On values of arithmetical functions at factorials I

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. The Smarandache function is a characterization of factorials, since $S(k!) = k$, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \geq 1 \text{ given}) \tag{1}$$

has $d(k!) - d((k-1)!)$ solutions, where $d(n)$ denotes the number of divisors of n. This follows from $\{x : \ S(x) = k\} = \{x : \ x|k!, \ x \nmid (k-1)!\}$. Thus, equation (1) always has at least a solution, if $d(k!) > d((k-1)!)$ for $k \geq 2$. In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions $\varphi, \sigma, d, \omega, \Omega$ at factorials. Here $\varphi(n)$ = Euler's arithmetical function, $\sigma(n)$ = sum of divisors of n, $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of total divisors of n. As it is well known, we have $\varphi(1) = d(1) = 1$, while $\omega(1) = \Omega(1) = 0$, and for $1 < \prod_{i=1}^{r} p_i^{a_i}$ ($a_i \geq 1$, p_i distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i,$$

$$d(n) = \prod_{i=1}^{r}(a_i + 1). \tag{2}$$

The functions φ, σ, d are multiplicative, ω is additive, while Ω is totally additive, i.e. φ, σ, d satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m,n) = 1$, while ω, Ω satisfy the equation $g(mn) = g(m) + g(n)$ for $(m,n) = 1$ in case of ω, and for all m, n is case of Ω (see [1]).

2. Let $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $n = \prod_{i=1}^{r} p_i^{\beta_i}$ $(\alpha_i, \beta_i \geq 0)$ be the canonical factorizations of m and n. (Here some α_i or β_i can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r}(\beta_i + 1)$$

with equality only if $\alpha_i = 0$ for all i. Thus:

$$d(mn) \geq d(n) \tag{3}$$

for all m, n, with equality only for $m = 1$.

Since $\prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \leq \prod_{i=1}^{r}(\alpha_i + 1)\prod_{i=1}^{r}(\beta_i + 1)$, we get the relation

$$d(mn) \leq d(m)d(n) \tag{4}$$

with equality only for $(n,m) = 1$.

Let now $m = k$, $n = (k-1)!$ for $k \geq 2$. Then relation (3) gives

$$d(k!) > d((k-1)!) \text{ for all } k \geq 2, \tag{5}$$

thus proving the assertion that equation (1) always has at least a solution (for $k = 1$ one can take $x = 1$).

With the same substitutions, relation (4) yields

$$d(k!) \leq d((k-1)!)d(k) \text{ for } k \geq 2 \tag{6}$$

Let $k = p$ (prime) in (6). Since $((p-1)!, p) = 1$, we have equality in (6):

$$\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime.} \tag{7}$$

3. Since $S(k!)/k! \to 0$, $\dfrac{S(k!)}{S((k-1)!)} = \dfrac{k}{k-1} \to 1$ as $k \to \infty$, one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

$$\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty. \tag{8}$$

In fact, this follows from $\sigma(k) = \sum_{d|k} d = \sum_{d|k} \frac{k}{d}$, so

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,$$

as it is known.

From the known inequality ([1]) $\varphi(n)\sigma(n) \leq n^2$ it follows

$$\frac{n}{\varphi(n)} \geq \frac{\sigma(n)}{n},$$

so $\dfrac{n!}{\varphi(n!)} \to \infty$, implying

$$\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{9}$$

Since $\varphi(n) > d(n)$ for $n > 30$ (see [2]), we have $\varphi(n!) > d(n!)$ for $n! > 30$ (i.e. $n \geq 5$), so, by (9)

$$\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{10}$$

In fact, much stronger relation is true, since $\dfrac{d(n)}{n^{\varepsilon}} \to 0$ for each $\varepsilon > 0$ $(n \to \infty)$ (see [1]). From $\dfrac{d(n!)}{n!} < \dfrac{\varphi(n!)}{n!}$ and the above remark on $\sigma(n!) > n! \log n$, it follows that

