Real prime definition via a proof to Riemann hypothesis

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Abstract

In this study, the Riemann problem is presented with highlights on history of the zeta function.

Thereafter, the real primes, which constitute a novelty, are defined.

It allows to generalize the Riemann hypothesis to the reals. A calculus of integral solves the problem.

The proof is generalized to the integers by an elementary development.

Keywords

Primes ; Reals ; Riemann hypothesis

Introduction

The Riemann conjecture is a conjecture which has been formulated in 1859 by Bernard Riemann in the subject of the Riemann function zeta or ζ. It is called the zeta Riemann function. This conjecture is one the seven problems of the millennium.

The first result is the divergence of the harmonic serie. The second consist of the results of Leonard Euler. Finally, Bernard Riemann has presented the conjecture.

The problem is reconsidered here as the real primes are defined for the first time. This definition will allow to generalize the hypothesis to the reals for which the solution is easier. Thereafter, the conjecture will be solved after an elementary development.

Materials or methods

The main novelty of the study consists on the definition of the real primes. In fact, it is a generalization of the concept of prime to the reals. The discovery of the real primes will make easier the resolution of the hypothesis since a calculus of integral will be sufficient.

The Riemann hypothesis

The zeta function of Riemann is defined as follows

$$\zeta(s)=\sum_{n=1}^{n=\infty}(\frac{1}{n^s})=1+\frac{1}{2^s}+\frac{1}{3^s}+...$$

The first result is the divergence of the harmonic serie

$$\zeta(1)=\sum_{n=1}^{n=\infty}(\frac{1}{n})=1+\frac{1}{2}+\frac{1}{3}+...$$

It has been proved in the middle age by Nicole Oresme.

In the XVIII century, Leonard Euler has discovered the main proprieties of the ζ function.

In the 1730's he conjectured after numerical calculus the following equality, which is often called the Basel problem.

$$\zeta(2)=\sum_{n=1}^{n=\infty}(\frac{1}{n^2})=1+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

Euler proved it in 1748 and introduced the ζ function. He calculated its value for the positive even numbers.

$$\zeta(2k)=\sum_{n=1}^{n=\infty}(\frac{1}{n^{2k}})=1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+...=\frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers.

Thereafter, he proved in 1744 the Euler idendity where prime numbers are related to the ζ function.

$$\zeta(s)=\sum_{n=1}^{n=\infty}(\frac{1}{n^s})=1+\frac{1}{2^s}+\frac{1}{3^s}+...=\prod_{primes}\frac{1}{1-p^{-s}}$$

Consequently he deduced the divergence of the serie of the inverse of primes.

With Bernard Riemann, s can be complex number. Riemann proved the following formula

$$\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{\frac{-(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Where

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$$

This formula demonstrates that this equation does not change if we replace s by 1-s. Thus it is symmetric | $s=\frac{1}{2}$

Riemann demonstrates that the only zeros in the $R(s)<0$ are the trivial zeros negative even numbers and that there is no zero in the $R(s)>1$.

The other zeros are the non trivial zeros. They are in the critical zone $0\le R(s)\le 1$. Riemann conjectured they are all in the critical line $R(s)=\frac{1}{2}$.

This conjecture is called the Riemann hypothesis.

They calculated numerically one billion zeros of the Riemann they are all located in the critical line.

Resolution of the Riemann hypothesis for the reals

Definition

A real number is compound if it can be written as $\prod_j p_j^{n_j}$ where p_j are primes and n_j are rationals. This decomposition in prime factors is unique. A prime real number or R-prime can be written only as p=p.1 . Thereafter, there are other real prime numbers like π, e, $\ln(2)$. Of course, it is a convention, because, if π^2 is prime π will be no more prime. It is equivalent in what will follow.

Thus $\sqrt[q]{p}=p^{\frac{1}{q}}$ is compound. Also $\sqrt[q]{p}+1=p^{\frac{1}{q}}+1$ is prime when p is prime and

$\sqrt[2^i]{p}-1=(p-1)(\sqrt[2^i]{p}+1)^{-1}(\sqrt[2^{i-1}]{p}+1)^{-1}...(\sqrt{p}+1)^{-1}$ is compound for p prime, for example.

The approach of the Riemann hypothesis

The Rieman hypothesis states that the non trivial zeros of the Riemann zeta function

$\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}$ lie on the critical line $\frac{1}{2}+iy$.

