THE SEMILATTICE WITH CONSISTENT RETURN

by **Ion Bălăcenoiu**, Department of Mathematics, University of Craiova,

1100, Romania

Let p be a prime number. In [5] is defined the function S_p as $S_p:N^* \to N^*, S_p(a) = k$, where k is the smallest positive integer so that p^a is a divizor for $k!$.

A Smarandache function of first kind is defined for each $n \in N^*$ in [1], as numerical function $S_n:N^* \to N^*$, so that:

i) if $n = u^i$, where $u = 1$ or $u = p$, then $S_n(a) = k$, k being the smallest positive integer with the property that $k! = M \cdot u^{ia}$.

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \cdots \cdot p_r^{i_r}$, then $\qquad S_n(a) = \max\limits_{1 \le j \le r} \left\{S_{p_j}(i_j a)\right\}$.

It is proved that:

$$\Sigma_1 \qquad \max\left\{S_n(a), S_n(b)\right\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

$$\Sigma_2 \qquad S_n(a+b) \le S_n(a) \cdot S_n(b)$$

In [2] is proved that:

i) the function S_n is monotonously increasing,

ii) the sequence of functions $\left\{S_{p^i}\right\}_{i \in N^*}$ is monotonously increasing.

iii) for p, q - prime numbers such that: $p < q \Rightarrow S_p < S_q$ and $\quad p \cdot i < q \Rightarrow S_{p^i} < S_q$, where $i \in N^*$

iv) if $n<p$, then $S_n < S_p$.

In [3] it is proved:

i) for $p \ge 5$, $S_p > \max\left\{S_{p-1}, S_{p+1}\right\}$

ii) for p, q - prime numbers, $i, j \in N^*$

$$p < q \ \text{ and } \ i \le j \ \Rightarrow \ S_{p^i} < S_{q^j}$$

iii) the sequence of functions $\left\{S_n\right\}_{n \in N^*}$ is generaly increasing boundled

iv) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \cdots \cdot p_r^{i_r}$, there are $k_1, k_2, ..., k_m \in \{1, 2, ..., r\}$ so that for each $t \in \overline{1, m}$ there is $q_t \in N^*$ so that

$$S_n(q_t) = S_{p_{k_t}^{i_{k_t}}}(q_t)$$

and for each $l \in N^*$ we have:

$$S_n(l)=\max_{1\le t\le m}\left\{S_{p_{k_t}^{i_{k_t}}}(l)\right\}.$$

We define the set $\left\{p_{k_t}^{i_{k_t}} \,\middle|\, t\in\overline{1,m}\right\}$ as the set of active factors of n and the others factors as the pasive factors.

Let $N_{p_1\cdot p_2\cdots p_r}=\left\{n=p_1^{i_1}\cdot p_2^{i_2}\cdots p_r^{i_r} \,\middle|\, i_1,i_2,\ldots,i_r\in N^*\right\}$, where $p_1<p_2<\cdots<p_r$ are prime numbers. Then

$$N^{p_1p_2\ldots p_r}=\left\{n\in N_{p_1p_2\ldots p_r} \,\middle|\, n \text{ has } p_1^{i_1},p_2^{i_2},\ldots,p_r^{i_r} \text{ as active factors}\right\}$$

is the S-active cone.

A Smarandache function of second kind is defined for each $k\in N^*$ in [1], as the function $S^k:N^*\to N^*$ where $S^k(n)=S_n(k)$.

It is proved that:

$$\Sigma_3 \qquad \max\left\{S^k(a),S^k(b)\right\}\le S^k(a\cdot b)\le S^k(a)+S^k(b)$$

$$\Sigma_4 \qquad S^k(a\cdot b)\le S^k(a)\cdot S^k(b)$$

In [4] it is proved that:

i) for $k,n\in N^*$ the formula $S^k(n)\le n\cdot k$ is true

ii) all prime numbers $p\ge 5$ are maximal points for S^k and

$$S^k(p)=p\left[k-i_p(k)\right],\quad \text{where}\quad 0\le i_p(k)\le\left[\frac{k-1}{p}\right]$$

iii) the function S^k has its relative minimum values for every $n=p!$, where p is a prime number and $p\ge\max\{3,k\}$

iv) the numbers kp for p prime number, $k\in N^*$ and $p>k$, are the fixed points of S^k

v) the function S^k have the following properties:

a) $S^k=0\quad(n^{1+\varepsilon})$, for $\varepsilon>0$

b) $\displaystyle\lim_{n\to\infty}\sup\frac{S^k(n)}{n}=k$

c) S^k is, "generally speaking", incresing, thus:

$$\forall n \in N^*, \exists m_0 \in N \text{ so that } \forall m \geq m_0 \Rightarrow S^k(m) \geq S^k(n)$$

