An elementary proof of Catalan-Mihailescu theorem and generalization to Fermat-Catalan conjecture

Jamel Ghanouchi

jamel.ghanouchi@live.com

Abstract

(MSC=11D04) We begin with an equation, for example : $Y^p = X^q \pm Z^c$ and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation; Approach)

Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

We have

$$X^{q-3}Y^2 - Y^{p-2}X^3 = A$$

And

$$Y^{p-2}Y^2 - X^{q-3}X^3 = Y^p - X^q = 1$$

If $A = 0$ then $X^{q-6} = Y^{p-4}$ leads, as $GCD(X,Y) = 1$, to $p = 4$ and $q = 6$. This case has been studied by Lebesgue in the XIX century, it has no solution. Thus $A \neq 0$.
And if $A = \pm 1$ then it means that both
$X^{q-4}Y^2 = \pm\frac{1}{X} + X^2Y^{p-2}$ and
$Y^{p-3}X^3 = \mp\frac{1}{X} + X^{q-3}Y$ are rationals

it means, as it is impossible, that $q = 3$ and $p = 2$.
We have

$$\frac{X^{q-3}}{A}Y^2 - \frac{Y^{p-2}}{A}X^3 = 1 = Y^{p-2}Y^2 - X^{q-3}X^3$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-3}}{A})Y^2 = (X^{q-3} - \frac{Y^{p-2}}{A})X^3$$

Or

$$(AY^{p-2} - X^{q-3})Y^2 = (AX^{q-3} - Y^{p-2})X^3$$

And

$$(Y^2 + \frac{X^3}{A})Y^{p-2} = (X^3 + \frac{Y^2}{A})X^{q-3}$$

Or

$$(AY^2 + X^3)Y^{p-2} = (AX^3 + Y^2)X^{q-3}$$

But X and Y are coprimes. It means that we have four cases with u and v integers

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$
$$Y^{p-2} = v(AX^3 + Y^2); \quad X^{q-3} = v(X^3 + AY^2)$$

Or

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 = -X^{q-3} + AY^{p-2}$$
$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

Or

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$
$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

Or

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 = -X^{q-3} + AY^{p-2}$$

$$Y^{p-2} = v(AX^3 + Y^2); \quad X^{q-3} = v(X^3 + AY^2)$$

First case

$$Y^p = uv(AX^q - Y^p + A(Y^2X^{q-3} - Y^{p-2}X^3))$$
$$= uv(A^2X^q - Y^p + A(A)) = uv(A^2X^q + A^2 - Y^p) = uv(A^2Y^p - Y^p)$$

Thus

$$uv = \frac{1}{A^2 - 1}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = 1$ or $A = \pm -1$ it means that $q = 3$ and $p = 2$.

Second case

$$uv\frac{Y^p}{A^2} = X^q - \frac{Y^p}{A^2} + \frac{Y^2X^{q-3} - Y^{p-2}X^3}{A}$$
$$= X^q - \frac{Y^p}{A^2} + 1 = X^q + 1 - \frac{Y^p}{A^2} = (\frac{A^2-1}{A^2})Y^p$$

Thus

$$uv = A^2 - 1$$

And

$$uv(Y^2X^{q-3} - X^3Y^{p-2}) = uvA = u(X^{2q-6} - Y^{2p-4})A = v(-X^6 + Y^4)A$$

Thus

$$u = -X^6 + Y^4; \quad v = X^{2q-6} - Y^{2p-4}$$
$$uv = A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4})$$

But

$$Y^p = X^q + 1 \Rightarrow Y^{p-2}X^3 = X^q\frac{X^3}{Y^2} + \frac{X^3}{Y^2}$$

And

$$Y^2X^{q-3} = X^q\frac{X^{q-3}}{Y^{p-2}} + \frac{X^{q-3}}{Y^{p-2}}$$

We deduce

$$A = Y^2X^{q-3} - X^3Y^{p-2} = (X^q + 1)(\frac{X^{q-3}}{Y^{p-2}} - \frac{X^3}{Y^2})$$

But

$$Y^p > X^q \Rightarrow A < 0 \Rightarrow \frac{X^{q-3}}{Y^{p-2}} < \frac{X^3}{Y^2}$$