For t integer, Euler has proved that $\zeta(z)=\sum_{t=1}^{\infty}\frac{1}{t^z}=\prod_{primes}\frac{1}{1-p^{-z}}=\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}})$, it is the Euler identity. For t real, it is still true and it becomes $\prod_{R-primes}(\sum_{\alpha\in Q}\frac{1}{p^{\alpha z}})=\int_1^{\infty}\frac{dt}{t^z}=\frac{1}{1-z}\left[t^{1-z}\right]_1^{\infty}$ But But let $\zeta_1(z)$ the Riemann function for the reals and $\zeta(z)$ the Riemann function for the integers. We will see below that

$\zeta(z)=0\Rightarrow\zeta_1(z)=0$

Thus

$$\zeta(-2k)=0\Rightarrow\zeta_1(-2k)=\lim_{t\to\infty}(\frac{t^{1+2k}-1}{1+2k})=0\Rightarrow\lim_{t\to\infty}(t^{1+2k})=1$$

$$t\to\infty\Rightarrow t\sim e^u;u\to\infty$$

$$k\in N:t^{-2ki\pi}\sim e^{-2kiu\pi}=(e^{2ki\pi})^{-u}=1^{-u}$$

$$\Rightarrow\lim_{t\to\infty}(t^{-2ki\pi})=1\Rightarrow\lim_{t\to\infty}(t^{3-6ki\pi})=1\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2}-ki\pi})=e^{\frac{i2m\pi}{6}}=e^{\frac{i2m'\pi}{6}}\lim_{t\to\infty}(\mathrm{t}^{-ki\pi})$$

If

$$m\neq m'\Rightarrow\lim_{t\to\infty}(t^{-ki\pi})=e^{\frac{i2(m-m')\pi}{6}};\forall k\in N\Rightarrow\lim_{t\to\infty}(t^{-i2k\pi})=1=e^{\frac{i2(m-m')\pi}{3}}=e^{i2\pi}=e^{i2m'\pi}$$

$$\Rightarrow m=m'+3=4m'\Rightarrow m'=1;m=4\Rightarrow\lim_{t\to\infty}(t^{-ik\pi})=-1;\forall k\in N$$

Let $k=2k'\Rightarrow 1=-1$ it is impossible $\Rightarrow m=m'$. Another proof :

$$k=2k'\Rightarrow\lim_{t\to\infty}(t^{-ik\pi})=1=e^{\frac{\mathrm{i}(m-m')\pi}{3}}=e^{\frac{i2(m-m')\pi}{3}}=e^{i2\pi}=e^{\frac{i6m'\pi}{3}}\Rightarrow m=m'$$

But $\lim_{t\to\infty}(t^6)=1=\lim_{t\to\infty}(t^5)\Rightarrow\lim_{t\to\infty}(t)=1\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2}})=\pm1$ and we know there exists

$\exists y_I;\zeta(\frac{1}{2}+iy_I)=0\Rightarrow\zeta_1(\frac{1}{2}+iy_I)=0\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2}+iy_I})=1=\lim_{t\to\infty}(t^{\frac{1}{2}})$ it means

$$\lim_{t\to\infty}(t^{\frac{1}{2}-iy\pi})=1;\forall y\in N$$

Let us suppose there exists m different of 2 verifying

$$\lim_{t\to\infty}(t^{\frac{1}{m}})=\lim_{t\to\infty}(t^{\frac{1}{2}})=1\Rightarrow\lim_{t\to\infty}(t^{\frac{m-2}{2m}})=1\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2m}})=1^{\frac{1}{m-2}}$$

$$\lim_{t\to\infty}(t^{\frac{m+2}{2m}})=1\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2m}})=1^{\frac{1}{m+2}}$$

$$\Rightarrow 1^{\frac{4}{m^2-4}}=1\Rightarrow 1^{\frac{1}{m^2-4}}=1^{\frac{1}{4}}=\lim_{t\to\infty}(t^{\frac{1}{4m}})=\lim_{t\to\infty}(t^{\frac{1}{8}})$$

Thus

$$\lim_{t\to\infty}(t^{\frac{1}{2m(m-2)}})=1^{\frac{1}{m^2-4}}=\lim_{t\to\infty}(t^{\frac{1}{2m(m+2)}})$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{2}{m(m^2-4)}})=1$$

And

$$\lim_{t\to\infty}(t^{\frac{1}{m}})=1\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{\frac{1}{2}m(m^2-4)}})=1=\lim_{t\to\infty}(t^{\frac{1}{M_1}})$$

Hence

$$2M_1=m(m^2-4)\geq 10\Rightarrow 2^2M_2=2M_1({M_1}^2-4)\geq 210$$

...

$$2^nM_n=2M_1({M_1}^2-4)...({M_{n-1}}^2-4)=m(m^2-4)({M_1}^2-4)...({M_{n-1}}^2-4)\geq m.P_n.2^n$$

$$\lim_{n\to\infty}(P_n)=\infty$$

And $\lim_{t\to\infty}(t^{\frac{1}{m}})=1=e^{i2u\pi}\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{M_n}})=1=e^{i2v\pi}=e^{\frac{i2^{n+1}u\pi}{(m^2-4)...({M_{n-1}}^2-4)}},\forall n\in N$ with

$$2^nu=m(m^2-4)...({M_{n-1}}^2-4)v\geq 2^nP_nv,\forall n\in N$$

It is impossible ! The initial hypothesis is false : m does not exist ! Hence m=2, is the only value for which $\lim_{t\to\infty}(t^{\frac{1}{2}+iy\pi})=1;\forall y\in N$.

