Studies On Vortexes

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Abstract
We check that the relation between the angle and the radius of the movement of an object following a logarithm spiral of $\sqrt[\mathbb{r}]{SO(2)}$ is constant.

Introduction

The Arm Lie groups theory (see [1]) gives new groups such as $\sqrt[p]{SO(2)}$. Next we know that $SO(2)$ is the group of rotation matrices and we dicover that $\sqrt[q]{SO(2)}$ is the group of spiral movement matrices.

We draw the curve of a point following this spirale movement : there is what we call a logarithm spirale. But this is not any of logarithm spirales, there is a special relation (1.8) between the angle and the radius of the movement. A point which follows this movements is described by a matrice (we call it the hurricane matrix) multiplied by the initial condition of the point. The parameter of this matrix is relied with the time of the experience (1.12). Having an expression of the time, we can calculate the 'speed' of the point and we finally find that this speed obey to the equation of vortex (1.17) in fluid mecanic.

Next I wanted to verify that those movements are 'used' by nature to show that they are not only mathematical objects. It is because why I wondered what are the movements in the nature which follow spirale movements. Furthermore, there is a lot of movements which follow this type of curve but I choose to study vertexes. So I took a recipient with a hole in the bottom, filled it with water and filmed the movement of small plastics balls turning down in the vortex created. On my computer, I noted the corresponding angle of each radius to check the relation (1.8). The results is that this relation is clearly constant for each movement. In fact, I think we can say that this relation is the same for same initial conditions.

To conclude with, I also check the relation on a picture of the famous vertex which is the hurricane Katrina. I checked that the relation (1.8) between the angle and the radius of its main cloud is constant.

In a first part, we calculate all the parameters of the movement and we express the relation (1.8) between the angle and the radius of the movement. In a second part, we check this relation on a small ball turning down in a vortex of water. Next, we check it in a cloud of the hurricane Katrina.

1 The Hurricane Matrix

We take the 'hurricane' matrix $h(s)$(c.f. [1]) :

$$\begin{aligned}
\exp\left(-s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\frac{1}{p}}\right) &= \exp\left(-s \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix}\right) \\
h(s) &= \exp\left(-s\ \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(-s\sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(-s\sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(-s\sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(-s\sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix}
\end{aligned}$$

which nothing else that the exponential of a root of the generating element of $\mathfrak{so}(2)$. We can see that if

$$X(s) = h(s)X(0) \tag{1.1}$$

with the initial condition

$$X(0) = \begin{pmatrix} r_0 \\ 0 \end{pmatrix} \tag{1.2}$$

then

$$X(s) = r_0 \exp\left(-s\ \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(-s\sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(-s\sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \tag{1.3}$$

is a point define by its radius

$$r(s) = r_0 \exp\left(-s\ \cos\left(\frac{\pi}{2p}\right)\right) \tag{1.4}$$

and its angle

$$\alpha(t) = -s\sin\left(\frac{\pi}{2p}\right) \tag{1.5}$$

Then we can see the relation between its angle and its radius which is

$$s^2 = \ln^2\left(\frac{r(t)}{r_0}\right) + \alpha^2(t) \tag{1.6}$$

or

$$s = \sqrt{\alpha^2(t) + \ln^2\left(\frac{r(t)}{r_0}\right)} \tag{1.7}$$

Here we call the parameter (1.7) t but we can remember that this is not the time.

From (1.5) and (1.4), we obtain

$$\frac{\alpha(s)}{\ln\left(\frac{r}{r_0}\right)} = \tan\left(\frac{\pi}{2p}\right) \tag{1.8}$$

or

$$\alpha(t) = \tan\left(\frac{\pi}{2p}\right)\ln\left(\frac{r}{r_0}\right) \tag{1.9}$$