1. **DEFINITION.** *Let* $\mathcal{M} = \{S_m(n) | n, m \in N^*\}$, *let* $A, B \in \mathcal{P}(N^*) \setminus \varnothing$ *and* $a = \min A$, $b = \min B, a^* = \max A, b^* = \max B$. *The set I is the set of the functions:*

$$I_A^B : N^* \to \mathcal{M}, \text{ with } I_A^B(n) = \begin{cases} S_a(b), n < \max\{a,b\} \\ S_{a_k}(b_k), \max\{a,b\} \leq n \leq \max\{a^k, b^k\} \\ \quad \text{where} \\ \quad a_k = \max_i \{a_i \in A | a_i \leq n\} \\ \quad b_k = \max_j \{b_j \in B | b_j \leq n\} \\ S_{a^*}(b^*), n > \max\{a^*, b^*\} \end{cases}$$

2. **EXAMPLES.**

a) $I_{\{3,8,10\}}^{\{6,10,12\}} : N^* \to \mathcal{M}$ and:

n	1	2	3	4	5	6	7	8	9	10	11	12	$n \geq 13$
$I_{\{3,8,10\}}^{\{6,10,12\}}$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_3(6)$	$S_8(6)$	$S_8(6)$	$S_{10}(10)$	$S_{10}(10)$	$S_{10}(12)$	$S_{10}(12)$

b) Let $A = \{1,3,5,\ldots,2k+1,\ldots\}$
$B = \{2,4,6,\ldots,2k,\ldots\}$

$I_A^B : N^* \to \mathcal{M}$ and:

n	1	2	3	4	5	6	...	2k	2k+1	...
I_A^B	$S_2(1)$	$S_2(1)$	$S_2(3)$	$S_4(3)$	$S_4(5)$	$S_6(5)$	...	$S_{2k}(2k-1)$	$S_{2k}(2k+1)$	...

c) Let $A = \{5,9,10\}$ and $I_A^A, I_{N^*}^{N^*} : N^* \to \mathcal{M}$ with

n	1	2	3	4	5	6	7	8	9	10	$n \geq 11$
I_A^A	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_5(5)$	$S_9(9)$	$S_{10}(10)$	$S_{10}(10)$
$I_{N^*}^{N^*}$	$S_1(1)$	$S_2(2)$	$S_3(3)$	$S_4(4)$	$S_5(5)$	$S_6(6)$	$S_7(7)$	$S_8(8)$	$S_9(9)$	$S_{10}(10)$	$S_n(n)$

It is easy to see that I_A^A is not the reduction of $I_{N^*}^{N^*}$ and $I_A^A(N^*) \subset I_{N^*}^{N^*}(N^*)$.

3. **REMARK.**

The functions whitch belongs to the set I have the folowing properties :

1) if $A_1 \subset A_2$ and $n \in A_1$, then $I_{A_1}^B(n) = I_{A_2}^B(n)$

1') if $B_1 \subset B_2$ and $n \in B_1$, then $I_A^{B_1}(n) = I_A^{B_2}(n)$

2) $I_{N^*}^{N^*}(n) = S_n(n) = S^n(n)$, the function $I_{N^*}^{N^*}$ is called the I - diagonal function and $I_{N^*}^{N^*}(N^*)$ is called the diagonal of $\mathcal{M}$.

3) for each $m \in N^*$ $I_{\{m\}}^{N^*} = S_m$ for $I_{\{m\}}^{N^*}(n) = S_m(n), \forall n \in N^*$.

3') for each $m \in N^*$ $I_{N^*}^{\{m\}} = S^m$ for $I_{N^*}^{\{m\}}(n) = S_n(m) = S^m(n), \forall n \in N^*$,

4) if $n \in A \cap B$, then $I_A^B(n) = I_{\{n\}}^{\{n\}}(n) = S_n(n)$.

