Stern-Gerlach experiment, quantum phase factor, and hidden-variables theories

Koji Nagata[1] and Tadao Nakamura[2]
[1] 1-17-107 West 19 South 3, Obihiro, Hokkaido 080-2469, Japan
[2] Department of Information and Computer Science, Keio University,
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
(Dated: March 24, 2014)

We discuss whether the Stern-Gerlach experiment accepts hidden-variables theories. We discuss that the existence of two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we detect $|\uparrow\rangle$, then measurement outcome is $+1$. If we detect $|\downarrow\rangle$, then measurement outcome is -1. This hidden-variables theory does not accept the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$. Therefore we have to give up the hidden-variables theory. This implies the Stern-Gerlach experiment cannot accept the specific hidden-variables theory. And we study whether quantum phase factor accepts hidden-variables theories. We use the transition probability for two spin-1/2 pure states $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$. It is $\cos^2(\theta/2)$. We discuss that the phase factor does not accept another specific hidden-variables theory. We explore the phase factor is indeed a quantum effect, not classical. Our research gives a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

PACS numbers: 03.65.Ta, 03.65.Ud, 03.67.-a

I. INTRODUCTION

As a famous physical theory, the quantum theory (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [6], hidden-variables interpretation of the quantum theory has been an attractive topic of research [2, 3].

Leggett-type non-local hidden-variables theory [7] is experimentally investigated [8–10]. The experiments report that the quantum theory does not accept Leggett-type non-local hidden-variables theory. These experiments are done in four-dimensional space (two parties) in order to study nonlocality of hidden-variables theories.

Recently, it is shown that the two expected values of a spin-1/2 pure state $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ rule out the existence of the actually measured results of von Neumann's projective measurement [11, 12]. More recently, it is also shown that the expected value of a spin-1/2 pure state $\langle\sigma_x\rangle$ rules out the existence of the actually measured results of von Neumann's projective measurement [13].

Many researches address non-classicality of observables. And non-classicality of quantum state itself is not investigated very much (however see [14]). Here we ask: Can the Stern-Gerlach experiment accept hidden-variables theories? Surprisingly the Stern-Gerlach experiment cannot accept specific hidden-variables theory.

We try to implement the Stern-Gerlach experiment. The Stern-Gerlach experiment, named after German physicists Otto Stern and Walther Gerlach, is an important experiment in quantum mechanics on the deflection of particles. This experiment, performed in 1922, is often used to illustrate basic principles of quantum mechanics. It can be used to demonstrate that electrons and atoms have intrinsically quantum properties, and how measurement in quantum mechanics affects the system being measured.

We see a single spin-1/2 pure state is used in quantum computation, quantum cryptography and so on. As for quantum computation, we are inputting non-classical information into quantum computer. As for quantum cryptography, we are exchanging non-classical information. Further, in various quantum information processing, we control quantum state by means of Pauli observables, which are non-classical. This manuscript gives new and important insight to quantum information theory, which can be implemented only by non-classical devices.

On the other hand, in quantum mechanics, a phase factor is a complex coefficient $e^{i\theta}$ that multiplies a ket $|\psi\rangle$ or bra $\langle\phi|$. It does not, in itself, have any physical meaning, since the introduction of a phase factor does not change the expectation values of a Hermitian operator. That is, the values of $\langle\phi|A|\phi\rangle$ and $\langle\phi|e^{-i\theta}Ae^{i\theta}|\phi\rangle$ are the same. However, differences in phase factors between two interacting quantum states can sometimes be measurable (such as in the Berry phase) and this can have important consequences. In optics, the phase factor is an important quantity in the treatment of interference.

We discuss that a quantum phase factor does not accept another specific hidden-variables theory. Thus, we explore the phase factor is indeed a quantum effect, not classical. Our research gives a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

In this paper, we discuss whether the Stern-Gerlach experiment accepts hidden-variables theories. We discuss that the existence of two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we detect $|\uparrow\rangle$, then measurement outcome is $+1$. If we detect $|\downarrow\rangle$, then measurement outcome is -1. This hidden-variables theory does not accept the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$.