Hypothesis H :

$$H:\exists n,m\,|\lim_{t\to\infty}(t^{\frac{n}{m}})=1=\lim_{t\to\infty}(t^{\frac{1}{2}})$$

$$2n\neq m, m\neq 2, 1\leq n<m$$

$$\lim_{t\to\infty}(t^{\frac{2n-m}{2m}})=1=\lim_{t\to\infty}(t^{\frac{2n+m}{2m}})\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{2m(2n-m)}})=\lim_{t\to\infty}(t^{\frac{1}{2m(2n+m)}})=1^{\frac{1}{4n^2-m^2}}$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{m}{n(4n^2-m^2)}})=1=\lim_{t\to\infty}(t)\Rightarrow\lim_{t\to\infty}(t^{\frac{1}{4n^2-m^2}})=\lim_{t\to\infty}(t^{\frac{n}{m}})=1$$

$$4n^2-m^2\neq 2\Rightarrow\lim_{t\to\infty}(t^{\frac{n}{m}})\neq 1$$

$$2n\neq m, m\neq 2, 1\leq n<m$$

Let the hypothesis H

$$H:\exists n,m\,|\lim_{t\to\infty}(t^{\frac{n}{m}})=a=\lim_{t\to\infty}(t^{-iy})$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{n}{m}+iy})=1=\lim_{t\to\infty}(t^{\frac{1}{2}+ik\pi})$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{2n-m}{2m}})=\lim_{t\to\infty}(t^{i(k\pi-y)})=a=\lim_{t\to\infty}(t^{\frac{n}{m}})$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{2n-m}{2n}})=\lim_{t\to\infty}(t^{1-\frac{m}{2n}})=1=\lim_{t\to\infty}(t^{\frac{-m}{2n}})$$

$$\Rightarrow\lim_{t\to\infty}(t^{\frac{m}{n}})=1=\lim_{t\to\infty}(t)\Rightarrow\lim_{t\to\infty}(t^{\frac{n}{m}})=1=a\Rightarrow a=1\Rightarrow\frac{2n}{m}=1$$

$$\Rightarrow t^{\frac{1}{2}+iy}=1;\forall y\in R$$

it means that $\zeta(\frac{1}{2}+iy)=\left[\frac{1}{1-\frac{1}{2}-iy}t^{1-\frac{1}{2}-iy}\right]_1^\infty=\lim_{t\to\infty}(\frac{1}{1-\frac{1}{2}-iy}(t^{\frac{1}{2}-iy}-1))=0$

Let

$$\zeta(x+iy)=0=\left[\frac{1}{1-x-iy}t^{1-x-iy}\right]_1^\infty=\lim_{t\to\infty}(\frac{1}{1-x-iy}(t^{\frac{1}{2}-iy+\frac{1}{2}-x}-1))=\lim_{t\to\infty}(\frac{t^{\frac{1}{2}-x}-1}{1-x-iy})=0\Rightarrow x=\frac{1}{2}$$

Thus the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis is proved for the real numbers. The Riemann hypothesis is important because it gives information about the zeros of the

Riemann function and the distribution of those zeros are related to real primes !

The generalization to the integers

As

$$\int_1^\infty \frac{dt}{t^z} = \frac{1}{1-z}\left[t^{1-z}\right]_1^\infty = \sum_{t=1}^\infty \frac{1}{t^z} + \sum_{\alpha\in Q^-}^\infty \frac{1}{t'^{\alpha z}} + \sum_{\alpha\in Q^+\setminus N}^\infty \frac{1}{t'^{\alpha z}}$$

$$= \sum_{t=1}^\infty \frac{1}{t^z}.B = \prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha z}}).B = \sum_{t=1}^\infty \frac{1}{t^z} + A$$

Let now

$$\prod_{primes}(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}}) = 0$$

If B is finite, the zeta function for the reals will be equal to zero. Else

Let $\zeta_1(z)$ the Riemann function for the reals and $\zeta(z)$ the Riemann function for the integers, if $B = +\infty$ and if A is finite and

$$A > 1$$

$$(A+\zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^n\zeta(z)+A)^2 = A^2 = A^2 + (A^n\zeta(z))^2 + 2A^{n+1}\zeta(z) \Rightarrow A^{n+1}\zeta(z) = 0, \forall n$$

$$\exists\alpha \mid B \sim A^\alpha < A^n \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^n\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 \geq A$$