And if

$$X^3 < Y^2 \Rightarrow X^{q-3} < Y^{p-2} \Rightarrow A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4} < 0$$

But

$$0 > Y^2(X^{q-3} - Y^{p-2}) = Y^2X^{q-3} - Y^p = Y^2X^{q-3} - X^q - 1 = X^{q-3}(Y^2 - X^3) - 1 > -1$$

And it is possible if $q - 3 = p - 2 = 0$.
And if

$$X^{q-3} > Y^{p-2} \Rightarrow X^3 > Y^2 \Rightarrow A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4} < 0$$

But

$$0 < Y^2(X^{q-3} - Y^{p-2}) = Y^2X^{q-3} - Y^p = Y^2X^{q-3} - X^q - 1 = X^{q-3}(Y^2 - X^3) - 1 < 0$$

And it is possible if $q - 3 = p - 2 = 0$

$$X^{q-3} > Y^{p-2} \Rightarrow X^{q-3}Y^2 > Y^p = X^q + 1 \Rightarrow Y^2 > X^3 + X^{3-q}$$

And

$$Y^2 - X^3 > X^{3-q} > 0$$

And $p-2=q-3=0$
Third case :
We have here

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$

$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

And

$$vY^p = u(A^2X^q - Y^p + A^2) = u(A^2-1)Y^p$$

$$v = u(A^2-1)$$

$$v(Y^2X^{q-3} - X^3Y^{p-2}) = vA = uvA(X^{2q-6} - Y^{2p-4}) = A(-X^6 + Y^4)$$

Thus

$$1 = u(X^{2q-6} - Y^{2p-4})$$

With u and A^2-1 integers, it means $A^2 = 1$! Fourth case :

$$u\frac{Y^2}{A} = X^{q-3} - \frac{Y^{p-2}}{A}; \quad u\frac{X^3}{A} = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$\frac{Y^{p-2}}{A} = v(X^3 + \frac{Y^2}{A}); \quad \frac{X^{q-3}}{A} = v(\frac{X^3}{A} + Y^2)$$

We have here

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 - AY^{p-2} = -X^{q-3}$$

And

$$Y^{p-2} = AX^{q-3} - uY^2 = (Y^2X^{q-3} - X^3Y^{p-2})X^{q-3} - uY^2$$

Hence

$$u\frac{Y^p}{A^2} = v(X^q - \frac{Y^p}{A^2} + 1) = v(1 - \frac{1}{A^2})Y^p$$

Thus

$$u = v(A^2-1)$$

$$u(Y^2X^{q-3} - X^3Y^{p-2}) = uA = A(X^{2q-6} - Y^{2p-4}) = uv(-X^6 + Y^4)A$$

Thus

$$1 = v(-X^6 + Y^4)$$

With v and A^2-1 integers, it means $A^2-1=0$!
The only solution, in all cases, in $p=2$ and $q=3$.
And $Y^2 = X^3 + 1$ whose solution is $(X,Y) = (2, \pm 3)$.

Resolution of Fermat equation

Let Fermat equation :

$$Y^n = X^n + Z^n$$

We have

$$X^{n-2}Y^2 - Y^{n-2}X^2 = A$$

And

$$Y^{n-2}Y^2 - X^{n-2}X^2 = Y^n - X^n = Z^n$$

If $A=0$ then $X^{n-4} = Y^{n-4}$ leads, as $GCD(X,Y)=1$, to $n=4$. This case has been studied by Fermat, it has no solution. Thus $A \neq 0$.
And if $A = \pm Z^n$ then it means that both
$X^{n-3}Y^2 = \pm\frac{Z^n}{X} + XY^{n-2}$ and
$Y^{n-3}X^2 = \mp\frac{Z^n}{Y} + X^{n-2}Y$ are rationals

it means, as it is impossible, that n is not greater than 2 or $n=2$.
We have

$$\frac{X^{n-2}Z^n}{A}Y^2 - \frac{Y^{n-2}Z^n}{A}X^2 = Z^n = Y^{n-2}Y^2 - X^{n-2}X^2$$

