

## Some Considerations Concerning the Sumatory Function Associated to Smarandache Function

by

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The Smarandache Function [4] is a numerical function  $S: N^* \rightarrow N^*$  defined by

$$S(n) = \min\{m \mid m! \text{ divisible by } n\}.$$

From the definition it results that if

$$n = p_1^{t_1} p_2^{t_2} \dots p_t^{t_t} \quad (1)$$

is the decomposition of  $n$  into primes, then

$$S(n) = \max\{S(p_i^{t_i}) \mid i = 1, 2, \dots, t\} \quad (2)$$

It is said that for every function  $f$  it can be attached the sumatory function

$$F(n) = \sum_{d \mid n} f(d) \quad (3)$$

If  $f$  is the Smarandache function and  $n = p^a$ , then

$$F_s(p^a) = \sum_{j=0}^a S(p^j) = \sum_{j=0}^a S_p(j) \quad (4)$$

In [2] it is proved that

$$S(p^j) = (p-1)j + \sigma_{[p]}(j) \quad (5)$$

where

$$j = \sum_{i=1}^{t_j} k_i^j a_i(p) \quad (6)$$

and

$$\sigma_{[p]}(j) = \sum_{i=1}^{t_j} k_i^j \quad (7)$$

is the sum of the digits of the integer  $j$ , written in the generalised scale

$$[p] : a_1(p), a_2(p), \dots, a_k(p), \dots$$

with

$$a_n(p) = \frac{p^n - 1}{p - 1}, \quad n = 1, 2, \dots$$

For example

$$p = p \cdot a_1(p);$$

$$p^p = (p-1) \cdot a_p(p) + 1 \cdot a_1(p);$$

$$p^j = (p-1) \cdot a_j(p) + 1;$$

and

$$\sigma_{[p]}(p^p) = p;$$

$$S(p^p) = p^2; \quad S(p^{p^p}) = (p-1)p^p + p.$$

In [3] it is proved that

$$F_s(p^a) = (p-1) \frac{a(a+1)}{2} + \sum_{j=1}^a \sigma_{[p]}(j) \quad (8)$$

In the following we give an algorithm to calculate the sumatory function, associated to the Smarandache function:

1. Calculating the generalised scale  $[p] : a_1(p), a_2(p), \dots, a_n(p), \dots$
2. Calculating the expression of  $a$  in the scale  $[p]$ . Let  $a_{[p]} = \overline{k_s k_{s-1} \dots k_1}$ .
3. For  $i = 1, 2, \dots, s$ 
  - 3.1. If  $k_i \neq 0$ 

then

    - 3.1.1.  $v_i = a - a_i(p) + 1$
    - 3.1.2.  $z_i = \left( \overline{k_s k_{s-1} \dots k_{i+1}} \right)_{u=a_i(p)}$
    - 3.1.3.  $h_i = v_i - z_i$

else

    - 3.1.4.  $b = \overline{k_s k_{s-1} \dots k_{i+1} - 1 p 00 \dots 0}$
    - 3.1.5.  $v_i = b - a_i(p) + 1$
    - 3.1.6.  $z_i = \left( \overline{k_s k_{s-1} \dots k_i} \right)_{u=a_i(p)}$
    - 3.1.7.  $h_i = v_i - z_i$
  - 3.2.  $A_i = \left[ \frac{h_i}{a_{i+1}(p) - a_i(p)} \right]$
  - 3.3.  $r_i = h_i - A_i (a_{i+1}(p) - a_i(p))$
  - 3.4.  $B_i = \left[ \frac{r_i}{a_i(p)} \right]$
  - 3.5.  $q_i = r_i - B_i * a_i(p)$
  - 3.6.  $S_i = A_i a_i(p) \frac{p(p-1)}{2} + A_i p + a_i(p) \frac{B_i (B_i + 1)}{2} + q_i (B_i + 1)$
4. Calculating  $F_s(p^a) = (p-1) \frac{a(a+1)}{2} + \sum_{i=1}^a S_i$ ,  $\left( \sum_{j=1}^a \sigma_{[p]}(j) = \sum_{i \geq 1} S_i(a) \right)$ .