Therefore we have to give up the hidden-variables theory. This implies the Stern-Gerlach experiment cannot accept the specific hidden-variables theory. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) is a single one-dimensional projector. In other word, a single one-dimensional projector does not have a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are commuting with each other. Our discussion shows that we cannot assign the specific definite values (+1 and −1) to the two commuting operators, simultaneously. And we study whether quantum phase factor accepts hidden-variables theories. We use the transition probability for two spin-1/2 pure states $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$. It is $\cos^2(\theta/2)$. We discuss that the phase factor does not accept another specific hidden-variables theory. We explore the phase factor is indeed a quantum effect, not classical. Our research gives a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

II. THE STERN-GERLACH EXPERIMENT AND HIDDEN-VARIABLES THEORIES

Let σ_z be Pauli observable of z-axis. We consider two quantum states $|\uparrow\rangle$ and $|\downarrow\rangle$, which can be described as an eigenvector of Pauli observable σ_z.

We consider a quantum expected value (the transition probability) as

$$|\langle\uparrow|\downarrow\rangle|^2 = 0. \tag{1}$$

We introduce specific hidden-variables theory for the quantum expected value of the transition probability. Then, the quantum expected value given in (1) can be

$$|\langle\uparrow|\downarrow\rangle|^2 = \int d\lambda \rho(\lambda) f(\lambda). \tag{2}$$

We introduce specific quantum measurement as follows. The possible values of $f(\lambda)$ are ± 1 (in $\hbar/2$ unit). If a particle passes one side, then the value of the result of measurement is $+1$. If a particle passes through another side, then the value of the result of measurement is -1.

We have the following from the formalism of the specific quantum measurement

$$-1 \le \int d\lambda \rho(\lambda) f(\lambda) \le +1. \tag{3}$$

Thus, we can assign the truth value "1" for the proposition (2). Assume the proposition (2) is true. We have the same quantum expected value

$$|\langle\uparrow|\downarrow\rangle|^2 = \int d\lambda' \rho(\lambda') f(\lambda'). \tag{4}$$

An important note here is that the value of the right-hand-side of (2) is equal to the value of the right-hand-side of (4) because we only change the label. Thus, we can assign the truth value "1" for the proposition (4).

We derive a necessary condition for the quantum expected value given in (2). We derive the possible value of the product $|\langle\uparrow|\downarrow\rangle|^4\delta(\lambda-\lambda')$ of the quantum expected value and a delta function. The quantum expected value is $|\langle\uparrow|\downarrow\rangle|^2$ given in (2). We have

$$\begin{aligned}
&|\langle\uparrow|\downarrow\rangle|^4\delta(\lambda-\lambda')\\
&= \int d\lambda\rho(\lambda)f(\lambda) \times \int d\lambda'\rho(\lambda')f(\lambda')\delta(\lambda-\lambda')\\
&= \int d\lambda\rho(\lambda)\int d\lambda'\rho(\lambda')f(\lambda)f(\lambda')\delta(\lambda-\lambda')\\
&= \int d\lambda\rho(\lambda)(f(\lambda))^2\\
&= \int d\lambda\rho(\lambda) = 1.
\end{aligned} \tag{5}$$

Here we use the fact

$$(f(\lambda))^2 = 1 \tag{6}$$

since the possible values of $f(\lambda)$ are ± 1. Hence we derive the following proposition if we assign the truth value "1" for the two propositions (2) and (4), simultaneously.

$$|\langle\uparrow|\downarrow\rangle|^4\delta(\lambda-\lambda') = 1. \tag{7}$$

We derive a necessary condition for the quantum expected value given in (1). We derive the possible value of the product

$$|\langle\uparrow|\downarrow\rangle|^2 \times |\langle\uparrow|\downarrow\rangle|^2 \times \delta(\lambda-\lambda') = |\langle\uparrow|\downarrow\rangle|^4\delta(\lambda-\lambda'). \tag{8}$$

