Hidden properties of quantum mechanics.

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Two unsolved problems in quantum mechanics are addressed: a source of randomness and an origin of entanglement. These problems being hidden in the Schrödinger equations became transparent in its Madelung version. Special attention is concentrated on equivalence between the Schrodinger and the Madelung equations.
It has been demonstrated that randomness in quantum mechanics has the same mathematical source as that in turbulence and chaos, and the origin of entanglement is the global constraint imposed by the normalization constraint of the probability density that becomes an additional variable in the Madelung version of the Schrodinger equation.

1.Introduction.

Quantum mechanics has introduced randomness into the basic description of physics via the uncertainty principle. In the Schrödinger equation, randomness is included in the wave function. But the Schrödinger equation does not *simulate* randomness: it rather describes its evolution from the prescribed initial (random) value, and this evolution is fully deterministic. The first purpose of this work is to trace down a mathematical origin of randomness in quantum mechanics, i.e. to find or build a "bridge" between the deterministic and random states. In order to do that, we will turn to the Madelung equation, [1]. For a particle mass *m* in a potential *F,* the Madelung equation takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\frac{\rho}{m} \nabla S) = 0 \qquad (1)$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + F - \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m\sqrt{\rho}} = 0 \qquad (2)$$

Here ρ and S are the components of the wave function $\psi = \sqrt{\rho} e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π. The last term in Eq. (1) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (2) is the Liouville equation that expresses continuity of the flow of probability density, and Eq. (1) is the Hamilton-Jacobi equation for the action S of the particle. Actually the quantum potential in Eq. (1), as a feedback from Eq. (2) to Eq. (1), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

Before starting the analysis of the Madelung equation, we have to notice that a physical equivalence of the Schrödinger and the Madelung equations is still under discussion. However, we will not make any comments on this discussion since our target is the mathematical equivalence between these two forms of the quantum formalism.

It should be emphasized again that the Madelung equations (1), and (2), as well as the corresponding Schrödinger equation, does ***not simulate*** randomness, but rather describe its evolution in terms of the probability density, and that description is fully deterministic.

Now let us divert our attention from the physical interpretation of these equations and consider a formal mathematical problem of solving differential equations (1) and (2) subject to some initial and boundary conditions. In order not to be bounded by the quantum scale, we will assume that $\hbar$ is not necessarily the Planck constant and it can be any positive number of a classical scale having the dimensionality of action. A particular question we will ask is the following: what happen if we simulate Eqs. (1), and (2) using, for instance, electrical circuits or optical devices, and how will deterministic initial conditions will generate randomness that is supposed to be present in the solutions?

2. Search for transition from determinism to randomness.

Turning to Eq. (2), we will start with some simplification assuming that $F = 0$. Rewriting Eq. (2) for the one-dimensional motion of a particle, and differentiating it with respect to x, one obtains

$$m\frac{\partial^2 x(X,t)}{\partial t^2} - \frac{\hbar^2}{2m}\frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}\right] = 0 \qquad (3)$$

where $\rho(X)$ is the probability distribution of x over its possible values X.

Without the last term, Eq. (3) would represent the second Newton's law applied to the inertial motions of infinite number of *independent* samples of a particle forming a continuum $x(X)$. The last term in Eq. (3), that is a feedback from the Liouville equation, introduces an additional "force" that depends upon the probability distribution of x over X, and thereby, it couples motions of all possible samples $x(X)$. (It should be noticed that from the viewpoint of usual interpretation of quantum mechanics, Eq. (3) is meaningless since it describes the particle trajectories that "cannot be detected").

Let us choose the following initial conditions for the deterministic state of the system:

$$x = 0,\ \rho = \delta(|x| \to 0), \qquad \dot{\rho} = 0 \quad at \qquad t = 0 \qquad (4)$$

We intentionally did not specify the initial velocity $\dot{x}$ expecting that the solution will comply with the uncertainty principle.