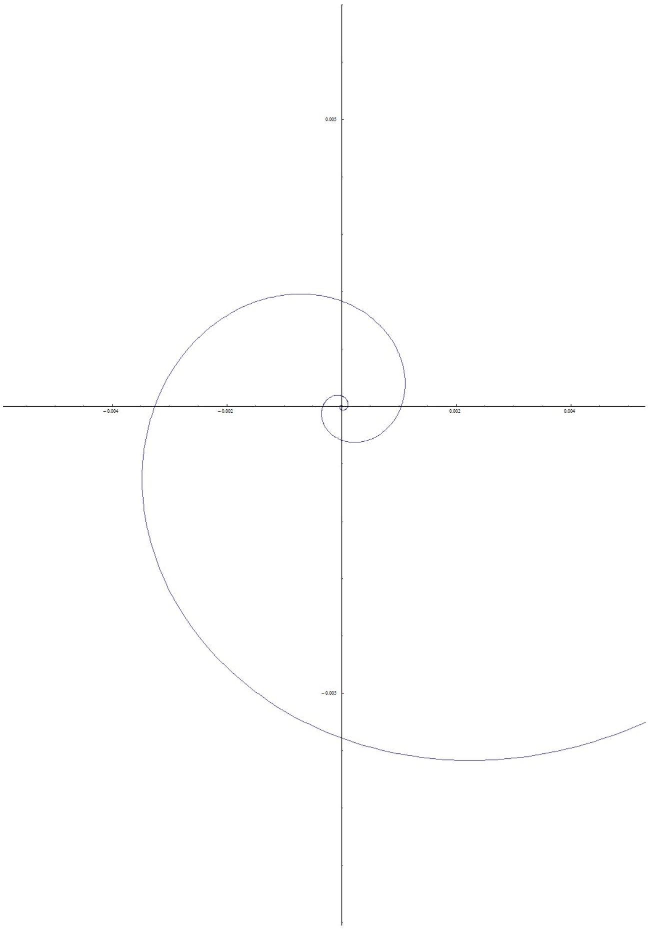

FIGURE 1 – Example of $X(s)$ for $p = 1.28$

If we put

$$dt = \cos\left(\frac{\pi}{2p}\right)\exp\left(-s\cos\left(\frac{\pi}{2p}\right)\right)ds \tag{1.10}$$

because

$$dt = ||X'(s)||ds \tag{1.11}$$

Then we can differentiate (1.12)

$$\frac{\partial t}{\partial s} = \cos\left(\frac{\pi}{2p}\right)\exp\left(-s\cos\left(\frac{\pi}{2p}\right)\right) \tag{1.12}$$

which gives

$$t = t_0 - \exp\left(-s\cos\left(\frac{\pi}{2p}\right)\right) \tag{1.13}$$

Using (1.14) and (1.3), we obtain

$$V(s) = \frac{ds}{dt}\frac{dX}{ds} = \frac{-r_0}{\cos(\frac{\pi}{2p})} \left(\begin{array}{c} \cos\left(\frac{\pi}{2p} - s\sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(\frac{\pi}{2p} - s\sin\left(\frac{\pi}{2p}\right)\right) \end{array} \right) \tag{1.14}$$

So we have the same type of equation than the vortex theory [2]

$$\frac{\partial V}{\partial t} = \frac{\alpha}{m}\,\Omega\, V \tag{1.15}$$

If we define $\Omega_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ as

$$\Omega = \tan\left(\frac{\pi}{2p}\right) \left(\begin{array}{cc} 0 & -\exp\left(s\cos\left(\frac{\pi}{2p}\right)\right) \\ \exp\left(s\cos\left(\frac{\pi}{2p}\right)\right) & 0 \end{array} \right) \tag{1.16}$$

and $\frac{\alpha}{m} = \frac{r_0}{\cos\left(\frac{\pi}{2p}\right)}$

2 Experimental verification

Here we try to see the validity of the relation (1.8) for vortexes which appears in natural conditions. I have test this relation on severals systems.

First we study this relation on a small colored ball turning in a vortex of water. We take a cylindric container, mine is about 5.25 cm radius and 10cm height. In this container, we perforate a hole in the bottom of the container of radius 4mm to create the vortex when we put water.
The hole in the bottom of the container have to be enough wide to create a good vortex. Next, we fill the container with water until the top of the container and we put a small colored plastic ball up far from the center of the vortex.
We also have to fill enough the container to ensure a Finally we film this ball turning up to the center of the vortex, I put the video on youtube please see :
`https://www.youtube.com/watch?v=VavnpNH9zjM`
I give in the following table the radius,the angle and the rate (1.8).