And we have simultaneously

$$(Y^{n-2} - \frac{X^{n-2}Z^n}{A})Y^2 = (X^{n-2} - \frac{Y^{n-2}Z^n}{A})X^2$$

Or

$$(AY^{n-2} - X^{n-2}Z^n)Y^2 = (AX^{n-2} - Y^{p-2}Z^n)X^2$$

And

$$(Y^2 + \frac{X^2Z^n}{A})Y^{n-2} = (X^2 + \frac{Y^2Z^n}{A})X^{n-2}$$

Or

$$(AY^2 + X^2Z^n)Y^{p-2} = (AX^2 + Y^2Z^n)X^{n-2}$$

We have four cases with u and v integers

$$\frac{Y^2}{A} = u(X^{n-2} - \frac{Y^{n-2}Z^n}{A}); \quad \frac{X^2}{A} = u(-\frac{X^{n-2}Z^n}{A} + Y^{p-2})$$

$$\frac{Y^{n-2}}{A} = v(X^2 + \frac{Y^2Z^n}{A}); \quad \frac{X^{n-2}}{A} = v(\frac{X^2Z^n}{A} + Y^2)$$

Or

$$u\frac{Y^2}{A} = X^{n-2} - \frac{Y^{n-2}Z^n}{A}; \quad u\frac{X^2}{A} = -\frac{X^{n-2}Z^n}{A} + Y^{n-2}$$

$$v\frac{Y^{n-2}}{A} = X^2 + \frac{Y^2Z^n}{A}; \quad v\frac{X^{n-2}}{A} = \frac{X^2Z^n}{A} + Y^2$$

Or

$$\frac{Y^2}{A} = u(X^{n-2} - \frac{Y^{n-2}Z^n}{A}); \quad \frac{X^2}{A} = u(-\frac{X^{n-2}Z^n}{A} + Y^{n-2})$$

$$v\frac{Y^{n-2}}{A} = X^2 + \frac{Y^2Z^n}{A}; \quad v\frac{X^{n-2}}{A} = \frac{X^2Z^n}{A} + Y^2$$

Or

$$u\frac{Y^2}{A} = X^{n-2} - \frac{Y^{n-2}Z^n}{A}; \quad u\frac{X^2}{A} = -\frac{X^{n-2}Z^n}{A} + Y^{n-2}$$

$$\frac{Y^{n-2}}{A} = v(X^2 + \frac{Y^2Z^n}{A}); \quad \frac{X^{n-3}}{A} = v(\frac{X^2Z^n}{A} + Y^2)$$

First case

$$Y^n = uv(A^2X^n - Y^nZ^{2n} + AZ^n(Y^2X^{n-2} - Y^{n-2}X^2))$$

$$= uv(A^2X^n - Y^nZ^{2n} + A(AZ^n)) = uv(A^2X^n + A^2Z^n - Y^nZ^{2n}) = uv(A^2Y^n - Z^{2n}Y^n)$$

Thus

$$uv = \frac{1}{A^2 - Z^{2n}}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = Z^{2n}$
it means that $p = 2$.
Second case

$$uv\frac{Y^n}{A^2} = X^n - \frac{Y^nZ^{2n}}{A^2} + Z^n\frac{Y^2X^{n-2} - Y^{n-2}X^2}{A}$$

$$= X^n - \frac{Y^nZ^{2n}}{A^2} + Z^n = X^n + Z^n - \frac{Y^nZ^{2n}}{A^2} = (\frac{A^2 - Z^{2n}}{A^2})Y^n$$

Thus

$$uv = A^2 - Z^{2n}$$

And

$$uv(Y^2X^{n-2} - X^2Y^{n-2}) = uvA = u(X^{2n-4} - Y^{2n-4})A = v(-X^4 + Y^4)A$$

Thus

$$u = -X^4 + Y^4; \quad v = X^{2n-4} - Y^{2n-4}$$

$$uv = A^2 - Z^{2n} = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4})$$

But

$$Y^n = X^n + Z^n \Rightarrow Y^{n-2}X^2 = X^n\frac{X^2}{Y^2} + Z^n\frac{X^2}{Y^2}$$

And

$$Y^2X^{n-2} = X^n\frac{X^{n-2}}{Y^{n-2}} + Z^n\frac{X^{n-2}}{Y^{n-2}}$$

We deduce

$$A = Y^2X^{n-2} - X^2Y^{n-2} = (X^n + Z^n)(\frac{X^{n-2}}{Y^{n-2}} - \frac{X^2}{Y^2})$$

