# Smarandache Magic Squares

*Sabin Tabirca**

*Bucks University College, Computing Department

The objective of this article is to investigate the existence of magic squares made with Smarandache's numbers [Tabirca, 1998]. Magic squares have been studied intensively and many aspects concerning them have been found. Many interesting things about magic squares can be found at the following WEB page http://www.pse.che.tohoko.ac.jp/~msuzuki/MagicSquares.html.

***Definition* 1.** A Smarandache magic square is a square matrix $a \in M_n(N)$ with the following properties:

a) $\{a_{i,j} \mid i,j = \overline{1,n}\} = \{S(i) \mid i = \overline{1,n^2}\}$ (1)

b) $(\forall\, j = \overline{1,n}) \sum_{i=1}^{n} a_{i,j} = k$ (2)

c) $(\forall\, i = \overline{1,n}) \sum_{j=1}^{n} a_{i,j} = k$ (3)

Therefore, a Smarandache magic square is a square matrix by order $n$ that contains only the elements $S(1), S(2), \ldots, S(n^2)$ [Smarandache, 1980] and satisfies the sum properties (2-3). According to these properties, the sum of elements on each row or column should be equal to the same number $k$. Obviously, this number satisfies the following equation $k = \frac{\sum_{i=1}^{n^2} S(i)}{n}$.

***Theorem* 1.** If the equation $n \mid \sum_{i=1}^{n^2} S(i)$ does not hold, then there is not a Smarandache magic square by order $n$.

**Proof**

This proof is obvious by using the simple remark $k = \frac{\sum_{i=1}^{n^2} S(i)}{n} \in N$. If $a \in M_n(N)$ is a Smarandache magic square, then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ should hold. Therefore, if this equation does not hold, there is no a Smarandache magic square. ♣

Theorem 1 provides simple criteria to find the non-existence of Smarandache magic square. All the numbers $1<n<101$ that do not satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$ can be found by using a simple computation [Ibstedt, 1997]. They are $\{2, 3, \ldots, 100\} \setminus \{6, 7, 9, 58, 69\}$. Clearly, a Smarandache magic square does not exist for this numbers. If $n$ is one of the numbers 6, 7, 9, 58, 69 then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ holds [see Table 1]. This does not mean necessarily that there is a Smarandache magic square. In this case, a Smarandache magic square is found using other techniques such us detailed analysis or exhaustive computation.

| n | S(n) | $\sum_{i=1}^{n^2} S(i)$ |
|---|---|---|
| 6 | 3 | 330 |
| 7 | 7 | 602 |
| 9 | 6 | 1413 |
| 58 | 29 | 1310162 |
| 69 | 23 | 2506080 |

**Table 1.** The values of $n$ that satisfy $n \mid \sum_{i=1}^{n^2} S(i)$.

An algorithm to find a Smarandache magic square is proposed in the following. This algorithm uses *Backtracking* strategy to complete the matrix $a$ that satisfies (1-3). The going trough matrix is done line by line from the left-hand side to the right-hand side.

The algorithm computes:

- Go trough the matrix
  - Find an unused element of the set $S(1), S(2), \ldots, S(n^2)$ for the current cell.
  - If there is no an unused element, then compute a step back.
  - If there is an used element, then
    - Put this element in the current cell.
    - Check the backtracking conditions.
    - If they are verified and the matrix is full, then a Smarandache magic square has been found.
    - If they are verified and the matrix is not full, then compute a step forward.

```
procedure Smar_magic_square(n);
begin
col:=1; row:=1;a[col, row]:=0;
while row>0 do begin
   while a[col, row]<n*n do begin
         a[col,row]:=a[col,row]+1;
        call check(col,row,n,a,cont);
        if cont=0 then exit;
    end
    if cont =0 then call back(col,row);
    if cont=1 and col=n and row=n
             then call write_square(n,a)
              else call forward(col,row);
end;
write('result negative');
end;

procedure back(col, row);
begin
   col:=col-1;
   if col=0 then  begin
          col:=n;row:=row-1;
   end;
end;

procedure forward(col, row);
begin
   col:=col+1;
   if col=n+1 then  begin
          col:=1;row:=row+1;
   end;
end;

procedure write_square(n,a);
begin
  for i:=1 to n do begin
     for j:=1 to n do write (S(a[i,j]),' ');
     writeln;
  end;
  stop;
end;

procedure check(col,row,n,,k,a,cont);
begin
cont:=1; sum:=0;
for i:=1 to col do sum:=sum+S(a[i,j]);
if (sum>k) or (col=n and sum<>k) then begin
          cont:=0;
```

```
        return;
end;
sum:=0
for j:=1 to row do sum:=sum+S(a[i,j]);
if (sum>k) or (row=n and sum<>k) then begin
                cont:=0;
                return;
        end;
        end;
```