Now let us rewrite the one-dimensional version of Eqs. (1) and (2) as

$$\frac{\partial^2\rho}{\partial t^2} + \frac{\hbar^2}{2m^2}\frac{\partial^4\rho}{\partial X^4} + \zeta = 0 \qquad at \qquad t \to 0 \qquad (5)$$

where ξ includes only lower order derivatives of ρ. For the first approximation, we ignore ξ (later that will be justified,) and solve the equation

$$\frac{\partial^2\rho}{\partial t^2} + a^2\frac{\partial^4\rho}{\partial X^4} = 0 \quad at \qquad t \to 0 \qquad\qquad a^2 = \frac{\hbar^2 T^2}{2m^2 L^4} \qquad (6)$$

subject to the initial conditions (4). The closed form solution to this problem is known from the theory of nonlinear waves, [2]

$$\rho = \frac{1}{\sqrt{4\pi t\dfrac{\hbar}{2m}}}\cos\left(\frac{x^2}{4t\dfrac{\hbar}{2m}} - \frac{\pi}{4}\right) \qquad at \qquad t \to 0 \qquad (7)$$

Based upon this solution, one can verify that $\xi \to 0$ at $t \to 0$, and that justifies the approximation (6) (for the proofs see the sub-section 2*). It is important to remember that the solution (7) is valid only for small times, and only during this period it is supposed to be positive and normalized.

Rewriting Eq. (3) in dimensionless form

$$\ddot{x} - a^2 \frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}} \frac{\partial^2 \sqrt{\rho(X)}}{\partial X^2}\right] = 0 \qquad (8)$$

and substituting Eq. (7) into Eq. (8)) at ***X* = *x***, after Taylor series expansion, simple differentiations and appropriate approximations, one arrives at the following differential equation instead of (8).

$$\ddot{x} = c\frac{x}{t^2}, \qquad c = -\frac{3}{8\pi^2 a^2} \qquad (9)$$

This is the Euler equation, and it has the following solution, [3]

$$x = C_1 t^{\frac{1}{2}+s} + C_2 t^{\frac{1}{2}-s} \qquad at \qquad 4c+1>0 \qquad (10)$$

$$x = C_1\sqrt{t} + C_2\sqrt{t}\ln t \qquad at \qquad 4c+1=0 \qquad (11)$$

$$x = C_1\sqrt{t}\cos(s\ln t) + C_2\sqrt{t}\sin(s\ln t) \qquad at \qquad 4c+1<0 \qquad (12)$$

where $2s = \sqrt{|4c+1|}$ (13)

Thus, the qualitative structure of the solution is uniquely defined by the dimensionless constant a^2 via the constants *c* and *s*, (see Eqs. (9) and (13). But the cases (11) and (12) should be disqualified at once since they are in a conflict with the approximations used for derivation of Eq. (9), (see sub-section 2*).

Hence, we have to stay with the case (10). This gives us the limits

$$0 < |c| < 0.25, \qquad (14)$$

In addition to that, we have to drop the second summand in Eq. (10) since it is in a conflict with the approximation used for derivation of Eq. (6) (see sub-section 2*). Therefore, instead of Eq. (10)) we now have

$$x = C_1 t^{\frac{1}{2}+s} \qquad at \qquad 4c+1>0 \qquad (15)$$

For illustration, let us evaluate the constant *c* based upon the following data:

$$\hbar = 10^{-34} m^2 kg/\sec, m = 10^{-30} kg, \ L = 2.8\times 10^{-15} m, \quad L/T = c = 3\times 10 m/\sec$$

where *m*- mass of electron, and *c*-speed of light. Then,

$$c = -1.5\times 10^{-4}, \qquad i.e. \quad |c| < 0.25$$

Hence, the value of *c* is within the limit (14). Thus, for the particular case under consideration, the solution (15) is

$$x = C_1 t^{0.9998} \qquad (16)$$

In the next sub-section, prior to analysis of the solution (15), we will present the proofs justifying the solution (7).

2*. Proofs.

1. Let us first justify the statement that $\xi \to 0 \; at \quad t \to 0$ (see Eq. (5).

For that purpose, consider the solution (7)

$$\rho = \frac{1}{\sqrt{4\pi at}}\cos(\frac{X^2}{4at} - \frac{\pi}{4}) \quad at \quad t \to 0 \tag{1*}$$

As follows from the solution (15),

$$\frac{x}{t} \approx o(t^{s-1/2}) \to \infty, \quad \frac{x^2}{t} \approx o(t^{2s}) \to 0 \quad at \quad t \to 0 \text{ since } 0 < s < 1/2 \tag{2*}$$