4. **DEFINITION.** *For each pair $m,n \in N^*$, $S_m(n)$ and $S^m(n)$ are called the simetrical numbers relative to the diagonal of $\mathcal{M}$.*

S_m and S^m are called the simmetrical functions relative to the I-diagonal function $I_{N^*}^{N^*}$. As a rule, I_A^B and I_B^A are called the simmetrical functions relative to the I-diagonal function $I_{N^*}^{N^*}$.

5. **DEFINITION.** *Let us consider the following rule $\top : I \times I \to I$, $I_A^B \top I_C^D = I_{A \cup C}^{B \cup D}$. It is easy to see that $\top$ is idempotent, commutative and associative, so that:*

i) $I_A^B \top I_A^B = I_A^B$

ii) $I_A^B \top I_C^D = I_C^D \top I_A^B$

iii) $\left(I_A^B \top I_C^D\right) \top I_E^F = I_A^B \top \left(I_C^D \top I_E^F\right)$, where $A,B,C,D,E,F \in \mathcal{P}(N^*) \setminus \varnothing$

6. **DEFINITION.** *Let us consider the following relative partial order relation ρ, where:*

$$\rho \subset I \times I,$$
$$I_A^B \rho I_C^D \Leftrightarrow A \subset C \text{ and } B \subset D.$$

It is easy to see that $(I, \top, \rho)$ is a semilattice.

7. **DEFINITION.** *The elements $u, v \in I$ are ρ - preceded if there is $w \in I$ so that:*

$$w \rho u \ \text{and} \ w \rho v.$$

8. **DEFINITION.** *The elements $u, v \in I$, are ρ - strictly preceded by w if:*

i) $w \rho u$ *and* $w \rho v$.

ii) $\forall x \in I \setminus \{w\}$ *so that* $x \rho u$ *and* $x \rho v \Rightarrow x \rho w$.

9. **DEFINITION.** *Let us defined:*

$$I^* = \{(u, v) \in I \times I | u, v \text{ are } \rho - \text{preceded}\}$$
$$I^\# = \{(u, v) \in I \times I | u, v \text{ are } \rho - \text{strictly preceded}\}.$$

It is evidently that $(u, v) \in I^* \Leftrightarrow (v, u) \in I^*$ and $(u, v) \in I^\# \Leftrightarrow (v, u) \in I^\#$.

10. **DEFINITION.** *Let us consider* $\mathrm{T}^\# = U \times U$, $U \subset I$ and let us consider the following rule: $\perp : I^\# \to W$, $W \subset I$, $I_A^B \perp I_C^D = I_{A \wedge C}^{B \wedge D}$ and the ordering partial relation $r \subset U \times U$ so that $I_A^B \, r \, I_C^D \Leftrightarrow I_C^D \rho I_A^B$.

The structure $(I^\#, \perp, r)$ is called the return of semilattice (I, T, ρ).

11. **DEFINITION.** *The following set*

$$\mathscr{B} = \{I_A^B \in I | A \cap B \neq \varnothing\}$$

is called the base of return $(I^\#, \perp, r)$.

12. **REMARK.** The base of return has the following properties:

i) if $I_A^B \in \mathscr{B} \Rightarrow I_B^A \in \mathscr{B}$

ii) for $\varnothing \neq X \subset N^*, I_X^X \in \mathscr{B}$

iii) for $I_A^B \in \mathscr{B}$ is true the following equivalence $\varnothing \neq X \subset C_{N^*}(A \wedge B) \Leftrightarrow$ non existence of $I_X^X \perp I_A^B$.

13. **PROPOSITION.** *For $I_A^B \in \mathscr{B}$ there exists $n \in N^*$ so that $I_A^B(n) = I_{N^*}^{N^*}(n)$.*

Proof. Because $A \cap B \neq \varnothing$ it results that there exists $n \in A \wedge B$ so that:

$$I_A^B(n) = S_n(n) = I_{N^*}^{N^*}(n).$$

It results that for $I_A^B \in \mathscr{B}$ then I_A^B has at least a point of contact with I-diagonal function.

