INEQUALITIES FOR THE INTEGER PART FUNCTION[1]

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In this article we will prove some inequalities for the integer part function, and we'll give some applications in the number theory.

Theorem 1. For any $x, y > 0$ we have the following inequality:

(1) $[5x]+[5y] \geq [3x+y]+[3y+x]$,

where [ ] means the integer part function.

Proof: We will use the notations: $x_1 = [x],\ y_1 = [y],\ u = \{x\},\ v = \{y\},\ x_1, y_1 \in \mathrm{N}$ and $u, v \in [0,1)$. We can write the inequality (1) as:

$$x_1 + y_1 + [5u] + [5v] \geq [3u+v] + [3v+u].$$

We distinguish the following cases:

α) Let $u \geq v$. If $u \leq 2v$, then $5v \geq 3v+u$ and $[5v] \geq [3v+u]$, analogously $5u \geq 3u+v$ and $[5u] \geq [3u+v]$, from where by addition we obtain (1).

β) If $u > 2v$ and $5u = a+b,\ 5v = c+d,\ a,c \in \mathrm{N},\ 0 \leq b < 1,\ 0 \leq d < 1$, then we have to prove the following inequality:

(2) $a+c+x_1+y_1 \geq \left[\dfrac{3a+c+3b+d}{5}\right] + \left[\dfrac{3c+a+3d+b}{5}\right].$

But, considering that $1 > u > 2v$, we obtain $5 > 5u > 10v$, from where, $5 > a+b > 2c+2d$, thus $a+b < 5$ and $a \leq 4$.

If $a < 2c$, then $a \leq 2c-1$ and $a+1-2c \leq 0$ thus $a+b-2c < 0$; contradiction with $a+b-2c > 2d$, thus $4 \geq a,\ a \geq 2c$, and $3b+d < 4,\ 3d+b < 4$.

From $4 \geq a \geq 2c$ we have the cases from the following table, and in each of the nine cases is verified the inequality (2).

a	4	4	4	3	3	2	2	1	0
c	2	1	0	0	1	1	0	0	0

Application 1. For any $m, n \in \mathrm{N}$, (5m)!(5n)! is divisible by

$$m!n!(3m+n)!(3a+m)!$$

Proof: If p is a prime number, the power exponent of p in decomposition of $m!$

[1]Together with Mihály Bencze and Florin Popovici

is

$$\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\ldots$$

It is sufficient to prove that

$$\left[\frac{5m}{r}\right]+\left[\frac{5n}{r}\right]\ge\left[\frac{m}{r}\right]+\left[\frac{n}{r}\right]+\left[\frac{3m+n}{r}\right]+\left[\frac{3n+m}{r}\right]$$

for any $r\in\mathrm{N}, r\ge 2$.

If $m=rm_1+x,\ \ n=rn_1+y$, where $0\le x<r,\ 0\le y<r$, $m,\ n\in\mathbf{Z}$, it is sufficient to prove that

$$\left[\frac{5x}{r}\right]+\left[\frac{5y}{r}\right]\ge\left[\frac{3x+y}{r}\right]+\left[\frac{3y+x}{r}\right],$$

but this inequality verifies the theorem 1.

Remark. If $x,y>0$, then we have the inequality:

$$5x \ + \ 5y \ \ge \ x \ + \ y \ + \ 3x+y \ + \ 3y+x \ .$$

Theorem 2. (Szilárd András). If $x,y,z\ge 0$, then we have the inequality:

$$3x \ + \ 3y \ + \ 3z \ \ge \ x \ + \ y \ + \ z \ + \ x+y \ + \ y+z \ + \ z+x \ .$$

Application 2. For any $a,\ b,\ c\in\mathrm{N}$, $(3a)!(3b)!(3c)!$ is divisible by

$$a!b!c!(a+b)!(b+c)!(c+a)!.$$

Proof: Let k_1,k_2,k_3 be the biggest power for which

$$p^{k_1}\mid\ 3a\ !,\ p^{k_2}\mid\ 3b\ !,\ p^{k_3}\mid\ 3c\ !$$

respectively, and $r_i,\quad i\in\ 1,2,3,4,5,6$ the biggest power for which

$$p^{r_1}\,|a!,\ p^{r_2}\,|b!,\ p^{r_3}\,|c!,\ p^{r_4}\mid\ a+b\ !,\ p^{r_5}\mid\ b+c\ !,\ p^{r_6}\mid\ c+a\ !$$

respectively, then

$$k_1+k_2+k_3=\left(\left[\frac{3a}{p}\right]+\left[\frac{3a}{p^2}\right]+\ldots\right)+\left(\left[\frac{3b}{p}\right]+\left[\frac{3b}{p^2}\right]+\ldots\right)+\left(\left[\frac{3c}{p}\right]+\left[\frac{3c}{p^2}\right]+\ldots\right)$$

and

$$\sum_{i=1}^{6}r_i\left(\left[\frac{a}{p}\right]+\left[\frac{a}{p^2}\right]+\ldots\right)+\left(\left[\frac{b}{p}\right]+\left[\frac{b}{p^2}\right]+\ldots\right)+\left(\left[\frac{c}{p}\right]+\left[\frac{c}{p^2}\right]+\ldots\right)+$$

$$+\left(\left[\frac{a+b}{p}\right]+\left[\frac{a+b}{p^2}\right]+\ldots\right)+\left(\left[\frac{b+c}{p}\right]+\left[\frac{b+c}{p^2}\right]+\ldots\right)+\left(\left[\frac{c+a}{p}\right]+\left[\frac{c+a}{p^2}\right]+\ldots\right).$$

We have to prove that $k_1+k_2+k_3\ge\sum_{i=1}^{6}r_i$, but this inequality reduces to theorem 2.