And if

$$1 \geq A > 0$$

$$(A^{-1}+\zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-n}\zeta(z)+A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n}\zeta(z))^2 + 2A^{-n-1}\zeta(z) \Rightarrow A^{-n-1}\zeta(z) = 0, \forall n$$

$$\exists\alpha \mid B \sim A^{-\alpha} < A^{-n} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{-n}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

Now if

$0 \geq A \geq -1$
$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$
$(A^{-2n-1}\zeta(z) + A^{-2})^2 = A^{-4} = A^{-4} + (A^{-2n-1}\zeta(z))^2 + 2A^{-2n-3}\zeta(z) \Rightarrow A^{-2n-3}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim -A^{-\alpha} < -A^{-2n-1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < -A^{-2n-1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$

And if

$A < -1$
$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$
$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim -A^{\alpha} < -A^{2n+1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$

If now $B = -\infty$ and if A is finite and

$A > 1$
$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$
$(A^n\zeta(z) + A)^2 = A^2 = A^2 + (A^n\zeta(z))^2 + 2A^{n+1}\zeta(z) \Rightarrow A^{n+1}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim -A^{\alpha} > -A^n \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^n\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 \geq A$

And if

$1 \geq A \geq 0$
$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$
$(A^{-n}\zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n}\zeta(z))^2 + 2A^{-n-1}\zeta(z) \Rightarrow A^{-n-1}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim -A^{-\alpha} > -A^{-n} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^{-n}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$

And if

$0 \geq A \geq -1$
$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$
$(A^{-2n-1}\zeta(z) + A^{-2})^2 = A^{-4} = A^{-4} + (A^{-2n-1}\zeta(z))^2 + 2A^{-2n-3}\zeta(z) \Rightarrow A^{-2n-3}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim A^{-\alpha} > A^{-2n-1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{-2n-1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$

And if

$A < -1$
$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$
$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$
$\exists \alpha \mid B \sim A^{\alpha} > A^{2n+1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$

And if $A = \pm\infty$: $B \sim A^{\alpha+1}$ hence

$$A^{\alpha m}\zeta(z)^{m} = 1$$
$$A^{n\alpha}\zeta(z)^{n+1} = 0$$
$$B = A^{\alpha+1} = A^{(m+n)\alpha} \Rightarrow A^{(m+n)\alpha}\zeta(z)^{m+n+1} = B\zeta(z)^{m+n+1} = A\zeta(z)^{m+n} \sim \frac{A^{m+n+1}}{A^{(\alpha+1)(m+n)}} = A^{1-(m+n)\alpha} = \frac{A}{B}$$
$$(m+n)\alpha = \alpha + 1$$
$$m+n-2 = \frac{1}{\alpha} - 1$$
$$B^{2}\zeta(z)^{m+n+1}A^{-1} = 1$$
$$\sim \frac{B^{2}A^{m+n}}{B^{m+n}} \Rightarrow \frac{BA}{B} = A = B^{-2+m+n}A^{1-m-n} = B^{\frac{1}{\alpha}-1}A^{-\frac{1}{\alpha}}$$
$$\Rightarrow (A^{-1}B)^{\frac{1}{\alpha}-1} = A^{1-\alpha} = A^{2} \Rightarrow B = A^{\alpha+1} = 1 < \infty$$

It is impossible !

$$\Rightarrow A \neq \infty, A \neq -\infty$$

Thus A=0 And

$$\int_{1}^{\infty}\frac{dt}{t^{x+iy}} = \frac{1}{1-x-iy}\left[t^{1-x-iy}\right]_{1}^{\infty} = \prod_{primes}\left(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}}\right).B = \prod_{primes}\left(\sum_{\alpha\in N}\frac{1}{p^{\alpha(x+iy)}}\right) + A = 0 \Rightarrow x = \frac{1}{2}$$

Thus the non trivial zeros of the Riemann funtion zeta lie in the critical line like for the reals ! It is the proof of the Riemann hypothesis !

Results

The concept of prime has been generalized to the reals. It allowed to reconsider the Riemann hypothesis under a new point of view.

Then, it has been proved for the reals after a calculus of integral and it has been generalized to the integers after an elementary calculs. Finally, the conjecture has been proved.

Discussion

As it has been said, the Riemann hypothesis is one of the most important problems of number theory. It is one of the seven problems of the millennium. Until now, the approaches never reconsidered the concept of prime as it has been done in this study.

In fact, this new approach will generalize the concept of prime to the reals. It constitutes a novelty. Therefore, the resolution will depend on a simple integral and an elementary calculus.

Conclusion

The generalization of the concept of prime is the main novelty of this study. It allows to generalize the hypothesis to the reals and to solve the conjecture.

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