THE SOLUTION OF SOME DIOPHANTINE EQUATIONS RELATED TO SMARANDACHE FUNCTION

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In the present note wesolve two diophantine eqations concerning the Smarandache function.

First, we try to solve the diophantine eqation :

$$S(x^y) = y^x \quad (1)$$

It is porposed as an open problem by F. Smarandache in the work [1], pp. 38 (the problem 2087).

Because $S(1) = 0$, the couple $(1,0)$ is a solution of eqation (1). If $x = 1$ and $y \geq 1$, the eqation there are no $(1,y)$ solutions. So, we can assume that $x \geq 2$. It is obvious that the couple $(2,2)$ is a solution for the eqation (1).

If we fix $y = 2$ we obtain the equation $S(x^2) = 2^x$. It is easy to verify that this eqation has no solution for $x \in \{3,4\}$, and for $x \geq 5$ we have $2^x > x^2 \geq S(x^2)$, so $2^x > S(x^2)$. Consequently for every $x \in N^* \setminus \{2\}$, the couple $(x,2)$ isn't a solution for the eqation (1).

We obtein the equation $S(2^y) = y^2$, $y \geq 3$, fixing $x = 2$. It is know that for p = prime number we have the ineqality:

$$S(p^r) \leq p \cdot r \quad (2)$$

Using the inequality (2) we obtein the inequality $S(2^y) \leq 2 \cdot y$. Because $y \geq 3$ implies $y^2 > 2y$, it results $y^2 > S(2^y)$ and we can assume that $x \geq 3$ and $y \geq 3$.

We consider the function $f: [3,\infty] \rightarrow R_+^*$ defined by $f(x) = \frac{y^x}{x^y}$, where $y \geq 3$ is fixed.

This function is derivable on the considered interval, and $f'(x) = \frac{y^x x^{y-1}(x \ln y - y)}{x^{2y}}$. In the point $x_0 = \frac{y}{\ln y}$ it is equal to zero, and $f(x_0) = f(\frac{y}{\ln y}) = y^{\frac{1}{\ln y}} (\ln y)^y$.

Because $y \geq 3$ it results that $\ln y > 1$ and $y^{\frac{1}{\ln y}} > 1$, so $f(x_0) > 1$. For $x > x_0$, the function f is strict incresing, so $f(x) > f(x_0) > 1$, that leads to $y^x > x^y \geq S(x^y)$, respectively $y^x > S(x^y)$. For $x < x_0$, the function f is strict decreasing, so $f(x) > f(x_0) > 1$ that lands to $y^x > S(x^y)$. There fore, the only solution of the eqaution (1) are the couples $(1,0)$ and $(2,2)$.