Then, finding the derivatives from Eq. (1') yields

$$|\frac{\partial^n \rho}{\partial X^n}| / |\frac{\partial^{n-1}\rho}{\partial X^{n-1}}| \approx o(t^{-1}) \to \infty \quad at \quad t \to 0 \tag{3*}$$

and that justifies the inequalities

$$|\frac{\partial^4 \rho}{\partial X^4}| >> |\frac{\partial^3 \rho}{\partial X^3}|, |\frac{\partial^2 \rho}{\partial X^2}|, |\frac{\partial \rho}{\partial X}|, \rho \tag{4*}$$

Similarly,

$$|\frac{\partial^n \rho}{\partial t^n}| / |\frac{\partial^{n-1}\rho}{\partial t^{n-1}}| \approx o(t^{-1}) \to \infty \quad at \quad t \to 0 \tag{5*}$$

and that justifies the inequalities

$$|\frac{\partial^2 \rho}{\partial t^2}| >> |\frac{\partial \rho}{\partial t}|, |\frac{\partial \rho}{\partial X}|^2$$

Also as follows from the solution (15)

$$\begin{aligned} &|\frac{\partial S}{\partial x}| \approx o(t^{S-0.5}), \qquad |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{-1}), \\ &|\frac{\partial S}{\partial x}| / |\frac{\partial^2 S}{\partial x^2}| \approx o(t^{S+0.5}) \to 0 \quad at \quad t \to 0 \end{aligned} \tag{6*}$$

It should be noticed that for Eq. (10), the evaluations (6*) do not go through, and that was the reason for dropping the second summand.

Finally, the inequalities (4*), (5*) and (6*) justify the transition from Eq. (5) to Eq. (7).

2. Next let us first prove the positivity of ρ in Eq. (7) for small times. Turning to the evaluation (2*)

$\frac{x^2}{t} \approx o(t^{2s}) \to 0 \quad at \quad t \to 0$, one obtains for small times

$$\rho = \frac{1}{\sqrt{4\pi a t}} \cos(-\frac{\pi}{4}) > 0 \quad at \quad t \to 0 \tag{7*}$$

In order to prove that ρ is normalized for small times, turn to Eq. (6) and integrate it over X

$$\int_{-\infty}^{\infty} \frac{\partial^2 \rho}{\partial t^2} dX + a^2 \int_{-\infty}^{\infty} \frac{\partial^4 \rho}{\partial X^4} dX = 0 \tag{8*}$$

Taking into account the initial conditions (4) and requiring that ρ and all its space derivatives vanish at infinity, one obtains

$$\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \rho \, dX = 0 \tag{9*}$$

But as follows from the initial conditions (4)

$$\int_{-\infty}^{\infty} \rho \, dX = 0, \quad \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho \, dX = 0 \quad at \quad t = 0 \tag{10*}$$

Combining Eqs. (9*) and (10*), one concludes that the normalization constraint is preserved during small times.

3. The solutions (10), (11) and (12) have been derived under assumption that

$$\frac{x^2}{t} \to 0 \quad at \quad t \to 0 \tag{17}$$

since this assumption was exploited for expansion of ρ in Eq. (7) in Taylor series. However, in the cases (11) and (12),

$$\frac{x^2}{t} \approx o(1) \quad at \quad t \to 0,$$

and that disqualify their derivation. Actually these cases require an additional analysis that is out of scope of this paper. For the same reason, Eq. ((10) has been truncated to the form (15).

3. Analysis of solution.

Turning to the solution (15), we notice that it satisfies the initial condition (4) i.e. *x=0* at *t=0* for *any* values of C_1: all these solutions co-exist in a superimposed fashion; it is also consistent with the sharp initial condition for the solution (7) of the corresponding Liouvile equation (1). The solution (7) describes the simplest *irreversible* motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of Lipchitz condition at *t=0,* Fig.2); then the solution splits into a continuous set of random samples representing a stochastic process with the probability density ρ controlled by Eq. (7). The irreversibility of the process follows from the fact that the backward motion obtained by replacement of *t* with *(-t)* in Eqs. (7) and (15) leads to imaginary values. Actually Fig. 1 illustrates a jump from determinism to a coherent state of superimposed solutions that is *lost in solutions of the Schrödinger equation.*

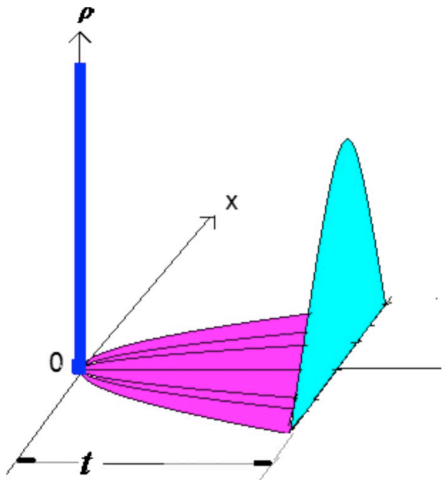

Figure 1. Hidden statistics of transition from determinism to randomness.