$$\limsup_{n \to \infty} \frac{d(n!)}{n!} \log n \leq 1. \tag{11}$$

These relations are obtained by very elementary arguments. From the inequality $\varphi(n)(\omega(n)+1) \geq n$ (see [2]) we get

$$\omega(n!) \to \infty \text{ as } n \to \infty \tag{12}$$

and, since $\Omega(s) \geq \omega(s)$, we have

$$\Omega(n!) \to \infty \text{ as } n \to \infty. \tag{13}$$

From the inequality $nd(n) \geq \varphi(n) + \sigma(n)$ (see [2]), and (8), (9) we have

$$d(n!) \to \infty \text{ as } n \to \infty. \tag{14}$$

This follows also from the known inequality $\varphi(n)d(n) \geq n$ and (9), by replacing n with $n!$. From $\sigma(mn) \geq m\sigma(n)$ (see [3]) with $n = (k-1)!$, $m = k$ we get

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \geq k \quad (k \geq 2) \tag{15}$$

and, since $\sigma(mn) \leq \sigma(m)\sigma(n)$, by the same argument

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \leq \sigma(k) \quad (k \geq 2). \tag{16}$$

Clearly, relation (15) implies

$$\lim_{k\to\infty} \frac{\sigma(k!)}{\sigma((k-1)!)} = +\infty. \tag{17}$$

From $\varphi(m)\varphi(n) \leq \varphi(mn) \leq m\varphi(n)$, we get, by the above remarks, that

$$\varphi(k) \leq \frac{\varphi(k!)}{\varphi((k-1)!)} \leq k, \quad (k \geq 2) \tag{18}$$

implying, by $\varphi(k) \to \infty$ as $k \to \infty$ (e.g. from $\varphi(k) > \sqrt{k}$ for $k > 6$) that

$$\lim_{k\to\infty} \frac{\varphi(k!)}{\varphi((k-1)!} = +\infty. \tag{19}$$

By writing $\sigma(k!) - \sigma((k-1)!) = \sigma((k-1)!)\left[\frac{\sigma(k!)}{\sigma((k-1)!)} - 1\right]$, from (17) and $\sigma((k-1)!) \to \infty$ as $k \to \infty$, we trivially have:

$$\lim_{k\to\infty} [\sigma(k!) - \sigma((k-1)!)] = +\infty. \tag{20}$$

In completely analogous way, we can write:

$$\lim_{k\to\infty} [\varphi(k!) - \varphi((k-1)!)] = +\infty. \tag{21}$$

4. Let us remark that for $k = p$ (prime), clearly $((k-1)!, k) = 1$, while for $k =$ composite, all prime factors of k are also prime factors of $(k-1)!$. Thus

$$\omega(k!) = \begin{cases} \omega((k-1)!k) = \omega((k-1)!) + \omega(k) & \text{if } k \text{ is prime} \\ \omega((k-1)!) & \text{if } k \text{ is composite } (k \geq 2). \end{cases}$$

Thus

$$\omega(k!) - \omega((k-1)!) = \begin{cases} 1, & \text{for } k = \text{prime} \\ 0, & \text{for } k = \text{composite} \end{cases} \tag{22}$$

Thus we have

$$\begin{aligned} &\limsup_{k\to\infty} [\omega(k!) - \omega((k-1)!)] = 1 \\ &\liminf_{k\to\infty} [\omega(k!) - \omega((k-1)!)] = 0 \end{aligned} \tag{23}$$

Let p_n be the nth prime number. From (22) we get

$$\frac{\omega(k!)}{\omega((k-1)!)} - 1 = \begin{cases} \frac{1}{n-1}, \text{if } k = p_n \\ 0, \text{if } k = \text{composite.} \end{cases}$$

Thus, we get

$$\lim_{k\to\infty} \frac{\omega(k!)}{\omega((k-1)!)} = 1. \tag{24}$$

The function Ω is totally additive, so

$$\Omega(k!) = \Omega((k-1)!k) = \Omega((k-1)!) + \Omega(k),$$

giving

$$\Omega(k!) - \Omega((k-1)!) = \Omega(k). \tag{25}$$

This implies

$$\limsup_{k\to\infty}[\Omega(k!) - \Omega((k-1)!)] = +\infty \tag{26}$$

(take e.g. $k = 2^m$ and let $m \to \infty$), and

$$\liminf_{k\to\infty}[\Omega(k!) - \Omega((k-1)!)] = 2$$

(take k = prime).

