Smarandache - Fibonacci Triplets

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We recall the definition of the Smarandache Function S(n):

S(n) = the smallest positive integer such that S(n)! is divisible by n.

and the Fibonacci recurrence formula:

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

This article is concerned with isolated occurrencies of triplets n, n-1, n-2 for which S(n)=S(n-1)+S(n-2). Are there infinitly many such triplets? Is there a method of finding such triplets that would indicate that there are in fact infinitely many of them.

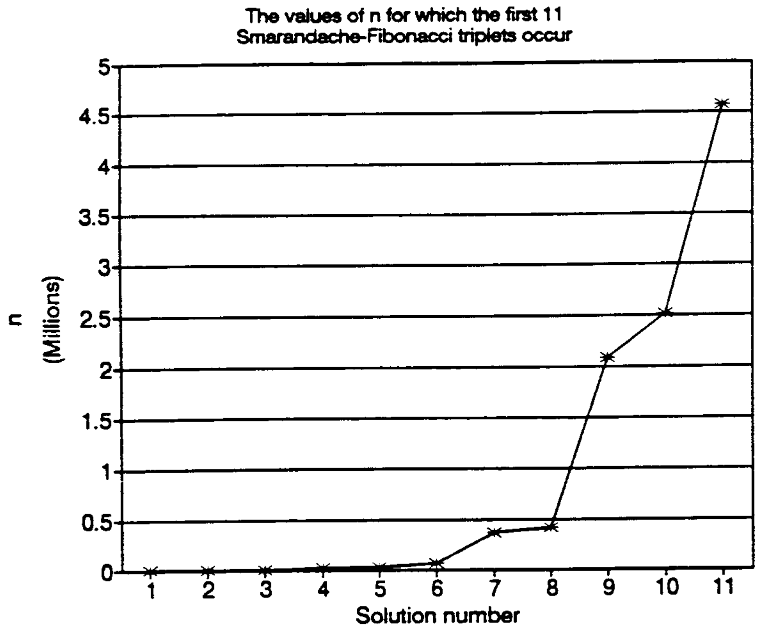
A straight forward search by applying the definition of the Smarandache Function to consecutive integers was used to identify the first eleven triplets which are listed in table 1. As often in empirical number theory this merely scratched the surface of the ocean of integers. As can be seen from diagram 1 the next triplet may occur for a value of n so large that a sequential search may be impractical and will not make us much wiser.

Table 1. The first 11 Smarandache-Fibonacci Triplets

#	n	S(n)	S(n-1)	S(n-2)
1	11	11	5	2*3
2	121	2*11	5	17
3	4902	43	29	2*7
4	26245	181	18	163
5	32112	223	197	2*13
6	64010	173	2*23	127
7	368140	233	2*41	151
8	415664	313	2*73	167
9	2091206	269	2*101	67
10	2519648	1109	2*101	907
11	4573053	569	2*53	463

However, an interesting observation can be made from the triplets already found. Apart from n=26245 the Smarandache-Fibonacci triplets have in common that one member is two times a prime number while the other two members are prime numbers. This observation

Diagram1.



leads to a method to search for Smarandache-Fibonacci triplets in which the following two theorems play a rôle:

I. If $n=ab$ with $(a,b)=1$ and $S(a)<S(b)$ then $S(n)=S(b)$.

II. If $n = p^a$ where p is a prime number and $a \le p$ then $S(p^a)=ap$.

The search for Smarandache-Fibonacci triplets will be restricted to integers which meet the following requirements:

$$n = xp^a \text{ with } a \le p \text{ and } S(x) < ap \tag{1}$$

$$n-1 = yq^b \text{ with } b \le q \text{ and } S(y) < bq \tag{2}$$

$$n-2 = zr^c \text{ with } c \le r \text{ and } S(z) < cr \tag{3}$$

p,q and r are primes. We then have $S(n)=ap$, $S(n-1)=bq$ and $S(n-2)=cr$. From this and by subtracting (2) from (1) and (3) from (2) we get

$$ap = bq + cr \tag{4}$$

$$xp^a - yq^b = 1 \tag{5}$$

$$yq^b - zr^c = 1 \tag{6}$$

TABLE 2. Smarandache - Fibonacci Triplets.