14. **REMARK.** From the 1. it results:

$$I_{\{n\}}^{B}(n)=S_n(b_n),\text{ where } b_n=\begin{cases} b,\ n<b=\min B \\ b_k,\ b\le n\le b^*=\max B \\ \qquad\text{where} \\ \qquad b_k=\max\{x\in B|x\le n\} \\ b^*\ ,n>b^* \end{cases}$$

and

$$I_{A}^{\{m\}}(m)=S^m(a_m),\text{ where } a_m=\begin{cases} a,\ m<a=\min A \\ a_k,\ a\le m\le a^*=\max A \\ \qquad\text{where} \\ \qquad a_k=\max\{x\in A|x\le m\} \\ a^*\ ,m>a^* \end{cases}$$

15. **PROPOSITION.** *There are true the following equivalences:*

$$\left(I_A^B,I_C^D\right)\in I^{\#}\Leftrightarrow I_A^C,I_B^D\in\mathscr{B}\Leftrightarrow\exists n,m\in N^*\ \textit{so that:}$$

$I_A^B(n)=I_{\{n\}}^B(n)=S_n(b_n)$, $I_C^D(n)=I_{\{n\}}^D(n)=S_n(d_n)$, $I_A^B(m)=I_A^{\{m\}}(m)=S^m(a_m)$, *and* $I_C^D(m)=I_C^{\{m\}}(m)=S^m(c_m)$ *where* a_m,b_n,c_m,d_n *are defined in the sense of 14.*

If $n\le m$, *then* $n\le a_m$, $c_m\le m$.

Proof. Evidently,
$\left(I_A^B,I_C^D\right)\in I^{\#}\Leftrightarrow A\cap C\ne\varnothing$ and $B\cap D\ne\varnothing\Leftrightarrow I_A^C,I_B^D\in\mathscr{B}$.

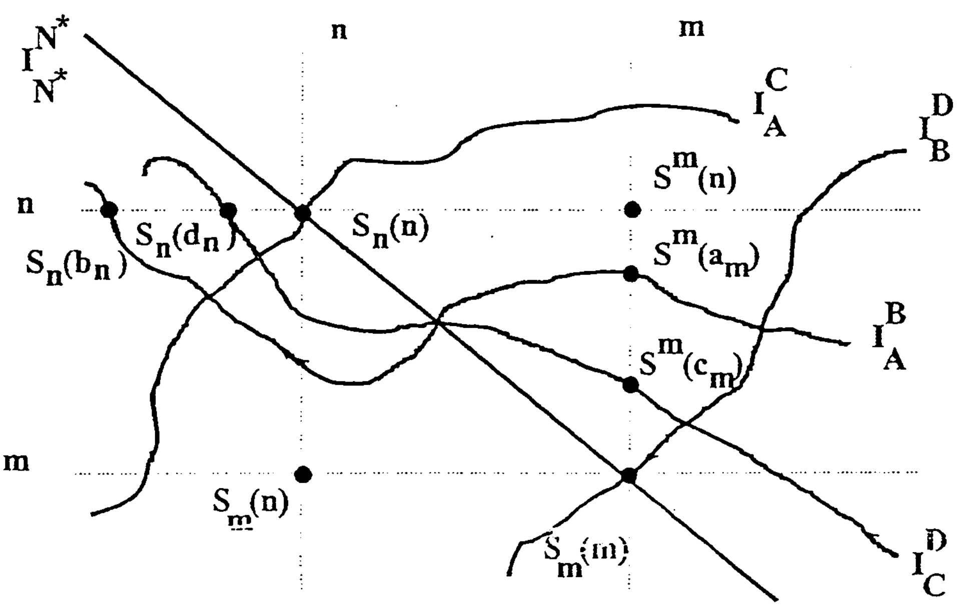
Because $A\cap C\ne\varnothing$ and $B\cap D\ne\varnothing$ it exists $n\in A\cap C$ and $m\in B\cap D$. Then:
$I_A^B(n)=I_{\{n\}}^B(n)=S_n(b_n)$, $I_C^D(n)=I_{\{n\}}^D(n)=S_n(d_n)$
$I_A^B(m)=I_A^{\{m\}}(m)=S^m(a_m)$, $I_C^D(m)=I_C^{\{m\}}(m)=S^m(c_m)$.