But

$$Y^n > X^n \Rightarrow A < 0 \Rightarrow \frac{X^{n-2}}{Y^{n-2}} < \frac{X^3}{Y^2}$$

And if

$$X^2 < Y^2 \Rightarrow X^{n-2} < Y^{n-2} \Rightarrow A^2 - 1 = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4} < 0$$

But

$$X^{n-2} < Y^{n-2} \Rightarrow X^{n-2}Y^2 < Y^n = X^n + Z^n \Rightarrow Y^2 < X^2 + Z^nX^{2-n}$$

And

$$0 < Y^2 - X^2 < Z^nX^{2-n}$$

and if $X^n > z^n$ it means $n = 2$. And if

$$X^{n-2} > Y^{n-2} \Rightarrow X^3 > Y^2 \Rightarrow A^2 - 1 = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4} > 0$$

But

$$X^{n-2} > Y^{n-2} \Rightarrow X^{n-2}Y^2 > Y^n = X^n + Z^n \Rightarrow Y^2 > X^2 + Z^nX^{2-n}$$

And

$$0 > Y^2 - X^2 > Z^nX^{2-n} > 0$$

And $n - 2 = 0$
Third case :
We have here

$$Y^2 = u(AX^{n-2} - Y^{n-2}Z^n); \quad X^2 = u(-X^{n-2}Z^n + AY^{n-2})$$
$$vY^{n-2} = AX^2 + Y^2Z^n; \quad vX^{n-2} = X^2Z^n + AY^2$$

And

$$vY^n = u(A^2X^n - Y^nZ^{2n} + A^2Z^n) = u(A^2 - Z^{2n})Y^n$$
$$v = u(A^2 - Z^{2n})$$
$$v(Y^2X^{n-2} - X^2Y^{p-2}) = vA = uvA(X^{2n-4} - Y^{2n-4}) = A(-X^4 + Y^4)$$

Thus

$$u(X^{2n-4} - Y^{2n-4}) = 1$$

With u and $A^2 - Z^{2n}$ integers, it means $A^2 = Z^{2n}$! Fourth case :

$$u\frac{Y^2}{A} = X^{n-2} - \frac{Y^{n-2}Z^n}{A}; \quad u\frac{X^2}{A} = -\frac{X^{n-2}Z^n}{A} + Y^{n-2}$$
$$\frac{Y^{n-2}}{A} = v(X^2 + \frac{Y^2Z^n}{A}); \quad \frac{X^{n-2}}{A} = v(\frac{X^2Z^n}{A} + Y^2)$$

Hence

$$u\frac{Y^n}{A^2} = v(X^n - \frac{Y^nZ2n}{A^2} + Z^n) = v(1 - \frac{Z^{2n}}{A^2})Y^n$$

Thus

$$u = v(A^2 - Z^{2n})$$
$$u(Y^2X^{n-2} - X^2Y^{n-2}) = uA = A(X^{2n-4} - Y^{2n-4}) = uv(-X^4 + Y^4)A$$
$$v(-X^4 + Y^4) = 1$$

With v and $A^2 - Z^{2n}$ integers, it means $A^2 - Z^{2n} = 0$!
The only solution, in all cases, in $n = 2$.

Resolution of Fermat- Catalan equation

Let Fermat-Catalan equation :

$$Y^p = X^q + aZ^c$$

$a = \pm 1$
We have

$$X^{q-w}Y^2 - Y^{p-2}X^w = aA$$

And

$$Y^{p-2}Y^2 - X^{q-w}X^w = Y^p - X^q = aZ^c$$

If $A = 0$ then $X^{q-2w} = Y^{p-4}$ leads, as $GCD(X,Y) = 1$, to $p = 4$ and $q = 2w$. Thus $A \neq 0$.
And if $A = \pm Z^c$ then it means that both
$X^{q-w-1}Y^2 = \pm\frac{aZ^c}{X} + X^{w-1}Y^{p-2}$ and
$Y^{p-3}X^w = \mp\frac{aZ^c}{Y} + X^{q-w}Y$ are rationals

it means, as i is imossble, that $q = w$ and $p = 2$.
We have

$$\frac{X^{q-w}Z^c}{A}Y^2 - \frac{Y^{p-2}Z^c}{A}X^w = aZ^c = Y^{p-2}Y^2 - X^{q-w}X^w$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-w}Z^c}{A})Y^2 = (X^{q-w} - \frac{Y^{p-2}Z^c}{A})X^w$$

