ON A SERIES INVOLVING $S(1) \cdot S(2) \dots \cdot S(n)$

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For any positive integer n let $S(n)$ be the minimal positive integer m such that $n \mid m!$. It is known that for any $\alpha > 0$, the series

$$\sum_{n \geq 1} \frac{n^\alpha}{S(1) \cdot S(2) \cdot \dots \cdot S(n)} \tag{1}$$

is convergent, although we do not know who was the first to prove the above statement (for example, the authors of [4] credit the paper [1] appeared in 1997, while the result appears also as Proposition 1.6.12 in [2] which was written in 1996).

In this paper we show that, in fact:

Theorem.

The series

$$\sum_{n \geq 1} \frac{x^n}{S(1) \cdot S(2) \cdot \dots \cdot S(n)} \tag{2}$$

converges absolutely for every x.

Proof

Write

$$a_n = \frac{|x|^n}{S(1) \cdot S(2) \cdot \dots \cdot S(n)}. \tag{3}$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{|x|}{S(n+1)}. \tag{4}$$

But for $|x|$ fixed, the ratio $|x|/S(n+1)$ tends to zero. Indeed, to see this, choose any positive real number m, and let $n_m = \lfloor m|x| + 1 \rfloor!$. When $n > n_m$, it follows that $S(n+1) > \lfloor m|x| + 1 \rfloor > m|x|$, or $S(n+1)/|x| > m$. Since m was arbitrary, it follows that the sequence $S(n+1)/|x|$ tends to infinity.

Remarks.

1. The convergence of (2) is certainly better than the convergence of (1). Indeed, if one fixes any $x > 1$ and any α, then certainly $x^n > n^\alpha$ for n large enough.

2. The convergence of (2) combined with the root test imply that

$$(S(1) \cdot S(2) \cdot \dots \cdot S(n))^{1/n}$$

diverges to infinity. This is equivalent to the fact that the average function of the logs of S, namely

$$LS(x) = \frac{1}{x} \sum_{n \leq x} \log S(n) \qquad \text{for } x \geq 1$$

tends to infinity with x. It would be of interest to study the order of magnitude of the function $LS(x)$. We conjecture that

$$LS(x) = \log x - \log\log x + O(1). \tag{5}$$

The fact that $LS(x)$ cannot be larger than what shows up in the right side of (5) follows from a result from [3]. Indeed, in [3], we showed that

$$A(x) = \frac{1}{x} \sum_{n \le x} S(n) < 2\frac{x}{\log x} \qquad \text{for } x \ge 64. \tag{6}$$

Now the fact that $LS(x) - \log x + \log\log x$ is bounded above follows from (6) and from Jensen's inequality for the log function (or the logarithmic form of the AGM inequality). It seems to be considerably harder to prove that $LS(x) - \log x + \log\log x$ is bounded below.

3. As a fun application we mention that for every integer $k \ge 1$, the series

$$\sum_{n \ge 1} \binom{n}{k} \cdot \frac{x^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)} \tag{7}$$

is absolutely convergent. Indeed, it is a straightforward computation to verify that if one denotes by $C(x)$ the sum of the series (2), then the series (7) is precisely

$$\frac{x^k}{k!} \cdot \frac{d^k C}{dx^k}. \tag{8}$$

When $k = x = 1$ series (7) becomes precisely series (1) for $\alpha = 1$.

4. It could be of interest to study the rationality of (2) for integer values of x. Indeed, if the function S is replaced with the identity in formula (2), then one obtains the more familiar e^x whose value is irrational (in fact, transcendental) at all integer values of x. Is that still true for series (2)?

References

[1] I. Cojocaru & S. Cojocaru, "On a function in number theory", *SNJ* **8** (1997), pp. 164-169.

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[3] F. Luca, "The average Smarandache function", preprint 1998.

[4] S. Tabirca & T. Tabirca, "The convergence of Smarandache Harmonic Series", *SNJ* **9** (1998) No. 1-2, pp. 27-35.