Let us show that this jump is triggered by instability of the deterministic state. Indeed, turning to the solution represented by Eq. (15) with $|C_1| \le 0.25$, we observe that for fixed values of C_1, the solution (15) is ***unstable*** since

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} \qquad > 0 \tag{18}$$

and therefore, an initial error always grows generating ***randomness***. Initially, at *t=0*, that growth is of *infinite rate* since the Lipchitz condition at this point is violated (such a point represents a ***terminal*** repeller,)

$$\frac{d\dot{x}}{dx} \to \infty \qquad at \qquad t \to 0 \tag{2.17}$$

This means that an *infinitesimal* initial error becomes finite in a bounded time interval. That kind of instability (similar to blow-up, or Hadamard, instability) has been analyzed in [4]. Considering first Eq. 15) at fixed C_1 as a sample of the underlying stochastic process (7), and then varying C_1, one arrives at the whole ensemble of one-parametrical random solutions characterizing that process, (see Fig.2). It should be stressed again that this solution is valid only during a small initial period representing a "bridge" between deterministic and random states, and that was essential for the derivation of the solutions (15), and (7).

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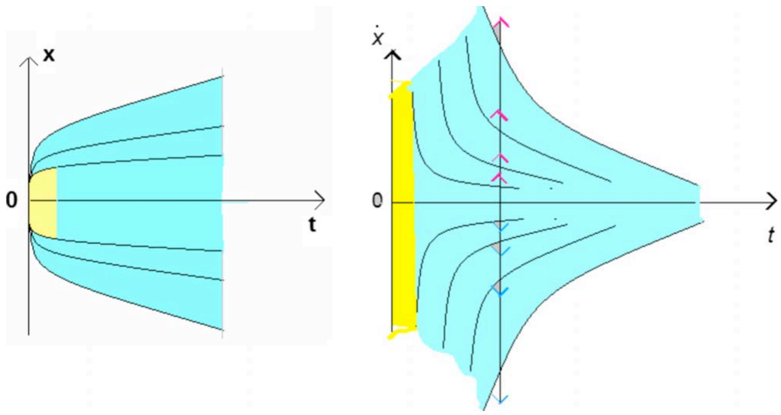

Figure 2. Family of random trajectories and particle velocities.

Returning to the quantum interpretation of Eqs. (1) and (2), one notice that during this transitional period, the quantum postulates are preserved. Indeed, as follows from Eq. (15),

$$\dot{x} \to \infty \quad at \quad t \to 0 \tag{20}$$

i.e. the initial velocity is *not defined*, (see the yellow areas in Fig. 2), and that confirms the uncertainty principle. It is interesting to note that an enforcement of the initial velocity would "blow-up" the solution (15); at the same time, the qualitative picture of the solution is not changed if the initial velocity is not enforced: the solution is composed of superposition of a family of random trajectories with the singularity (20) at the origin. Next, the solution (15) justifies the *belief* sheared by the most physicists that particle trajectories do not exist, although, to be more precise, as follows from Eq. (15), *deterministic* trajectories do not exist: each run of the solution (15) produces different trajectory that occurs with probability governed by Eq. (7). It is easily verifiable that the transition of motion from one trajectory to another is very sensitive to errors in initial conditions in the neighborhood of the deterministic state. Indeed, as follows from Eq. (15),

$$C_1 = x_0 t_0^{-(s+0.5)}, \qquad \frac{\partial C_1}{\partial x_0} = t_0^{-(s+0.5)} \to \infty \quad as \qquad t_0 \to 0 \tag{21}$$

where x_0 and t_0 are small errors in initial conditions.