Or

$$(AY^{p-2} - X^{q-w}Z^c)Y^2 = (AX^{q-w} - Y^{p-2}Z^c)X^w$$

And

$$(Y^2 + \frac{X^wZ^c}{A})Y^{p-2} = (X^w + \frac{Y^2Z^c}{A})X^{q-w}$$

Or

$$(AY^2 + X^wZ^c)Y^{p-2} = (AX^w + Y^2Z^c)X^{q-w}$$

We have four cases with u and v integers

$$\frac{Y^2}{A} = u(X^{q-w} - \frac{Y^{p-2}Z^c}{A}) \quad \frac{X^w}{A} = u(-\frac{X^{q-w}Z^c}{A} + Y^{p-2})$$

$$\frac{Y^{p-2}}{A} = v(X^w + \frac{Y^2Z^c}{A}) \quad \frac{X^{q-w}}{A} = v(\frac{X^wZ^c}{A} + Y^2)$$

Or

$$\frac{uY^2}{A}) = X^{q-w} - \frac{Y^{p-2}Z^c}{A} \quad \frac{uX^w}{A}) = -\frac{X^{q-w}Z^c}{A} + Y^{p-2}$$

$$\frac{vY^{p-2}}{A}) = X^w + \frac{Y^2Z^c}{A} \quad \frac{vX^{q-w}}{A} = \frac{X^wZ^c}{A} + Y^2$$

Or

$$\frac{Y^2}{A} = u(X^{q-w} - \frac{Y^{p-2}Z^c}{A}) \quad \frac{X^w}{A} = u(-\frac{X^{q-w}Z^c}{A} + Y^{p-2})$$

$$v\frac{Y^{p-2}}{A} = X^w + \frac{Y^2Z^c}{A} \quad v\frac{X^{q-w}}{A} = \frac{X^wZ^c}{A} + Y^2$$

Or

$$u\frac{Y^2}{A} = X^{q-w} - \frac{Y^{p-2}Z^c}{A} \quad u\frac{X^w}{A} = -\frac{X^{q-w}Z^c}{A} + Y^{p-2}$$

$$\frac{Y^{p-2}}{A} = v(X^w + \frac{Y^2Z^c}{A}) \quad \frac{X^{q-w}}{A} = v(\frac{X^wZ^c}{A} + Y^2)$$

First case

$$Y^p = uv(A^2X^q - Y^pZ^{2c} + AZ^c(Y^2X^{q-w} - Y^{p-2}X^w))$$

$$= uv(A^2X^q - Y^pZ^{2c} + aA(A)Z^c) = uv(A^2X^q + A^2Z^c - Y^pZ^{2c}) = uv(A^2Y^p - Y^pZ^{2c})$$

Thus

$$uv = \frac{1}{A^2 - Z^{2c}}$$

As uv is integer, it means that it is impossible thus $u=0$ and $A^2=Z^{2c}$ or $A=\pm{-}Z^c$ it means that $q=w$ and $p=2$.
Second case

$$uv\frac{Y^p}{A^2}=X^q-\frac{Y^pZ^{2c}}{A^2}+Z^c\frac{Y^2X^{q-w}-Y^{p-2}X^w}{A}$$

$$=X^q-\frac{Y^pZ^{2c}}{A^2}+aZ^c=X^q+aZ^c-\frac{Y^pZ^{2c}}{A^2}=(\frac{A^2-Z^{2c}}{A^2})Y^p$$