$\delta(\lambda-\lambda')$ is the delta function. $|\langle\uparrow|\downarrow\rangle|^2$ is the quantum expected value given in (1). We have the following proposition since $|\langle\uparrow|\downarrow\rangle|^2 = 0$

$$|\langle\uparrow|\downarrow\rangle|^4\delta(\lambda-\lambda') = 0. \tag{9}$$

We do not assign the truth value "1" for two propositions (7) and (9), simultaneously. We are in a contradiction. We give up assigning the truth value "1" for the four propositions (2), (4), (7), and (9) simultaneously. That is, we cannot assign the truth value "1" for the four propositions simultaneously

- Proposition (2) concerning hidden-variables theory
- Proposition (4) concerning hidden-variables theory with changing the label accepting proposition (2)
- Proposition (7) concerning mathematical calculations accepting two propositions (2) and (4)
- Proposition (9) concerning quantum theory

Assume we give up proposition (2). We cannot assign the specific definite values (+1 and −1) for the quantum state $|\uparrow\rangle$ and for the quantum state $|\downarrow\rangle$ simultaneously. It turns out that the single spin-1/2 pure state $|\uparrow\rangle$ and the single spin-1/2 pure state $|\downarrow\rangle$ does not have counterparts in such physical reality simultaneously. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) is a single one-dimensional projector. In other word, a single one-dimensional projector does not have a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are commuting with each other. Our discussion shows that we cannot assign the specific definite values ($+1$ and -1) to the two commuting operators, simultaneously.

In sum, we give up the following situation

$$\overbrace{|\uparrow\rangle\langle\uparrow|}^{\text{observable}} \rightarrow \overbrace{+1}^{\text{physical reality}} \quad \text{and} \quad \overbrace{|\downarrow\rangle\langle\downarrow|}^{\text{observable}} \rightarrow \overbrace{-1}^{\text{physical reality}} . \tag{10}$$

III. QUANTUM PHASE FACTOR AND HIDDEN-VARIABLES THEORIES

We study whether quantum phase factor accepts another hidden-variables theories. This discussion gives us general result of the previous section. We use the transition probability for two spin-1/2 pure states

$$\begin{aligned} |0\rangle &= (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}, \\ |\theta\rangle &= (|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2} \end{aligned} \tag{11}$$

We consider the following transition probability

$$|\langle 0|\theta\rangle|^2 = \cos^2(\theta/2). \tag{12}$$

We introduce another specific hidden-variables theory in order to explain the value of the transition probability. Then, the transition probability given in (12) is

$$|\langle 0|\theta\rangle|^2 = \int d\lambda \rho(\lambda) f(\lambda). \tag{13}$$

The possible values of $f(\lambda)$ are ± 1 (in $\hbar/2$ unit).

We have the following from the formalism of the specific hidden-variables theory

$$-1 \leq \int d\lambda \rho(\lambda) f(\lambda) \leq +1. \tag{14}$$

Thus, we can assign the truth value "1" for the proposition (13). Assume the proposition (13) is true. We have the same value of the transition probability

$$|\langle 0|\theta\rangle|^2 = \int d\lambda' \rho(\lambda') f(\lambda'). \tag{15}$$

An important note here is that the value of the right-hand-side of (13) is equal to the value of the right-hand-side of (15) because we only change the label. Thus, we can assign the truth value "1" for the proposition (15).