Actually Eq. (17) represents a hidden statistics of the underlying Schredinger equation. As pointed out above, the cause of the randomness is non-Lipchitz instability of Eq. (17) at *t=0*. Therefore, trajectories of quantum particles have the same "status" as trajectories of classical particles in a turbulent or chaotic motion with the only difference that the "choice" of the trajectory is made only at $t_0 \to 0$. It should be emphasized again that the transition (17) is *irreversible*. However, as soon as the difference between the current probability density and its initial sharp value becomes finite, one arrives at the conventional quantum formalism described by the Schrödinger, as well as the Madelung equations. Thus, in the conventional quantum formalism, the *transition* from the classical to the quantum state has been *lost*, and that created a major obstacle to interpretation of quantum mechanics as an extension of the Newtonian mechanics. However, as demonstrated above, the quantum and classical worlds can be reconciliated via the more subtle mathematical treatment of the *same equations*. This result is generalizable to multi-dimensional case as well as to case with external potentials.

2.5. Comments on equivqlence of Schrödinger and Madelung equations.

Equivalence of Schrödinger and Madelung equations was questioned by some quantum physicists on the ground that to recover the Schrödinger equation from the Madelung equation, one must add by hand a quantization condition, as in the old quantum theory. However, this argument has been challenged by other physicists. We will not go into details of this discussion since we will be more interested in *mathematical* rather than physical equivalence of Schrödinger and Madelung equations. Firstly we have to notice that the Schrödinger equation is more attractive for computations due to its linearity, while the Madelung equations have a methodological advantage: they allow one to trace down the Newtonian origin of the quantum physics. Indeed, if one drops the Planck's constant, the Madelung equations degenerate into the Hamilton-Jacobi equation supplemented by the Liouville equation. However despite the fact that these two forms of the same governing equations of quantum physics can be obtained from one another (in an open interval t > 0) without a violation of any of mathematical rules, there is more significant difference between them, and this difference is associated with the concept of stability. Indeed, as demonstrated in Sections 2 and 3 of this paper, the solution of the Madelung equations with deterministic initial condition (4) is unstable, and it describes the jump from the determinism to randomness. This illuminates the origin of randomness in quantum physics. However the Schrödinger equation does not have such a solution; moreover, it does not "allow" posing such a problem and that is why the randomness in quantum mechanics had to be postulated. So what happens with mathematical equivalence of Schrödinger and Madelung equations? In order to answer this question, let us turn to the concept of stability. It should be recalled that stability is not an invariant of a physical model. It is an attribute of its mathematical description: it depends upon the frame of reference, upon the class of functions in which the motion is presented, upon the metrics of configuration space, and in particular, upon the way in which the distance between the basic and perturbed solutions is defined. As an example, consider an inviscid stationary flow with a *smooth* velocity field, [5]

$$v_x = A\sin z + C\cos y,\ v_y = B\sin x + A\cos z,\ v_z = C\sin y + b\cos x \tag{22}$$

Surprisingly, the trajectories of individual particles of this flow are unstable (Lagrangian turbulence). It means that this flow is stable in the Eulerian representation, but unstable in the Lagrangian one. The same happens with stability in Hilbert space (Schrödinger equation), and stability in physical space (Madelung equations). One should recall that stability analysis is based upon a departure from the basic state into a perturbed state, and such departure requires an expansion of the basic space. However, Schrödinger and Madelung equations in the ***expanded*** spaces are not necessarily equivalent any more, and that explains the difference in the concept of stability of the same solution as well as the interpretation of randomness in quantum mechanics.

There is another "mystery" in quantum mechanics that can be clarified by transfer to the Madelung space: a ***belief*** that a particle trajectory does not exist. Indeed, let us turn to Eq. (15). For any particular value of the arbitrary constant C_1, it presents the corresponding particle's trajectory. However as a result of Lipchitz instability at $t = 0$, this constant is supersensitive to infinitesimal disturbances, and actually it becomes random at $t=0$. That makes random the choice of the whole trajectory, while the randomness is controlled by Eq. (1). Actually this provides a justification for the ***belief*** that a particle can occupy any place at any time: it is due to randomness of its trajectory. However it should be emphasized that the particle makes random choice only once: at $t = 0$. After that it stays on the chosen trajectory. Therefore in our interpretation this belief does not mean that a trajectory does not exist: it means only that the trajectory exists, but it is unstable. Based upon that, we can extract some deterministic information about the particle trajectory by posing the following question: find such a trajectory that has the highest probability to appear. The solution of this problem is straight forward: in the process of collecting statistics for the arbitrary constant C_1 find such its value that has the highest frequency to appear. Then the corresponding trajectory will have the highest probability to appear as well.