A Pascal program has been designed to the calculus of  $F_s(p^a)$  :

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uses dos,crt;
      type tablou=array[1..100] of real;
      var a,k,amare,bmare,niu,z,alfaa,betaa,ro,r,s:tablou;
alfa,p,ik,amax,beta,suma,fsuma,u:real;
i,dim,max,j:longint;
hour,min,sec,sec100:word;
{*****}
{Calc. scale p right}
procedure bazapd(var b:tablou;var p:real;var a:real; var dim:longint);
var i:longint;
begin
  for i:=1 to 100 do
    bi]:=0;
  b[1]:=1;
  i:=0;

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repeat
  i:=i+1;
  b[i]:=b[i-1]*p+1;
until b[i]>a;
dim:=i;
end;

{*****}
{write alfa in the scale p right}
procedure nrbazapd(var a:tablou; var p:real;
var alfa:real; var k:tablou; var max:longint);
var m,i:longint;
d,r,prod:real;
begin
for i:=1 to 100 do
  k[i]:=0;
d:=alfa;
max:=trunc(ln((p-1)*d+1)/ln(p));
repeat
  m:=trunc(ln((p-1)*d+1)/ln(p));
  k[m]:=trunc(d/a[m]);
  r:=d-a[m]*k[m];
  d:=r ;
until r<p;
if r<>0 then
  k[1]:=r;
end;
{*****}
{calc. z for given i }

procedure calcz(var k:tablou;var a:tablou;
var i:longint;var u:real; var z:tablou; var p:real);
var j,i1,ind:longint;
prod:real;
begin
z[i]:=0;
ind:=1;
for j:=i+1 to max do
begin
if k[j]<>0 then
begin
prod:=1;
if ind>1 then
begin
for i1:=1 to ind-1 do
prod:=prod*p; {****}
prod:=prod*u+a[ind-1];
end
else
prod:=u;
z[i]:=z[i]+k[j]*prod;

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    end;
    ind:=ind+1;
end;
end;
{*****}
begin
clrscr;
write('    give p=');
readln(p);
write('    give alfa=');
readln(alfa);
gettime(hour,min,sec,sec100);
writeln('    Timp Start:',hour,':',min,':',sec,':',sec100);
bazapd(a,p,alfa,dim);
nrbazapd(a,p,alfa,k,max);
for i:=1 to max do
begin
if k[i]<>0 then
begin
niu[i]:=alfa-a[i]+1;
u:=a[i];
calcz(k,a,i,u,z,p);
alfaa[i]:=niu[i]-z[i];
end
else
begin
for j:=1 to max do
betaa[j]:=k[j];
betaa[i]:=p;
betaa[i+1]:=betaa[i+1]-1;
for j:=1 to i-1 do
betaa[j]:=0;
{ Write beta in the scale 10}
beta:=0;
for j:=1 to max do
beta:=beta+betaa[j]*a[j];
niu[i]:=beta-a[i]+1;
u:=a[i];
calcz(betaa,a,i,u,z,p);
alfaa[i]:=niu[i]-z[i];
end;
amare[i]:=int(alfaa[i]/(a[i+1]-a[i]));
r[i]:=alfaa[i]-amare[i]*(a[i+1]-a[i]);
bmare[i]:=int(r[i]/a[i]);
ro[i]:=r[i]-bmare[i]*a[i];
s[i]:=amare[i]*a[i]*(p*(p-1)/2)+amare[i]*p;
s[i]:=s[i]+a[i]*(bmare[i]*(bmare[i]+1)/2);
s[i]:=s[i]+ro[i]*(bmare[i]+1);
end;
suma:=0;