Thus

$$uv=A^2-Z^{2c}$$

And (with $a>0$, the proof is similar for $a<0$)

$$uv(Y^2X^{q-w}-X^wY^{p-2})=uvaA=u(X^{2q-2w}-Y^{2p-4})A=v(-X^{2w}+Y^4)A$$

Thus

$$ua=-X^{2w}+Y^4;\quad va=X^{2q-2w}-Y^{2p-4}$$

$$uv=A^2-Z^{2c}=(-X^{2w}+Y^4)(X^{2q-2w}-Y^{2p-4})$$

$$Y^b=X^a+Z^c\Rightarrow Y^{b-2}X^w=X^a\frac{X^w}{Y^2}+Z^n\frac{X^w}{Y^2}$$

And

$$Y^2X^{a-w}=X^a\frac{X^{a-w}}{Y^{b-2}}+Z^c\frac{X^{a-w}}{Y^{b-2}}$$

We deduce

$$A=Y^2X^{a-w}-X^wY^{b-2}=(X^a+Z^c)(\frac{X^{a-w}}{Y^{b-2}}-\frac{X^w}{Y^2})$$

But

$$Y^b>X^a\Rightarrow A<0\Rightarrow\frac{X^{a-w}}{Y^{b-2}}<\frac{X^w}{Y^2}$$

And if

$$X^w<Y^2\Rightarrow X^{a-w}<Y^{b-2}\Rightarrow A^2-1=(-X^{2w}+Y^4)(X^{2a-2w}-Y^{2b-4}<0$$

But

$$X^{a-w}<Y^{b-2}\Rightarrow X^{a-w}Y^2<Y^n=X^a+Z^c\Rightarrow Y^2<X^w+Z^cX^{w-a}$$

And

$$0<Y^2-X^w<Z^cX^{w-a}$$

and if $X^a>Z^c$ it means $b=2$. And if

$$X^{a-w}>Y^{b-2}\Rightarrow X^w>Y^2\Rightarrow A^2-1=(-X^{2w}+Y^4)(X^{2a-2w}-Y^{2b-4}>0$$

But

$$X^{a-w}>Y^{b-2}\Rightarrow X^{a-w}Y^2>Y^b=X^a+Z^c\Rightarrow Y^2>X^w+Z^cX^{w-a}$$

And

$$0>Y^2-X^w>Z^cX^{w-a}>0$$

And $b-2=0$
Third case :
We have here

$$Y^2=u(AX^{q-w}-Y^{p-2}Z^c);\quad X^w=u(-X^{q-w}Z^c+AY^{p-2})$$

$$vY^{p-2}=AX^w+Y^2Z^c;\quad vX^{q-w}=X^wZ^c+AY^2$$

And

$$vY^p=u(A^2X^q-Y^pZ^{2c}+aA^2Z^c)=u(A^2-Z^{2c})Y^p$$

$$v=u(A^2-Z^{2c})$$

$$v(Y^2X^{q-w}-X^wY^{p-2})=vaA=uvA(X^{2q-2w}-Y^{2p-4})=A(-X^{2w}+Y^4)$$

Thus $au(X^{2q-2w}-Y^{2p-4})=1$: it is possible only if $p=2$ and $q=w$. Fourth case :

$$u\frac{Y^2}{A}=X^{q-w}-\frac{Y^{p-2}Z^c}{A};\quad u\frac{X^w}{A}=-\frac{X^{q-w}Z^c}{A}+Y^{p-2}$$

$$\frac{Y^{p-2}}{A}=v(X^w+\frac{Y^2Z^c}{A});\quad \frac{X^{q-w}}{A}=v(\frac{X^wZ^c}{A}+Y^2)$$

We have here

$$uY^2=AX^{q-w}-Y^{p-2}Z^c;\quad uX^w-AY^{p-2}=-X^{q-w}Z^c$$

And

$$Y^{p-2}=AX^{q-w}-uY^2Z^c=(Y^2X^{q-w}-X^wY^{p-2})X^{q-w}-uY^2Z^c$$

Hence

$$u\frac{Y^p}{A^2}=v(X^q-\frac{Y^pZ^{2c}}{A^2}+aZ^c)=v(1-\frac{Z^{2c}}{A^2})Y^p$$

Thus

$$u=v(A^2-Z^{2c})$$

$$u(Y^2X^{q-w}-X^wY^{p-2})=uaA=A(X^{2q-2w}-Y^{2p-4})=uv(-X^{2w}+Y^4)A$$

Thus $-av(X^{2w}-Y^4)=1$: it is possible only if $u=0$ and $p-2=q-w=0$
The only solution, in all cases, is $p=2$.
And $Y^2=X^q+aZ^c$.

Conclusion

Fermat-Catalan equation $Y^p=X^q\pm Z^c$ has solutions only for $p=2$. We have shown a way to solve it.

References

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