**Figure 1.** Detailed algorithm for Smarandache magic squares.

The backtracking conditions are the following:

$$\left(\forall j=\overline{1,n}\right)\sum_{i=1}^{col} a_{i,j} \le k \text{ and } \left(\forall j=\overline{1,n}\right)\sum_{i=1}^{n} a_{i,j} = k \tag{4a}$$

$$\left(\forall i=\overline{1,n}\right)\sum_{j=1}^{row} a_{i,j} \le k \text{ and } \left(\forall i=\overline{1,n}\right)\sum_{j=1}^{n} a_{i,j}' = k . \tag{4b}$$

$$(\forall (i,j) < (row,col))\, a_{i,j} \ne a_{row,col} \tag{5}$$

These conditions are checked by the procedure check. A detailed algorithm is presented in a pseudo-cod description in Figure 1.

***Theorem* 2.** If there is a Smarandache magic square by order *n*, then the procedure Smar_magic_square finds it.

**Proof**

This theorem establishes the correctness property of the procedure Smar_magic_square. The *Backtracking* conditions are computed correctly by the procedure check that verifies if the equations (4-5) hold. The correctness this algorithm is given by the correctness of the *Backtracking* strategy. Therefore, this procedure finds a Smarandache magic square.

♣

***Theorem* 3.** The complexity of the procedure complexity Smar_magic_square is $O(n^{2\cdot n^2+1})$.

**Proof**

The complexity of the procedure Smar_magic_square is studied in the worst case when there is not a Smarandache magic square. In this case, this procedure computes all the checking operations for the *Backtracking* strategy. Therefore, all the values

$S(1), S(2), \ldots, S(n^2)$ are gone through for each cell. For each value put into a cell, at most *O(n)* operations are computed by the procedure check. Therefore, the complexity is $O\left(n \cdot \left(n^2\right)^{n^2}\right) = O\left(n^{2 \cdot n^2+1}\right)$. ♣

***Remark* 1.** The complexity $O(n^{2 \cdot n^2+1})$ is not polynomial. Moreover, because this is a very big complexity, the procedure Smar_magic_square can be applied only for small values of $n$. For example, this procedure computes at most $6^{73} > 10^{56}$ operations in the case $n$=6. The above procedure has been translated into a C program that has been run on a Pentium MMX 233 machine. The execution of this program has taken more than 4 hours for $n$=6. Unfortunately, there is not a Smarandache magic square for this value of $n$. The result of computation for $n$=7 has not been provided by computer after a twelve hours execution. This reflects the huge number of operations that should be computed ($7^{99} > 10^{83}$).

According to these negative results, we believe that Smarandache magic squares do not exist. If $n$ is a big number that satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$, then we have many possibilities to change, to permute and to arrange the numbers $S(1), S(2), \ldots, S(n^2)$ into a square matrix. In spite of that Equations (2-3) cannot be satisfied. Therefore, we may conjecture the following: ***"There are not Smarandache magic squares"***. In order to confirm or infirm this conjecture, we need more powerful method than the above computation. Anyway, the computation looks for a particular solution, therefore it does not solve the problem.

**References**


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Smarandache, F (1980) A Function in Number Theory, *Analele Univ. Timisoara*, **XVIII**.

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