For $\Omega(k!)/\Omega((k-1)!)$ we must evaluate

$$\frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}.$$

Since $\Omega(k) \leq \frac{\log k}{\log 2}$ and by the theorem of Hardy and Ramanujan (see [1]) we have

$$\sum_{n\leq x} \Omega(n) \sim x \log\log x \quad (x \to \infty)$$

so, since $\frac{\log k}{(k-1)\log\log(k-1)} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \tag{27}$$

5. Inequality (18) applied for $k = p$ (prime) implies

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!)} = 1. \tag{28}$$

This follows by $\varphi(p) = p - 1$. On the other hand, let $k > 4$ be composite. Then, it is known (see [1]) that $k|(k-1)!$. So $\varphi(k!) = \varphi((k-1)!k) = k\varphi((k-1)!)$, since $\varphi(mn) = m\varphi(n)$ if $m|n$. In view of (28), we can write

$$\lim_{k\to\infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \tag{29}$$

For the function σ, by (15) and (16), we have for $k = p$ (prime) that $p \le \frac{\sigma(p!)}{\sigma((p-1)!)} \le \sigma(p) = p + 1$, yielding

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p-1)!)} = 1. \tag{30}$$

In fact, in view of (15) this implies that

$$\liminf_{k\to\infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k-1)!)} = 1. \tag{31}$$

By (6) and (7) we easily obtain

$$\limsup_{k\to\infty} \frac{d(k!)}{d(k)d((k-1)!)} = 1. \tag{32}$$

In fact, inequality (6) can be improved, if we remark that for $k = p$ (prime) we have $d(k!) = d((k-1)!) \cdot 2$, while for $k =$ composite, $k > 4$, it is known that $k|(k-1)!$. We apply the following

Lemma. *If* $n|m$, *then*

$$\frac{d(mn)}{d(m)} \le \frac{d(n^2)}{d(n)}. \tag{33}$$

Proof. Let $m = \prod p^{\alpha} \prod q^{\beta}$, $n = \prod p^{\alpha'}$ $(\alpha' \le \alpha)$ be the prime factorizations of m and n, where $n|m$. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha+\alpha'+1)\prod(\beta+1)}{\prod(\alpha+1)\prod(\beta+1)} = \prod\left(\frac{\alpha+\alpha'+1}{\alpha+1}\right).$$

Now $\frac{\alpha+\alpha'+1}{\alpha+1} \le \frac{2\alpha'+1}{\alpha'+1} \Leftrightarrow \alpha' \le \alpha$ as an easy calculations verifies. This immediately implies relation (33).

By selecting now $n = k$, $m = (k-1)!$, $k > 4$ composite we can deduce from (33):

$$\frac{d(k!)}{d((k-1)!)} \le \frac{d(k^2)}{d(k)}. \tag{34}$$

By (4) we can write $d(k^2) < (d(k))^2$, so (34) represents indeed, a refinement of relation (6).

References

[1] T.M. Apostol, *An introduction to analytic number theory*, Springer Verlag, 1976.

[2] J. Sándor, *Some diophantine equations for particular arithmetic functions* (Romanian), Univ. Timişoara, Seminarul de teoria structurilor, No.53, 1989, pp.1-10.

[3] J. Sándor, *On the composition of some arithmetic functions*, Studia Univ. Babeş-Bolyai Math. **34**(1989), 7-14.