# Some Elementary Algebraic Considerations Inspired by Smarandache Type Functions (II)

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**Abstract**

The paper presents new properties for some functions constructed similarly to the function $\eta : \mathbf{N}^* \to \mathbf{N}^*$, the Smarandache function, defined by $\forall n \in \mathbf{N}^*$, $\eta(n) = \min\{k | k! \text{ is divisible by } n\}$ – "Smarandache's type function".

The Smarandache $\eta$ function and its principal properties are already known in the literature of speciality. Other functions were built analogously, among which the following ones.

The function $\eta_1$. Starting from a sequence of positive integers $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ satisfying the condition

$$\forall n \in \mathbf{N}^*, \exists m_n \in \mathbf{N}^*, \forall m \geq m_n \Rightarrow n/\sigma(m) \tag{1}$$

an associated function was built $\eta_1 : \mathbf{N}^* \to \mathbf{N}^*$, defined by

$$\eta_1(n) = \min\{m_n | m_n \text{ is given by } (1)\}, \forall n \in \mathbf{N}^*. \tag{2}$$

Such sequences - possibly satisfying an extra condition - considered by G. Christol to generalise the p-adic numbers were called also multiplicative convergent to zero (m.c.z.). An example is $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ with $\sigma(n) = n!$. For $n = 6$, there is $m_6' = 4$ such that $\forall m \geq 4 \Rightarrow 6/m!$ ( $6/4!$ for $m = 4$; $6/5!$ for $m = 5$) but there is and $m_6'' = 7$ such that $\forall m \geq 7 \Rightarrow 6/m!$; because the smallest of them is $m_6 = 3$ such that $\forall m \geq 3 \Rightarrow 6/3!$, it results $\eta_1(6) = 3$.

We note that for $\sigma(n) = n!$ the associated function $\eta_1$ is just the $\eta$ function - from where the ideea of building the $\eta_i$ functions (by generalization of the sequence).

The function $\eta_2$. A sequence of positive integers $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ is called "of divisibility sequence (d.s.)" if:

$$m/n \Rightarrow \sigma(m)/\sigma(n), \tag{3}$$

and "of strong divisibility sequence (s.d.s.)" if

$$\sigma((m,n)) = (\sigma(m), \sigma(n)), \ \forall m, n \in \mathrm{N}^*, \tag{4}$$

$(m,n)$ being the greatest common factor.

(Strong divisibility sequences are studied for instance by N. Jensen in [5]. It is known that the Fibonacci sequence is a s.d.s.).

Starting from a sequence $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ satisfying the condition

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \in \mathrm{N}^*, \ m_n/m \Rightarrow n/\sigma(m) \tag{5}$$

an associated function was built that is $\eta_2 : \mathrm{N}^* \to \mathrm{N}^*$ defined by

$$\eta_2(n) = \min\{m_n | m_n \text{ is given by } (5)\}, \forall n \in \mathrm{N}^*. \tag{6}$$

If the sequence $\sigma$ is d.s. or s.d.s., the function $\eta_2$ has new properties with interesting algebraic interpretations.

We observe that in (1) appeared both the natural order ($m \geq m_n$) and the divisibility as relation of order on $\mathrm{N}^*$ ($n/\sigma(m)$) and in (5), only the divisibility as relation of order on $\mathrm{N}^*$. From the alternation of the two relations of order on $\mathrm{N}^*$ can be defined analogously two more functions $\eta_3$ and $\eta_4$. (see [1])

Starting from a sequence $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ satisfying the condition

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \in \mathrm{N}^*, m_n/m \Rightarrow n \leq \sigma(m) \tag{7}$$

an associated function was built that is $\eta_3 : \mathrm{N}^* \to \mathrm{N}^*$, defined by

$$\eta_3(n) = \min\{m_n | m_n \text{ is given by } (7)\}, \forall n \in \mathrm{N}^*. \tag{8}$$

Also, starting from a sequence $\sigma : \mathrm{N}^* \to \mathrm{N}^*$ satisfying the condition

$$\forall n \in \mathrm{N}^*, \exists m_n \in \mathrm{N}^*, \forall m \in \mathrm{N}^*, m_n \leq m \Rightarrow n \leq \sigma(m) \tag{9}$$

an associated function was built that is $\eta_4 : \mathrm{N}^* \to \mathrm{N}^*$, defined by

$$\eta_4(n) = \min\{m_n | m_n \text{ is given by } (9)\}, \forall n \in \mathrm{N}^*. \tag{10}$$

The principal properties of the functions above are divided in three groups:

I The arithmetical properties of the proper function.