Thus, strictly speaking, the Schrödinger and Madelung equations are equivalent only in the open time interval

$$t > 0 \qquad (23),$$

since the Schrödinger equation does not include the infinitesimal area around the singularity at

$$t = 0 \qquad (24)$$

while the Madelung equation exists in the closed interval

$$t \geq 0 \qquad (25)$$

But all the "machinery" of randomness emerges precisely in the area around the singularity (24). That is why the source of randomness is missed in the Schrödinger equation, and the randomness had to be postulated.

Hence although historically the Schrödinger equation was proposed first, and only after a couple of months Madelung introduced its hydrodynamic version that bears his name, strictly speaking, the foundations of quantum mechanics would be saved of many paradoxes had it be based upon the Madelung equation.

4. Mystery of entanglement.

In this Section we will comment on a fundamental and still mysterious property that was predicted theoretically and corroborated experimentally in quantum systems: entanglement. Quantum entanglement is a phenomenon in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated. This leads to correlations between observable physical properties of the systems. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it. Different views of what is actually occurring in the process of quantum entanglement give rise to different interpretations of quantum mechanics.

We will start with general characteristic of entanglement in physics.

1. Criteria for non-local interactions. Based upon analysis of all the known interactions in the Universe and defining them as local, one can formulate the following criteria of *non*-local interactions: they are *not* mediated by another entity, such as a particle or field; their actions are *not* limited by the speed of light; the strength of the interactions does not drop off with distance. All of these criteria lead us to the concept of the global constraint as a starting point.

2. Global constraints in physics. It should be recalled that the concept of a global constraint is one of the main attribute of Newtonian mechanics. It includes such *idealizations* as a rigid body, an incompressible fluid, an inextensible string and a membrane, a non-slip rolling of a rigid ball over a rigid body, etc. All of those idealizations introduce geometrical or kinematical restrictions to positions or velocities of particles and provides "instantaneous" speed of propagation of disturbances. Let us discuss the role of the reactions of these constraints. One should recall that in an incompressible fluid, the reaction of the global constraint $\nabla \cdot v \geq 0$ (expressing non-negative divergence of the velocity *v*) is a non-negative pressure $p \geq 0$; in inextensible flexible (one- or two-dimensional) bodies, the reaction of the global constraint $g_{ij} \leq g^0_{ij}, i,j = 1,2$ (expressing that the components of the metric tensor cannot exceed their initial values) is a non-negative stress tensor $\sigma_{ij} \geq 0$, i,j=1,2. It should be noticed that all the known forces in physics (the gravitational, the electromagnetic, the strong and the weak nuclear forces) are local. However, the reactions of the global constraints listed above do not belong to any of these local forces, and therefore, they are non-local. Although these reactions are being successfully applied for engineering approximations of theoretical physics, one cannot relate them to the origin of ***entanglement*** since they are result of idealization that ignores the discrete nature of the matter.

However, there is another type of the global constraint in physics: the normalization constraint imposed upon the probability density

$$\int_{-\infty}^{\infty} \rho \, dV = 1 \tag{26}$$

This constraint is fundamentally different from those listed above for two reasons. Firstly, it is not an idealization, and therefore, it cannot be removed by taking into account more subtle properties of matter such as elasticity, compressibility, discrete structure, etc. Secondly, it imposes restrictions not upon positions or velocities of particles, but upon the *probabilities* of their positions or velocities, and that is where the ***entanglement*** comes from. Indeed, if the Liouville equation is coupled with equations of motion as in quantum mechanics, the normalization condition imposes a global constraint upon the state variables, and that is the origin of quantum entanglement. In quantum physics, the reactions of the normalization constraints can be associated with the energy eigenvalues that play the role of the Lagrange multipliers in the conditional extremum formulation of the Schrödinger equation, [6.].

