surface) of the prism $AM_1M_2M_3BN_1N_2N_3$ is by drawing a perpendicular (or a geodesic) from P onto the closest prism's face, and denoting by P' the intersection between the perpendicular (geodesic) and the prism's face.

And similarly if one has an arbitrary finite body B in the $3D$-space, bounded by surfaces. One computes as in classical mathematics:

$$d(P, B) = \operatorname*{Inf}_{Q \in B} |PB|. \tag{23}$$

18 Linear Attraction Point Principle in 3D-Space

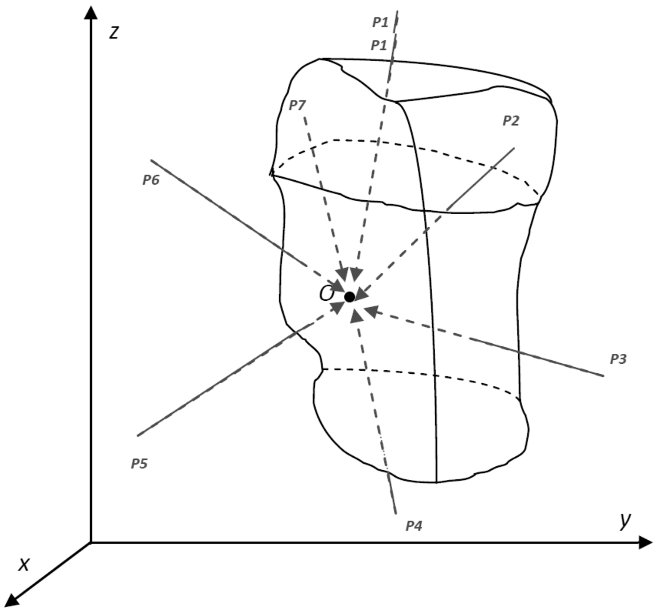

Fig. 9: Linear Attraction Point Principle for any bounded 3D-body.

19 Non-Linear Attraction Point Principle in 3D-Space, and in $(n - D)$-Space

There might be spaces where the attraction phenomena undergo not linearly by upon some specific non-linear curves. Let's see below such example for points P_i whose trajectories of attraction towards the optimal point follow some non-linear 3D-curves.

20 $(n - D)$-Space

In general, in a universe of discourse U, let's have an $(n - D)$-set S and a point P. Then the Extension Linear $(n - D)$-Distance between point P and set S, is:

$$\rho(P, S) = \begin{cases} -\underset{P' \in \mathrm{Fr}(S)}{d(P, P')}, & P \neq 0,\ P \in |OP'| \\ \underset{P' \in \mathrm{Fr}(S)}{d(P, P')}, & P \neq 0,\ P' \in |OP| \\ -\max\limits_{P' \in \mathrm{Fr}(S)} d(P, M), & P = 0 \end{cases} \tag{24}$$

where O is the optimal point (or linearly attraction point); $d(P, P')$ means the classical linearly $(n - D)$-distance between two points P and P'; $\mathrm{Fr}(S)$ means the frontier of set S; and $|OP'|$ means the line segment between the points O and P' (the extremity points O and P' included), therefore $P \in |OP'|$ means that P lies on the line OP', in between the points O and P'.

For P coinciding with O, one defined the distance between the optimal point O and the set S as the negatively maximum distance (to be in concordance with the 1D-definition).

And the Extension Non-Linear $(n - D)$-Distance between point P and set S, is:

$$\rho_c(P, S) = \begin{cases} -\underset{P' \in \mathrm{Fr}(S)}{d_c(P, P')}, & P \neq 0,\ P \in c\,(OP') \\ \underset{P' \in \mathrm{Fr}(S)}{d_c(P, P')}, & P \neq 0,\ P' \in c\,(OP) \\ -\max\limits_{P' \in \mathrm{Fr}(S),\ M \in c(O)} d_c(P, M), & P = 0 \end{cases} \tag{25}$$

where means the extension distance as measured along the curve c; O is the optimal point (or non-linearly attraction point); the points are attracting by the optimal point on trajectories described by an injective curve c; $dc\,(P, P')$ means the non-linearly $(n - D)$-distance between two points P and P', or the arc length of the curve c between the points P and P'; $\mathrm{Fr}(S)$ means the frontier of set S; and $c\,(OP')$ means the curve segment between the points O and P' (the extremity points O and P' included), therefore $P \in (OP')$ means that P lies on the curve c in between the points O and P'.

