A GENERALIZATION OF A PROBLEM OF STUPARU

by L. Seagull, Glendale Community College

Let n be a composite integer ≥ 48. Prove that between n and $S(n)$ there exist at least 5 prime numbers.

Solution:

T. Yau proved that Smarandache function has the following property:

$S(n) \leq n/2$ for any composite number $n \geq 10$,

because:

if $n = pq$, with $p < q$ and $(p, q) = 1$, then:

$S(n) = \max\{S(p), S(q)\} = S(q) \leq q = n/p \leq n/2$;

if $n = p^r$, with p prime and r integer ≥ 2, then:

$S(n) \leq pr \leq (p^r)/2 = n/2$.

(Inequation $pr \leq (p^r)/2$ doesn't hold:

for $p = 2$ and $r = 2, 3$;

as well as for $p = 3$ and $r = 2$;

but in either case $n = p^r$ is less than 10.

For $p = 2$ and $r = 4$, we have $8 \leq 16/2$;

therefore for $p = 2$ and $r \geq 5$, inequality holds because the right side is exponentially increasing while the left side is only linearly increasing,

i.e. $2r \leq (2^r)/2$ for $r \geq 4$ (1)

Similarly for $p = 3$ and $r \geq 3$,

i.e. $3r \leq (3^r)/2$ for $r \geq 3$. (2)

Both of these inequalities can be easily proved by induction.

For $p = 5$ and $r = 2$, we have $10 \leq 25/2$;

and of course for $r \geq 3$ inequality $5r \leq (5^r)/2$ will hold.

If $p \geq 7$ and $r = 2$, then $p2 \leq (p^2)/2$,

which can be also proved by induction.)

Stuparu proved, using Bertrand/Tchebychev postulate/theorem, that there exists at least one prime between n and $n/2$ {i.e. between n and $S(n)$}.

But we improve this if we apply Breusch's Theorem, which says that between n and $(9/8)n$ there exists at least one prime.

Therefore, between n and $2n$ there exist at least 5 primes,

because $(9/8)^5 = 1.802032470703125\ldots < 2$,

while $(9/8)^6 = 2.027286529541016\ldots > 2$.

References:

I. M. Radu, "Mathematical Spectrum", Vol. 27, No. 2, p. 43, 1994/5.

D. W. Sharpe, Letters to the Author, 24 February & 16 March, 1995.