THE SECOND CONSTANT OF SMARANDACHE

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In the present note we prove that the sum of remarcable series $\sum\limits_{n\geq 2} \frac{S(n)}{n!}$, which implies the Smarandache function is an irrational number(second constant of Smarandache).

Because $S(n) \leq n$, it results $\sum\limits_{n\geq 2} \frac{S(n)}{n!} \leq \sum\limits_{n\geq 2} \frac{1}{(n-1)!}$. Therefore the serie $\sum\limits_{n\geq 2} \frac{S(n)}{n!}$ is convergent to a number f.

Proposition. The sum f of the series $\sum\limits_{n\geq 2} \frac{S(n)}{n!}$ is an irrational number.

Proof. From the precedent lines it results that $\lim\limits_{n\to\infty} \sum\limits_{i=2}^{n} \frac{S(n)}{n!} = f$. Against all reson we assume that $f \in Q$, $f > 0$. Therefore it exists $a,b \in N$, $(a,b) = 1$, so that $f = \frac{a}{b}$.

Let p be a fixed prime number, $p > b$, $p \geq 3$. Obviously, $\frac{a}{b} = \sum\limits_{i=2}^{p-1} \frac{S(i)}{i!} + \sum\limits_{i\geq p} \frac{S(i)}{i!}$ which leads to:

$$\frac{(p-1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i\geq p} \frac{(p-1)!S(i)}{i!}$$

Because $p > b$ it results that $\frac{(p-1)!a}{b} \in N$ and $\sum\limits_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in N$. Consequently we have $\sum\limits_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$ too.

Be $\alpha = \sum\limits_{i\geq p} \frac{(p-1)!S(i)}{i!} \in N$. So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} + \frac{(p-1)!S(p+2)}{(p+2)!} + \dots .$$

Because p is a prime number it results $S(p) = p$.

So

$$\alpha = 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)} + \dots > 1 \qquad (1)$$

We know that $S(p+i) \leq p+i$ $(\forall) i \geq 1$, with equality only if the number $p+i$ is prime. Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \ldots < 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \ldots = \frac{p}{p-1} < 2 \tag{2}$$

From the inequalities (1) and (2) it results that $1 < \alpha < 2$, impossible, because $\alpha \in N$. The proposition is proved.

REFERENCES

[1] **Smarandache Function Journal** , Vol.1 (1990), Vol. 2-3 (1993), Vol. 4-5 (1994), Number Theory Publishing, Co., R. Emller Editor, Phoenix, New York, Lyon.

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