For P coinciding with O, one defined the distance between the optimal point O and the set S as the negatively maximum curvilinear distance (to be in concordance with the 1D-definition).

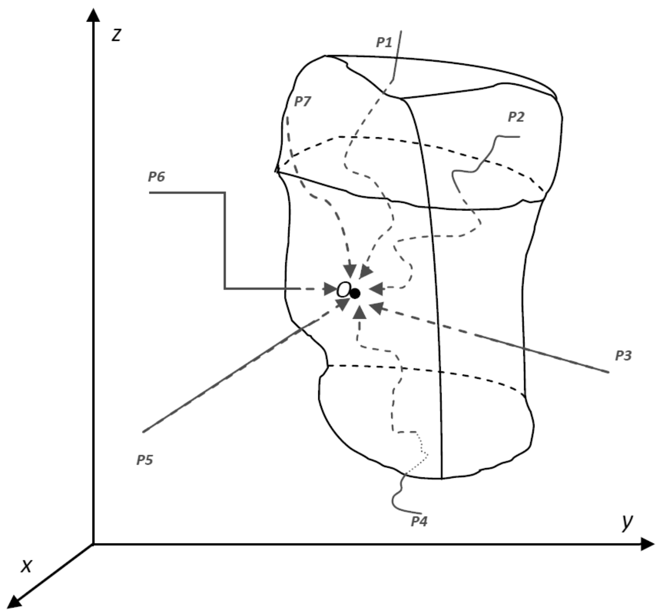

Fig. 10: Non-Linear Attraction Point Principle for any bounded 3D-body.

In general, in a universe of discourse U, let's have a nest of two $(n-D)$-sets, $S_1 \subset S_2$, with no common end points, and a point P. Then the Extension Linear Dependent $(n-D)$-Function referring to the point $P(x_1, x_2, \dots, x_n)$ is:

$$K_{nD}(P) = \frac{\rho(P, S_2)}{\rho(P, S_2) - \rho(P, S_1)}, \tag{26}$$

where is the previous extension linear $(n-D)$-distance between the point P and the $(n-D)$-set S_2.

And the Extension Non-Linear Dependent $(n-D)$-Function referring to point $P(x_1, x_2, \dots, x_n)$ along the curve c is:

$$K_{nD}(P) = \frac{\rho_c(P, S_2)}{\rho_c(P, S_2) - \rho_c(P, S_1)}, \tag{27}$$

where is the previous extension non-linear $(n-D)$-distance between the point P and the $(n-D)$-set S_2 along the curve c.

21 Remark 3

Particular cases of curves c could be interesting to studying, for example if c are parabolas, or have elliptic forms, or arcs of circle, etc. Especially considering the geodesics would be for many practical applications. Tremendous number of applications of Extenics could follow in all domains where attraction points would exist; these attraction points could be in physics (for example, the earth center is an attraction point), economics (attraction towards a specific product), sociology (for example attraction towards a specific life style), etc.

22 Conclusion

In this paper we introduced the *Linear and Non-Linear Attraction Point Principle*, which is the following:

Let S be an arbitrary set in the universe of discourse U of any dimension, and the optimal point $O \in S$. Then each point $P(x_1, x_2, \dots, x_n)$, $n \geqslant 1$, from the universe of discourse (linearly or non-linearly) tends towards, or is attracted by, the optimal point O, because the optimal point O is an ideal of each point.