Conversely, if there exist $n\in N^*$ so that $I_A^B(n)=S_n(b_n)$ and $I_C^D(n)=S_n(d_n)$, then because $I_A^B(n)=S_n(b_n)$ it results $n=a_k=\max\limits_i\left\{a_i\in A|a_i\le n\right\}$, so that $n\in A$. Because $I_C^D(n)=S_n(d_n)$ it results $n\in C$.

Therefore $A\cap C\ne\varnothing$, thus, finally, $I_A^C\in\mathscr{B}$. It is also proved $I_B^D\in\mathscr{B}$ in the some way.

If $n\le m$, because $n\in A\cap C$ it results that $n\in\left\{x\in A|x\le m\right\}$ and $n\in\left\{y\in C|y\le m\right\}$, therefore $n\le a_m\le m$ and $n\le c_m\le m$.

This is presented in the following scheme:



16. **DEFINITION**. *The return* $(L^{\#},\perp,r)$ *of semillatice* $(L,\top,\rho)$ *is:*

a) null, ,if $L^{\#}=\{(u,u)|u\in L\}=\Delta_L$.

b) weak, if $\mathrm{card}L^{\#}<\mathrm{card}(L\times L\setminus L^{\#})$

c) consistent, if $\mathrm{card}L^{\#}=\mathrm{card}(L\times L-L^{\#})$

d) vigour, if $\mathrm{card}L^{\#}>\mathrm{card}(L\times L-L^{\#})$

e) total, if $L^{\#}=L\times L$.

17. **PROPOSITION**. *The return* $(I^{\#},\perp,r)$ *of the semilattice* $(I,\top,\rho)$ *is consistent.*

Proof. Evidently, $\mathrm{card}(\mathscr{P}(N^*)\setminus\varnothing)=\aleph$, $\mathrm{card}\,I=\mathrm{card}\left[(\mathscr{P}(N^*)-\varnothing)\times(\mathscr{P}(N^*)-\varnothing)\right]=\aleph$ and $\mathrm{card}(I\times I)=\aleph$.

Let us consider $\mathscr{F}=\left\{(A,C)\middle|A,C\in\mathscr{P}(N^*)-\varnothing,A\cap C=\varnothing\right\}$ and $\overline{\mathscr{F}}=\{(A,C)|A,C\in\mathscr{P}(N^*)-\varnothing,$ $A\cap C\neq\varnothing\}$.

$\mathrm{card}\mathscr{F}=\mathrm{card}\overline{\mathscr{F}}=\aleph$. Indeed, if $A\cap C=\varnothing$ it results that $C_N\cdot A\cup C_N\cdot C=N^*$; because for every $X\in P(N^*)-\varnothing$ $\exists Y=N^*\setminus X$ so that $X\cup Y=N^*$ then it results $\mathrm{card}\mathscr{F}=\mathrm{card}\mathscr{P}(N^*)=\aleph$. Because for each (A,C), $A,C\in\mathscr{P}(N^*)-\varnothing,A\cap C=\varnothing$, it exist at least two $(A_1,C_1),(A_2,C_2)$ with $A_1\cap C_1\neq\varnothing,A_2\cap C_2\neq\varnothing$ it results $\mathrm{card}\overline{\mathscr{F}}\geq\mathrm{card}\mathscr{F}=\aleph$.

Since $\operatorname{card}\overline{\mathscr{F}} \le \operatorname{card}\left[(\mathscr{P}(N^*)-\varnothing)\times(\mathscr{P}(N^*)-\varnothing)\right]=\aleph$ finally $\operatorname{card}\overline{\mathscr{F}}=\aleph$. Because $\operatorname{card}I^{\#}=\operatorname{card}(\overline{\mathscr{F}}\times\overline{\mathscr{F}})=\aleph$ and $\operatorname{card}(I\times I)-I^{\#}=\operatorname{card}(\mathscr{F}\times\mathscr{F})=\aleph$ it results that $(I^{\#},\perp,r)$ is a return consistent.

18. **REMARK.** Generaly, it is interesting the folowing problems:

i) what relations, operations, structures can be defined on

$$M=\left\{S_m(n)\middle| n,m\in N^*\right\}?$$

ii) what relations, operations, structures can be defined on

$$\mathscr{H}=\{f\,|\,f:N^*\to\mathscr{M}\}?$$

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