II The properties of sumatory function associated to each of the numerical functions above. (see [3])

III The algebraical properties of the proper function. Thanks to the arithmetical properties, every function can be viewed as morphism (endomorphism) between certain universal algebras (we can be obtain several situations considering various operations of $\mathrm{N}^*$ ). (see [2], [4])

This paper presents a construction from group III which guides to a prolongation $s_4$ of the function $\eta_4$ for more complexe universal algebras.

If the initial sequence is s.d.s., the associated function $\eta_4$ has a series of important properties from which we retain:

$$\eta_4(\max\{a,b\}) = \max\{\eta_4(a), \eta_4(b)\}\ ; \tag{11}$$

$$\eta_4(\min\{a,b\}) = \min\{\eta_4(a), \eta_4(b)\}\, \forall a, b \in \mathrm{N}^* . \tag{12}$$

We may stand out, from other possible structures on $\mathrm{N}^*$, the universal algebra $(\mathrm{N}^*, \Omega)$ where the set of operations is $\Omega = \{\vee, \wedge, \psi_0\}$ with $\vee, \wedge : (\mathrm{N}^*)^2 \to \mathrm{N}^*$ defined by $a \vee b = \sup\{a,b\}$, $a \wedge b = \inf\{a,b\}, \forall a, b \in \mathrm{N}^*$ ( $\mathrm{N}^*$ is a lattice with the natural order) and $\psi_0 : (\mathrm{N}^*)^0 \to \mathrm{N}^*$ - a null operation that fixes 1, the unique particular element with the role of neutral element for "$\vee$": $1 = e_\vee$.

Therefore, the universal algebra $(\mathrm{N}^*, \Omega)$ is of type $\tau = \begin{pmatrix} \vee & \wedge & \psi_0 \\ 2 & 2 & 0 \end{pmatrix} = (2,2,0)$.

With the properties (11) and (12) the function $\eta_4$ is endomorphism for the universal algebra above. It can be stated

**Teorema 1** *If $\eta_4 : \mathrm{N}^* \to \mathrm{N}^*$ is the function defined by (10), endomorphism for the universal algebra $(\mathrm{N}^*, \Omega)$ and $I$ is a set, then there is a unique $s_4 : (\mathrm{N}^*)^I \to (\mathrm{N}^*)^I$, endomorphism for the universal algebra $\left((\mathrm{N}^*)^I, \Omega\right)$ so that $p_i \circ s_4 = \eta_4 \circ p_i, \forall i \in I$, where $p_j : (\mathrm{N}^*)^I \to \mathrm{N}^*$ with $\forall a = \{a_i\}_{i \in I} \in (\mathrm{N}^*)^I$, $p_j(a) = a_j, \forall j \in I$, are the canonical projections, morphisms between $\left((\mathrm{N}^*)^I, \Omega\right)$ and $(\mathrm{N}^*, \Omega)$.*

The proof can be done directly: it is shown that the correspondence $\eta_4$ is a function, endomorphism and complies with the required conditions. The operations of $\Omega$ for the universal algebra $\left((\mathbf{N}^*)^I, \Omega\right)$ are made "on components".

The algebraic properties of $s_4$ - the prolongation to more ampler universal algebra of the function $\eta_4$ - for its restriction to $\mathbf{N}^*$, could bring new properties for the function $\eta_4$ that we considered above.

The paper contents, in completion, a formula of calcul for the sumatory function $F_{\eta_2}$ of function $\eta_2$.

If the initial sequence is s.d.s., this formula is:

$$F_{\eta_2}(n) = \eta_2(1) + \sum_{\substack{h,t=1 \\ h \neq t}}^{k} [\eta_2(p_h), \eta_2(p_t)] + \sum_{\substack{h,t,q=1 \\ h \neq t \neq q}}^{k} [\eta_2(p_h), \eta_2(p_t), \eta_2(p_q)] +$$

$$+ \cdots + \eta_2(n)\,,\ \forall n = p_1 \cdot p_2 \cdots p_k,\ p_i,\ i = \overline{1,k} \text{ - prime numbers}$$

and $\eta_2(p^a) = F_{\eta_2}(p^a) - F_{\eta_2}(p^{a-1})$.

# References


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