α (rad)	$\frac{r}{r_0}$	$\ln\left(\frac{r}{r_0}\right)$	$\frac{\alpha}{\ln\left(\frac{r}{r_0}\right)}$
1,57	1	0	
3,14	0,7230769231	-0,3242396682	-9,6841944651
4,71	0,5076923077	-0,6778797084	-6,9481354014
6,28	0,4153846154	-0,8785504039	-7,1481385384
7,85	0,2769230769	-1,284015512	-6,1136333063

We can see in this table that the rate $\frac{\alpha}{\ln\left(\frac{r}{r_0}\right)}$ seems to be constant.
So we can see that

$$\tan\left(\frac{\pi}{2p}\right) \simeq 6.68 \tag{2.17}$$

which gives

$$p \simeq 1.10 \tag{2.18}$$

We do the same experiment in filling again the container and leaving the colored ball on the top. We check this rate in this similar experiment which gives

α (rad)	$\frac{r}{r_0}$	$\ln\left(\frac{r}{r_0}\right)$	$\frac{\alpha}{\ln\left(\frac{r}{r_0}\right)}$
4,71	1	0	
6,28	0,6527777778	-0,4265185173	-14,7238624941
7,85	0,5138888889	-0,6657482064	-11,7912446851
9,42	0,4583333333	-0,7801585575	-12,0744685921
10,99	0,3888888889	-0,9444616088	-11,6362591101
12,56	0,3472222222	-1,0577902941	-11,8738090806

Again we can see that the last column seems to be constant. So we can see that

$$\tan\left(\frac{\pi}{2p}\right) \simeq 11.87 \tag{2.19}$$

which gives

$$p \simeq 1.05 \tag{2.20}$$

Now we verify the relation (1.8) on the picure of the vortex of a hurricane Katrina. This picture is taken from `http://www.livescience.com/15805-calm-hurricane-eye.html`

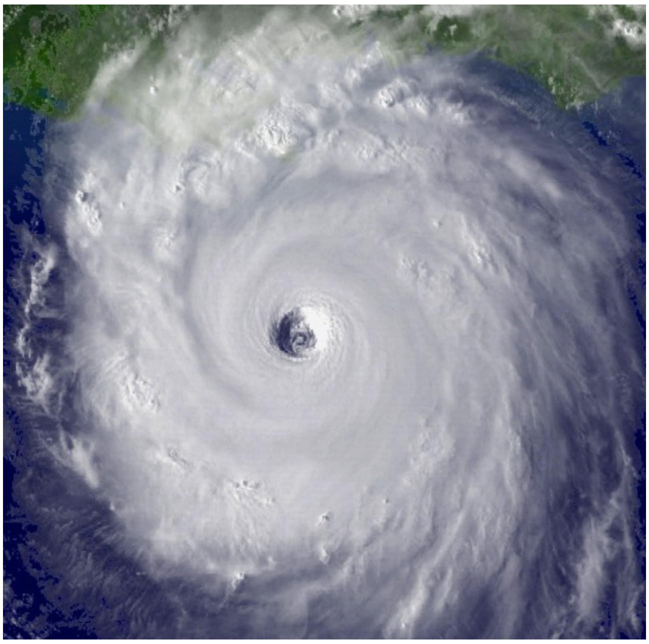

FIGURE 2 – Example of $X(t)$ for $p = 1.28$

I measured for four angle the corresponding radius and we obtain the following table

α (rad)	$\frac{r}{r_0}$	$\ln\left(\frac{r}{r_0}\right)$	$\frac{\alpha}{\ln\left(\frac{r}{r_0}\right)}$
1,57	1	0	
3,14	0,6363636364	-0,4519851237	-6,9471312994
4,71	0,4	-0,9162907319	-5,140289906
6,28	0,2727272727	-1,2992829841	-4,8334351151
7,85	0,1818181818	-1,7047480922	-4,6047859128

Again we can see that the last column seems to be constant. So we can see that

$$\tan\left(\frac{\pi}{2p}\right) \simeq 4.83 \tag{2.21}$$

which gives

$$p \simeq 1.15 \tag{2.22}$$

Références

[1] Arm B. N., The Arm Lie Group Theory

[2] Hadjesfandiari Ali R., Vortex theory of electromagnetism and its non-Euclidean character