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for i:=1 to max do
  suma:=suma+s[i];
fsuma:=(p-1)*((alfa*(alfa+1))/2)+suma;
writeln('    fsuma=',fsuma);
gettime(hour,min,sec,sec100);
writeln('    Timp Stop:',hour,':',min,':',sec,':',sec100);
end.

```

We applied the algorithm for  $p = 3$  and  $a = 300$  we obtain

TIMES START: 10:34:1:56

TIMES STOP: 10:34:1:57

We applied the formulas [4] for  $p = 3$  and  $a = 300$  we obtain

TIMES START : 10:33:31:2

TIMES STOP: 10:33:31:95

A consequence of this work is that the proposed algorithm is faster then formula [4] .

From the Legendre formula it results that [1]

$$S_p(j) = p \binom{j - i_p(j)}{p} \text{ with } 0 \leq i_p(j) \leq \left\lfloor \frac{j-1}{p} \right\rfloor,$$

and

$$F_s(p^a) = \sum_{j=0}^a p \binom{j - i_p(j)}{p} = p \sum_{j=0}^a j - p \sum_{j=0}^a i_p(j),$$

consequently

$$F_s(p^a) = \frac{pa(a+1)}{2} - p \sum_{i=0}^a i_p(j) \quad (9)$$

In [1] it is showed that

$$i_p(j) = \frac{j - \sigma_{[p]}(j)}{p}.$$

In particular,

$$i_p(p) = 0; \quad i_p(p^t) = p^{t-1} - 1$$

and

$$\sum_{j=0}^a i_p(j) = \frac{1}{p} \left[ \sum_{j=0}^a j - \sum_{i=1}^i k_i^j \right].$$

If  $p \geq a$ , then

$$j = j \cdot a_1(p), \quad \sigma_{[p]}(j) = j, \quad (j = 1, 2, \dots, a), \quad S(p^a) = pa$$

and

$$\sum_{j=0}^a \sigma_{[p]}(j) = \frac{a(a+1)}{2}; \quad \sum_{j=0}^a i_p(j) = 0$$

$$F_s(p^a) = \frac{pa(a+1)}{2} \quad (10)$$

For example

$$F_s(11^3) = s(1) + S(11) + S(11^2) + S(11^3) = 66 \text{ or}$$

$$F_s(11^3) = \frac{11 \cdot 3 \cdot 4}{2} = 66.$$

In particular,

$$F_s(p^p) = \frac{p^2(p+1)}{2} \quad (11)$$

If  $p \leq a$ , then  $a = pQ + R$  with  $0 \leq R \leq p$ , and

$$\begin{aligned} \sum_{j=0}^a i_p(j) &= \sum_{j=0}^a \left\{ \left[ \frac{j}{p} \right] - \left[ \frac{\sigma_p(j)}{p} \right] \right\} \Rightarrow \\ \sum_{j=0}^a i_p(j) &= \frac{pQ(Q-1)}{2} + Q(R+1) - \sum_{j=p}^a \left[ \frac{\sigma_p(j)}{p} \right], \end{aligned}$$

consequently,

$$F_s(p^a) = \frac{pa(a+1)}{2} - \frac{p^2Q(Q-1)}{2} - pQ(R+1) + p \sum_{j=0}^a \left[ \frac{\sigma_{[p]}(j)}{p} \right] \quad (12)$$

In particular, for  $p = a$  then  $Q = 1$ ,  $R = 0$  and

$$F_s(p^p) = \frac{p^2(p+1)}{2} \quad (13)$$

For example,

$$F_s(3^3) = 18; \quad F_s(5^5) = 75;$$

$$F_s(n^n) = \frac{2(2^p+1)^2(2^{p-2}+1)}{27}, \text{ for } n = \frac{2^p+1}{3}, \text{ with } 3 < p \leq 31 \text{ and } p \text{ prime.}$$

