MOMENTS OF THE SMARANDACHE FUNCTION

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Given a positive integer n, let $P(n)$ denote the largest prime factor of n and $S(n)$ denote the smallest integer m such that n divides $m!$

This paper extends earlier work [1] on the average value of the Smarandache function $S(n)$ and is based on a recent asymptotic result [2]:

$$\left|\{n \le N: P(n) < S(n)\}\right| = o\left(\frac{N}{\ln(N)^j}\right) \quad \text{for any positive integer j}$$

due to Ford. We first prove:

Theorem 1. $$E(S(N)^k) = \frac{1}{N}\cdot\sum_{n=1}^{N} S(n)^k = \frac{\zeta(k+1)}{k+1}\cdot\frac{N^k}{\ln(k)} + O\left(\frac{N^k}{\ln(N)^2}\right)$$

where $\zeta(x)$ is the Riemann zeta function. In particular,

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot E(S(N)) = \frac{\pi^2}{12} = 0.82246703\ldots$$

$$\lim_{N\to\infty}\frac{\ln(N)}{N^2}\cdot Var(S(N)) = \frac{\zeta(3)}{3} = 0.40068563\ldots$$

Sketch of Proof. On one hand,

$$L(k) = \lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E(P(n)^k) \le \lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E(S(n)^k) = \lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\sum_{n=1}^{N} S(n)^k$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\left(\sum_{P(n)=S(n)} P(n)^k + \sum_{P(n)<S(n)} S(n)^k\right)$$

The second part vanishes:

$$\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot\left(\sum_{P(n)<S(n)}\left(\frac{S(n)}{N}\right)^k\right)\le\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot\left(\sum_{P(n)<S(n)}1\right)=\lim_{N\to\infty}\frac{\ln(N)}{N}\cdot o\left(\frac{N}{\ln(N)}\right)=0$$

while the first part is bounded from above:

$$\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\left(\sum_{P(n)=S(n)}P(n)^k\right)\le\lim_{N\to\infty}\frac{\ln(N)}{N^{k+1}}\cdot\sum_{n=1}^{N}P(n)^k=\lim_{n\to\infty}\frac{\ln(n)}{n^k}\cdot E\left(P(n)^k\right)=L(k)$$

A formula for $L(k)$ was found by Knuth and Trabb Pardo [3] and the remaining second-order details follow similarly.

Observe that the ratio $\sqrt{Var(S(N))}\,/\,E(S(N))\to\infty$ as $N\to\infty$, which indicates that the traditional sample moments are unsuitable for estimating the probability distribution of $S(N)$. An alternative estimate involves the relative number of digits in the output of S per digit in the input. A proof of the following is similar to [1]; the integral formulas were discovered by Shepp and Lloyd [4].

Theorem 2.

$$\lim_{N\to\infty}E\left(\left\{\frac{\ln(S(N))}{\ln(N)}\right\}^k\right)=\int_0^\infty\frac{x^{k-1}}{k!}\cdot\exp\left(-x-\int_x^\infty\frac{e^{-y}}{y}dy\right)dx=\begin{cases}0.62432998 & \text{if} \quad k=1\\0.42669576 & \text{if} \quad k=2\\0.31363067 & \text{if} \quad k=3\\0.24387660 & \text{if} \quad k=4\\0.19792289 & \text{if} \quad k=5\end{cases}$$

The mean output of S hence has asymptotically 62.43% of the number of digits of the input, with a standard deviation of 19.21%. A web-based essay on the Golomb-Dickman constant 0.62432998... appears in [5] and has further extensions and references.

References

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4. L. A. Shepp and S. P. Lloyd, Ordered cycle lengths in a random permutation, *Trans. Amer. Math. Soc.* 121 (1966) 350-557.
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