It is a king of convergence/attraction of each point towards the optimal point. There are classes of examples and applications where such attraction point principle may apply.

If this principle is good in all cases, then there is no need to take into consideration the center of symmetry of the set S, since for example if we have a $2D$ factory piece which has heterogeneous material density, then its center of weight (barycenter) is different from the center of symmetry.

Then we generalized in the track of Cai Wen's idea to extend $1D$-set to an extension $(n-D)$-set, and thus defined the *Linear* (or *Non-Linear*) *Extension* $(n-D)$-*Distance* between a point $P(x_1, x_2, \dots, x_n)$ and the $(n-D)$-set S as $\rho[(x_1, x_2, \dots, x_n), S]$ on the linear (or non-linear) direction determined by the point P and the optimal point O (the line PO, or respectively the curvilinear PO) in the following way:

1) $\rho[(x_1, x_2, \dots, x_n), S]$ is the negative distance between P and the set frontier, if P is inside the set S;
2) $\rho[(x_1, x_2, \dots, x_n), S] = 0$, if P lies on the frontier of the set S;
3) $\rho[(x_1, x_2, \dots, x_n), S]$ is the positive distance between P and the set frontier, if P is outside the set.

We got the following properties:

4) It is obvious from the above definition of the extension $(n-D)$-distance between a point P in the universe of discourse and the extension $(n-D)$-set S that:
   i) Point $P(x_1, x_2, \dots, x_n) \in \mathrm{Int}(S)$ if $\rho[(x_1, x_2, \dots x_n), S] < 0$;
   ii) Point $P(x_1, x_2, \dots, x_n) \in \mathrm{Fr}(S)$ if $\rho[(x_1, x_2, \dots x_n), S] = 0$;
   iii) Point $P(x1, x2, \dots, x_n) \notin S$ if $\rho[(x_1, x_2, \dots x_n), S] > 0$.
5) Let S_1 and S_2 be two extension sets, in the universe of discourse U, such that they have no common end points, and $S_1 \subset S_2$. We assume they have the same optimal points $O1 \equiv O2 \equiv O$ located in their center of symmetry. Then for any point $P(x_1, x_2, \dots, x_n) \in U$ one has:

$$\rho[(x_1, x_2, \dots x_n), S_2] \geqslant \rho[(x_1, x_2, \dots x_n), S_1]. \tag{28}$$

Then we proceed to the generalization of the dependent function from $1D$-space to Linear (or Non-Linear) $(n-D)$-space Dependent Function, using the previous notations.

The *Linear* (or *Non-Linear*) *Dependent* $(n-D)$-*Function* of point $P(x_1, x_2, \dots, x_n)$ along the curve c, is:

$$K_{nD}(x_1, x_2, \dots, x_n) = \Big(\rho_c[(x_1, x_2, \dots x_n), S_2]\Big) \times \\ \times \Big(\rho_c[(x_1, x_2, \dots x_n), S_2] - \rho_c[(x_1, x_2, \dots x_n), S_1]\Big)^{-1} \tag{29}$$

(where c may be a curve or even a line) which has the following property:

6) If point $P(x_1, x_2, \dots, x_n) \in \mathrm{Int}(S1)$, then $K_{nD}(x_1, x_2, \dots, x_n) > 1$;
7) If point $P(x_1, x_2, \dots, x_n) \in \mathrm{Fr}(S1)$, then $K_{nD}(x_1, x_2, \dots, x_n) = 1$;
8) If point $P(x_1, x_2, \dots, x_n) \in \mathrm{Int}(S2 - S1)$, then $K_{nD}(x_1, x_2, \dots, x_n) \in (0, 1)$;
9) If point $P(x_1, x_2, \dots, x_n) \in \mathrm{Int}(S2)$, then $K_{nD}(x_1, x_2, \dots, x_n) = 0$;
10) If point $P(x_1, x_2, \dots, x_n) \notin \mathrm{Int}(S2)$, then $K_{nD}(x_1, x_2, \dots, x_n) < 0$.

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