We derive a necessary condition for the transition probability given in (13). We derive the possible value of the product

$$|\langle 0|\theta\rangle|^2 \times |\langle 0|\theta\rangle|^2 \times \delta(\lambda - \lambda') \tag{16}$$

where $\delta(\lambda - \lambda')$ is a delta function. We have

$$\begin{aligned} &|\langle 0|\theta\rangle|^4 \delta(\lambda - \lambda') \\ &= \int d\lambda \rho(\lambda) f(\lambda) \times \int d\lambda' \rho(\lambda') f(\lambda') \delta(\lambda - \lambda') \\ &= \int d\lambda \rho(\lambda) \int d\lambda' \rho(\lambda') f(\lambda) f(\lambda') \delta(\lambda - \lambda') \\ &= \int d\lambda \rho(\lambda) (f(\lambda))^2 \\ &= \int d\lambda \rho(\lambda) = 1. \end{aligned} \tag{17}$$

Here we use the fact

$$(f(\lambda))^2 = 1 \tag{18}$$

since the possible value of $f(\lambda)$ is ± 1. Hence we derive the following proposition if we assign the truth value "1" for the two propositions (13) and (15), simultaneously

$$|\langle 0|\theta\rangle|^4 \delta(\lambda - \lambda') = 1. \tag{19}$$

We derive a necessary condition for the transition probability given in (12). We derive the possible value of the product

$$|\langle 0|\theta\rangle|^2 \times |\langle 0|\theta\rangle|^2 \times \delta(\lambda - \lambda'). \tag{20}$$

$\delta(\lambda - \lambda')$ is the delta function. We have the following proposition since the transition probability is $\cos^2(\theta/2)$

$$|\langle 0|\theta\rangle|^4 \delta(\lambda - \lambda') = \cos^4(\theta/2) \delta(\lambda - \lambda'). \tag{21}$$

We do not assign the truth value "1" for two propositions (19) and (21), simultaneously. We are in a contradiction. We have to give up assigning the truth value "1" for the four propositions (13), (15), (19), and (21) simultaneously. That is, we cannot assign the truth value "1" for the four propositions simultaneously

- Proposition (13) concerning hidden-variables theory
- Proposition (15) concerning hidden-variables theory with changing the label accepting proposition (13)
- Proposition (19) concerning mathematical calculations accepting two propositions (13) and (15)
- Proposition (21) concerning quantum theory

Assume we give up proposition (13) and we accept quantum theory. We have to give up the hidden-variables theory in order to explain the value of the transition probability $\cos^2(\theta/2)$. Thus, the quantum phase factor does not accept the specific hidden-variables theory. In the case that

$$\theta = \pi \tag{22}$$

we get the result of the previous section.

In short, we give up the following general situation

$$\overbrace{|0\rangle\langle 0|}^{\text{observable}} \to \overbrace{+1}^{\text{physical reality}} \quad \text{and} \quad \overbrace{|\theta\rangle\langle \theta|}^{\text{observable}} \to \overbrace{-1}^{\text{physical reality}} . \tag{23}$$

IV. CONCLUSIONS

In conclusions, we have discussed whether Stern-Gerlach experiment accepts hidden-variables theories. We have discussed that the existence of two spin-1/2 pure states $|\uparrow\rangle$ and $|\downarrow\rangle$ rules out the existence of probability space of specific quantum measurement. If we detect $|\uparrow\rangle$, then measurement outcome has been $+1$. If we detect $|\downarrow\rangle$, then measurement outcome has been -1. This hidden-variables theory has not accepted the transition probability $|\langle\uparrow|\downarrow\rangle|^2 = 0$. Therefore we have had to give up the hidden-variables theory. This has implied the Stern-Gerlach experiment cannot accept the specific hidden-variables theory. A single spin-1/2 pure state (e.g., $|\uparrow\rangle\langle\uparrow|$) has been a single one-dimensional projector. In other word, a single one-dimensional projector does not have had a counterpart in such physical reality, in general. The one-dimensional projectors $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ have been commuting with each other. Our discussion has shown that we cannot assign the specific definite values ($+1$ and -1) to the two commuting operators, simultaneously. And we have studied whether quantum phase factor accepts hidden-variables theories. We have used the transition probability for two spin-1/2 pure states $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)/\sqrt{2}$. It has been $\cos^2(\theta/2)$. We have discussed that the phase factor does not accept another specific hidden-variables theory. We have explored the phase factor is indeed a quantum effect, not classical. Our research has given a new insight to the quantum information processing which relies on quantum phase factor, such as Deutsch's algorithm.

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