If  $n = p^a q^b$  with  $p < q$  and  $p^a < q$ , then

$$F_s(p^a q^b) = \sum_{d|p^a q^b} S(d) = \sum_{i=0}^a \sum_{j=0}^b S(p^i q^j) = (a+1) \sum_{j=0}^b S(q^j) = (a+1) F_s(q^b)$$

Then :

I. If  $q \geq b$ ,

$$F_s(p^a q^b) = \frac{qb(a+1)(b+1)}{2} \quad (14)$$

II. If  $q < b$ ,

$$\begin{aligned} F_s(p^a q^b) &= \frac{(a+1)qb(b+1)}{2} - \frac{(a+1)q^2 \bar{Q}(\bar{Q}-1)}{2} - q(a+1) \bar{Q}(\bar{R}+1) + \\ & \quad q(a+1) \sum_{j=q}^b \left[ \frac{\sigma_{[q]}(j)}{q} \right] \end{aligned} \quad (15)$$

where  $b = q\bar{Q} + \bar{R}$ , with  $0 \leq \bar{R} \leq q$ .

If  $n = p^a q$ , then

$$F_s(p^a q) = \sum_{i=0}^a S(p^i) + \sum_{i=0}^a S(qp^i).$$

For  $p > q$ , then  $p^i > q$  and  $S(qp^i) = S(p^i)$  with  $i \geq 1$  consequently,

$$F_s(p^a q) = 2F_s(p^a) + S(q) - 1 \quad (16)$$

For  $p < q$ , there exists  $x < a$  with  $p^{x-1} < q < p^x$  and

$$S(qp^i) = \{S(q), \quad i = 0, 1, \dots, x-1$$

$$S(p^i), \quad i = x, \dots, a$$

consequently,

$$F_s(p^a q) = \sum_{i=0}^{x-1} S(p^i) + xS(q) + 2 \sum_{i=x}^a S(p^i)$$

or

$$F_s(p^a q) = F_s(p^{x-1}) + xS(q) + (a+x)(a-x+1)(p-1) + 2 \sum_{j=x}^a \sigma_{[p]}(j) \quad (17)$$

For example, if  $p \geq a$ , then

$$F_s(p^a q) = F_s(p^{x-1}) + xS(q) + p(x+a)(a-x+1)$$

If  $n = p^a q^k$  with  $p > q$ , then

$$F_s(p^a q^2) = \sum_{i=0}^a S(p^i) + \sum_{i=0}^a S(qp^i) + \sum_{i=0}^a S(q^2 p^i).$$

But  $S(q^k p^i) = S(p^i)$  for  $i \geq k$ , because  $\max(S(p^i), S(q^k)) = S(p^i)$  for  $i \geq k$  consequently,

$$\begin{aligned} F_s(p^a q^2) &= F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1 \\ &= 3F_s(p^a) + S(q) + S(q^2) + S(q^2 p) - p - 2 \end{aligned}$$

In short

$$\begin{aligned} F_s(p^a q) &= 2F_s(p^a) + S(q) - 1 \\ F_s(p^a q^2) &= F_s(p^a q) + F_s(p^a) + S(q^2) + S(q^2 p) - p - 1 \\ F_s(p^a q^3) &= F_s(p^a q^2) + F_s(p^a) + S(q^3) + S(q^3 p) + S(q^3 p^2) - p - 2p - 1 \\ &\dots\dots\dots \\ F_s(p^a q^k) &= F_s(p^a q^{k-1}) + F_s(p^a) + S(q^k) + S(q^k p) + S(q^k p^2) + \\ &\quad + \dots + S(q^k p^{k-1}) - p - 2p - \dots - (k-1)p - 1 \end{aligned}$$

Hence

$$\begin{aligned} F_s(p^a q^k) &= (k+1)F_s(p^a) + \sum_{i=1}^k s(q^i) + \sum_{i=2}^k S(q^i p) + \sum_{i=3}^k S(q^i p^2) + \\ &\quad + \dots + \sum_{i=k-1}^k S(q^i p^{k-2}) + S(q^k p^{k-1}) - k - \frac{pk(k^2-1)}{6} \end{aligned} \quad (18)$$

### References

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