#	N	S(N)	S(N-1)	S(N-2)	t
1	4	4 *	3	2 *	0
2	11	11	5	6 *	0
3	121	22 *	5	17	0
4	4902	43	29	14 *	-4
5	32112	223	197	26 *	-1
6	64010	173	46 *	127	-1
7	368140	233	82 *	151	-1
8	415664	313	167	146 *	-8
9	2091206	269	202 *	67	-1
10	2519648	1109	202 *	907	0
11	4573053	569	106 *	463	-3
12	7783364	2591	202 *	2389	0
13	79269727	2861	2719	142 *	10
14	136193976	3433	554 *	2879	-1
15	321022289	7589	178 *	7411	5
16	445810543	1714 *	761	953	-1
17	559199345	1129	662 *	467	-5
18	670994143	6491	838 *	5653	-1
19	836250239	9859	482 *	9377	1
20	893950202	2213	2062 *	151	0
21	937203749	10501	10223	278 *	-9
22	1041478032	2647	1286 *	1361	-1
23	1148788154	2467	746 *	1721	3
24	1305978672	5653	1514 *	4139	0
25	1834527185	3671	634 *	3037	-5
26	2390706171	6661	2642 *	4019	0
27	2502250627	2861	2578 *	283	-1
28	3969415464	5801	1198 *	4603	-2
29	3970638169	2066 *	643	1423	-6
30	4652535626	3506 *	3307	199	0
31	6079276799	3394 *	2837	557	-1
32	6493607750	3049	1262 *	1787	5
33	6964546435	2161	1814 *	347	-4
34	11329931930	3023	2026 *	997	-4
35	11695098243	12821	1294 *	11527	2
36	11777879792	2174 *	1597	577	6
37	13429326313	4778 *	1597	3181	1
38	13849559620	6883	2474 *	4409	1
39	14298230970	2038 *	1847	191	7
40	14988125477	3209	2986 *	223	2
41	17560225226	4241	3118 *	1123	-2
42	18704681856	3046 *	1823	1223	4
43	23283250475	4562 *	463	4099	-10
44	25184038673	5582 *	1951	3631	-2
45	29795026777	11278 *	8819	2459	0
46	69481145903	6301	3722 *	2579	3
47	107456166733	10562 *	6043	4519	-1
48	107722646054	8222 *	6673	1549	-1
49	122311664350	20626 *	10463	10163	0
50	126460024832	6917	2578 *	4339	11
51	155205225351	8317	4034 *	4283	-5
52	196209376292	7246 *	3257	3989	-5
53	210621762776	6914 *	1567	5347	11
54	211939749997	16774 *	11273	5501	0
55	344645609138	7226 *	2803	4423	9
56	484400122414	16811	12658 *	4153	-1
57	533671822944	21089	18118 *	2971	0
58	620317662021	21929	20302 *	1627	0
59	703403257356	13147	10874 *	2273	-2
60	859525157632	14158 *	3557	10601	-5
61	898606860813	19973	13402 *	6571	1
62	972733721905	10267	10214 *	53	-4
63	1185892343342	18251	12022 *	6229	-2
64	1225392079121	12202 *	9293	2909	-4
65	1294530625810	17614 *	5807	11807	-3
66	1517767218627	11617	8318 *	3299	-8
67	1905302845042	22079	21478 *	601	-1
68	2679220490034	11402 *	7459	3943	11
69	3043063820555	14951	12202 *	2749	5
70	6098616817142	24767	20206 *	4561	2
71	6505091986039	31729	19862 *	11867	2
72	13666465868293	28099	16442 *	11657	7

The greatest common divisor $(p^a, q^b)=1$ obviously divides the right hand side of (5). This is the condition for (5) to have infinitely many solutions for each solution (p,q) to (4). These are found using Euclid's algorithm and can be written in the form:

$$x = x_0 + q^b t, \qquad y = y_0 - p^a t \tag{5'}$$

where t is an integer and (x_0,y_0) is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for z.

$$z = (yq^b - 1)/r^c \tag{6'}$$

Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and (a,b,c)=(1,1,2) with the parameter t restricted to the interval $-11 \le t \le 11$. The result is shown in table 2. Since the correctness of these calculations are easily verfied from factorisations of S(n), S(n-1), and S(n-2) these are given in table 3 for two large solutions taken from an extension of table 2.

Table 3. Factorisation of two Smarandache-Fibonacci Triplets.

n=	$16{,}738{,}688{,}950{,}356=2^2\cdot3\cdot31\cdot193\cdot\underline{15{,}269}^2$	S(n)=	$\underline{2\cdot15{,}269}$
n-1=	$16{,}738{,}688{,}950{,}355=5\cdot197\cdot1{,}399\cdot1{,}741\cdot\underline{6{,}977}$	S(n-1)=	$\underline{6{,}977}$
n-2=	$16{,}738{,}688{,}950{,}354=2\cdot7^2\cdot19\cdot23\cdot53\cdot313\cdot\underline{23{,}561}$	S(n-2)=	$\underline{23{,}561}$
n=	$19{,}448{,}047{,}080{,}036=2^2\cdot3^2\cdot43^2\cdot\underline{17{,}093}^2$	S(n)=	$\underline{2\cdot17{,}093}$
n-1=	$19{,}448{,}047{,}080{,}035=5\cdot7\cdot19\cdot37\cdot61\cdot761\cdot\underline{17{,}027}$	S(n-1)=	$\underline{17{,}027}$
n-2=	$19{,}448{,}047{,}080{,}034=2\cdot97\cdot1{,}609\cdot3{,}631\cdot\underline{17{,}159}$	S(n-1)=	$\underline{17{,}159}$

Conjecture. There are infinitely many triplets n, n-1, n-2 such that S(n)=S(n-1)+S(n-2).

Questions:

1. It is interesting to note that there are only 7 cases in table 2 where S(n-2) is two times a prime number and that they all occur for relatively small values of n. Which is the next one?

2. The solution for n=26245 stands out as a very interesting one. Is it a unique case or is it a member of family of Smarandache-Fibonacci triplets different from those studied in this article?

References:

C. Ashbacher and M. Mudge, *Personal Computer World*, October 1995, page 302.

M. Mudge, in a Letter to R. Muller (05/14/96), states that: "John Humphries of Hulse Ground Farm, Little Faringdo, Lechlade, Glovcester, GL7 3QR, U.K., has found a set of three numbers, greater than 415662, whose Smarandache Function satisfies the Fibonacci Recurrence, i.e. S(2091204) = 67, S(2091205) = 202, S(2091206) = 269, and 67 + 202 = 269.