3. Speed of action propagation. Further illumination of the concept of quantum entanglement follows from comparison of quantum and Newtonian systems. Such a comparison is convenient to perform in terms of the Madelung version of the Schrödinger equation, see Eqs. (1) and (1). As follows from these equations, the Newtonian mechanics ($\hbar = 0$), in terms of the S and ρ as state variables, is of a hyperbolic type, and therefore, any discontinuity propagates with the *finite* speed *S/m*, i.e. the Newtonian systems do not have non-localities. But the quantum mechanics ($\hbar \neq 0$) is of a parabolic type. This means that any disturbance of S or ρ in one point of space instantaneously transmitted to the whole space, and this is the mathematical origin of non-locality. But is this a unique property of quantum evolution? Obviously, it is not. Any parabolic equation (such as Navier-Stokes equations or Fokker-Planck equation) has exactly the same non-local properties. However, the difference between the quantum and classical non-localities is in their physical interpretation. Indeed, the Navier-Stokes equations are derived from simple laws of Newtonian mechanics, and that is why a physical interpretation of non-locality is very simple: If a fluid is incompressible, then the pressure plays the role of a reaction to the geometrical constraint $\nabla \cdot v \geq 0$, and it is transmitted instantaneously from one point to the whole space (the Pascal law). One can argue that the incompressible fluid is an idealization, and that is true. However, it does not change our point: Such a model has a lot of engineering applications, and its non-locality is well understood. The situation is different in quantum mechanics since the Schrodinger equation has never been derived from Newtonian mechanics: It has been postulated. In addition to that, the solutions of the Schrodinger equation are random, while the ***origin*** of the randomness does not follow from the Schrodinger formalism. That is why the physical origin of the same mathematical phenomenon cannot be reduced to simpler concepts such as "forces": It should be accepted as an attribute of the Schrodinger equation.

*4. Origin of randomness in physics***.** Since entanglement in quantum systems as well as in L-particle models are exposed via instantaneous propagation of changes in the probability density, it is relevant to ask what is the origin of randomness in physics. The concept of randomness has a long history. Its philosophical aspects first were raised by Aristotle, while the mathematical foundations were introduced and discussed much later by Henry Poincare who wrote: “A very slight cause, which escapes us, determines a considerable effect which we cannot help seeing, and then we say this effect is due to chance”. Actually Poincare suggested that the origin of randomness in physics is the dynamical instability, and this viewpoint has been corroborated by theory of turbulence and chaos. However, the theory of dynamical stability developed by Poincare and Lyapunov revealed the main flaw of physics: its fundamental laws do not discriminate between stable and unstable motions. But unstable motions cannot be realized and observed, and therefore, a special mathematical analysis must be added to find out the existence and observability of the motion under consideration. However, then another question can be raised: why turbulence as a post-instability version of an underlying laminar flow can be observed and measured? In order to answer this question, we have to notice that the concept of stability is an attribute of mathematics rather than physics, and in mathematical formalism, stability must be referred to the corresponding class of functions. For example: a laminar motion with sub-critical Reynolds number is *stable* in the class of deterministic functions. Similarly, a turbulent motion is *stable* in the class of random functions. Thus the same physical phenomenon can be unstable in one class of functions, but stable in another, enlarged class of functions.

Thus, we are ready now to the following conclusion: any stochastic process in *Newtonian* dynamics describes the physical phenomenon that is unstable in the class of the deterministic functions.
This elegant union of physics and mathematics has been disturbed by the discovery of quantum mechanics that complicated the situation: Quantum physicists claim that quantum randomness is the "true" randomness unlike the "deterministic" randomness of chaos and turbulence. Richard Feynman in his "Lectures on Physics" stated that randomness in quantum mechanics in *postulated*, and that closes any discussions about its origin. However, recent result disproved existence of the "true" randomness. Indeed, as shown in Chapter 2, the origin of randomness in quantum mechanics can be traced down to *instability* generated by quantum potential at the point of departure from a deterministic state if for dynamical analysis one transfer from the Schrödinger to the Madelung equation. As demonstrated there, the instability triggered by failure of the Lipchitz condition splits the solution into a continuous set of random samples representing a "bridge" to quantum world. Hence, now we can state that any stochastic process in physics describes the physical phenomenon that is unstable in the class of the deterministic functions. Actually this statement can be used as a definition of randomness in physics.

Summary.

Reformulation of quantum mechanics using the Madelung equation allows one to clarify the origin of randomness and justify the belief that a particle can occupy any position at any time. The clarifications are based upon the blow-up instability of a deterministic state due to failure of Lipchitz condition. This property does not exist in Hilbert space formulation.
The paper clarifies the physics origin of entanglement by associating it with the global constraint imposed upon the state variables by the normalization condition to be satisfied by the probability density.

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