Theorem 3. If $x,\ y,\ z\ge 0$, then we have the inequality:

$$2x \; + \; 2y \; + \; 2z \; \leq \; x \; + \; y \; + \; z \; + \; x+y+z \; .$$

Application 3. If $a, b, c \in \mathrm{N}$, then $a!b!c!(a+b+c)!$ is divisible by $(2a)!(2b)!(2c)!$.

Theorem 4. If $x, y \geq 0$ and $n, k \in \mathrm{N}$ such that $n \geq k \geq 0$, then we have the inequality:

$$nx \; + \; ny \; \geq k \; x \; + k \; y \; + \; n-k \quad x+y \; .$$

Application 4. If $a, b, n, k \in \mathrm{N}$ and $n \geq k$, then $(na)!(nb)!$ is divisible by

$$a!^{\;k} \; b!^{\;k} \quad a+b \; !^{\;n-k} .$$

Theorem 5. If $x_k \geq 0,\ (k = 1,2,...,n)$, then we have the inequality:

$$2\sum_{k=1}^{n} \; 2x_k \; \geq 2\sum_{k=1}^{n} \; x_k \; + \; x_1 + x_2 \; + \; x_2 + x_3 \; + ... + \; x_n + x_1 \; .$$

Application 5. If $a_k \in \mathrm{N},\ (k = 1,2,...,n)$, then $\prod_{k=1}^{n} \; 2a_k \; !^{\;2}$ is divisible by

$$\prod_{k=1}^{n} (a_k!)^2 (a_1 + a_2)!(a_2 + a_3)!...(a_n + a_1)! .$$

Theorem 6. If $x_k \geq 0,\ (k = 1,2,...,n)$, then we have the inequality:

$$m\sum_{k=1}^{n}[2x_k] + n\sum_{p=1}^{m}[2x_p] \geq m\sum_{k=1}^{n}[x_k] + n\sum_{p=1}^{m}[x_p] + \sum_{k=1}^{n}\sum_{p=1}^{m}[x_k + x_p].$$

Application 6. If $a_k \in \mathrm{N},\ (k = 1,2,...,n)$, then $\prod_{k=1}^{n}(2a_k!)^m \prod_{p=1}^{m}(2a_p!)^n$ is divisible

by

$$\prod_{k=1}^{n} \; a_k!^{\;m} \prod_{p=1}^{m} \; a_p!^{\;n} \prod_{k=1}^{n}\prod_{p=1}^{m} \quad a_k + a_p \;\; ! \; .$$

Theorem 7. If $x, y \geq 1$, then we have the inequality:

$$\left[\sqrt{x}\right] + \left[\sqrt{y}\right] + \left[\sqrt{x+y}\right] \geq \left[\sqrt{2x}\right] + \left[\sqrt{2y}\right]$$

Proof: By the concavity of the square root function:

$$\sqrt{x+y} = \sqrt{\frac{2x+2y}{2}} \geq \frac{1}{2}\sqrt{2x} + \frac{1}{2}\sqrt{2y} \geq \left[\frac{1}{2}\sqrt{2x}\right] + \left[\frac{1}{2}\sqrt{2y}\right],$$

it follows that:

$$\left[\sqrt{x+y}\right] \geq \left[\frac{1}{2}\sqrt{2x}\right] + \left[\frac{1}{2}\sqrt{2y}\right].$$

Therefore, it is sufficient to show that

$$\left[\sqrt{x}\right]+\left[\frac{1}{2}\sqrt{2x}\right]\geq\left[\sqrt{2x}\right] \text{ for } x\geq 1.$$

The identity $[x]+\left[x+\frac{1}{2}\right]$ has a straightforward proof. We'll use it to replace $\left[\frac{1}{2}\sqrt{2x}\right]$ with

$$\left[\sqrt{2x}\right]-\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right].$$

This yields $\left[\sqrt{x}\right]\geq\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right]$, for $x\geq 1$.

This last inequality followed by the fact that $x\geq 4$ implies

$$2-\sqrt{2}\ \ \sqrt{x}>1 \text{ or } \left[\sqrt{x}\right]>\left[\frac{1}{2}\sqrt{2x}+\frac{1}{2}\right]$$

and $1\leq x<4$ implies

$$\frac{1}{2}\sqrt{2x}+\frac{1}{2}<2.$$

Application 7. If $a,b\in\mathrm{N}$, then $a!b!\left[\sqrt{a^2+b^2}\,!\right]$ is divisible by $\left[a\sqrt{2}\right]!\left[b\sqrt{2}\right]!$.

["Octogon", Braşov, Vol. 5, No. 2, 60-2, October 1997.]