ON THE M-POWER COMPLEMENT NUMBERS

Zhang Xiaobeng
Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the m-power complement numbers, and give an interesting asymptotic formula for it.

§1. Introduction and results

Let $n \geq 2$ is any integer, $a_m(n)$ is called a m-power complement about n if $a_m(n)$ is the smallest integer such that $n \times a_m(n)$ is a perfect m-power. For example $a_m(2) = 2^{m-1}$, $a_m(3) = 3^{m-1}$, $a_m(4) = 2^{m-2}$, $a_m(2^m) = 1, \cdots$. The famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3, \cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of m-power complement number sequence. About this problem, some authors have studied it before. See [4]. In this paper, we use the elementary method to study the mean value properties of m-power complement number sequence, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *Let $x \geq 1$ be any real number and $m \geq 2$, then we have the asymptotic formula*

$$\sum_{n \leq x} a_m(S(n)) = \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right).$$

§2. Proof of the theorem

To complete the proof of the theorem, we need some lemmas.

Lemma 1. *If $p(n) > \sqrt{n}$, then $S(n) = p(n)$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} p(n)$; so we have

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} < \sqrt{n}$$

then

$$p_i^{\alpha_i} \mid p(n)!, \quad i = 1, 2, \cdots, r.$$

So $n \mid p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $S(n) = p(n)$.

This completes the proof of the lemma 1.

Lemma 2. *If $x \geq 1$ be any real number and $m \geq 2$, then we have the two asymptotic formulae:*

$$\sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S^{m-1}(n) = O\left(x^{\frac{m+1}{2}} \ln^{m-1} x\right);$$

$$\sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S^{m-1}(n) = \frac{x^m \zeta(m)}{m \ln x} + O\left(\frac{x^m}{\ln^2 x}\right).$$

Proof. First, from the Euler summation formula [2] we can easily get

$$\begin{aligned} & \sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S^{m-1}(n) \ll \sum_{n \leq x} (\sqrt{n} \ln n)^{m-1} \\ = & \int_1^x (\sqrt{t} \ln t)^{m-1} dt + \int_1^x (t - [t]) \left((\sqrt{t} \ln t)^{m-1}\right)' dt + (\sqrt{x} \ln x)^{m-1}(x - [x]) \\ = & \frac{m+3}{m+1} x^{\frac{m+1}{2}} \ln^{m-1} x + O\left(x^{\frac{m}{2}} \ln^{m-1} x\right). \end{aligned}$$

And then, we have

$$\begin{aligned} \sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S^{m-1}(n) &= \sum_{\substack{np \leq x \\ p > \sqrt{np}}} S^{m-1}(np) = \sum_{\substack{n \leq \sqrt{x} \\ \sqrt{n} < p \leq \frac{x}{n}}} p^{m-1} \\ &= \sum_{n \leq \sqrt{x}} \sum_{\sqrt{n} < p \leq \frac{x}{n}} p^{m-1}. \end{aligned}$$

Let $\pi(x)$ denote the number of the primes up to x. From [3], we have

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Using Abel's identity [2], we can write

$$\begin{aligned} & \sum_{\sqrt{x} < p \leq \frac{x}{n}} p^{m-1} = \pi(\frac{x}{n})(\frac{x}{n})^{m-1} - \pi(\sqrt{x})(\sqrt{x})^{m-1} - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(t)(t^{m-1})' dt \\ = & \left(\frac{x^m}{n^m(\ln x - \ln n)} + O\left(\frac{x^m}{n^m(\ln x - \ln n)^2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{2x^{\frac{m}{2}}}{\ln x} + O\left(\frac{4x^{\frac{m}{2}}}{\ln^2 x}\right)\right) - (m-1)\int_{\sqrt{x}}^{\frac{x}{n}}\left(\frac{t^{m-1}}{\ln t} + O\left(\frac{t^{m-1}}{\ln^2 x}\right)\right) dt \\
&= \frac{x^m}{mn^m \ln x} + O\left(\frac{x^m}{n^m \ln^2 x}\right).
\end{aligned}
$$

According to [2], we know that

$$
\sum_{n\le x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad if\ s > 0, s \neq 1.
$$

so we have

$$
\sum_{n\le\sqrt{x}}\ \sum_{\sqrt{n}<p\le\frac{x}{n}} p^{m-1} = \frac{x^m\zeta(m)}{m\ln x} + O\left(\frac{x^m}{\ln^2 x}\right).
$$

This completes the proof of the lemma 2.

3 Proof of the Theorem

In this section, we complete the proof of the Theorem. Combining Lemma 1, Lemma 2 and the definition of $a_m(n)$ it is clear that

$$
\begin{aligned}
\sum_{n\le x} a_m(S(n)) &= \sum_{\substack{n\le x\\ p(n)>\sqrt{n}}} p^{m-1} + O\left(\sum_{\substack{n\le x\\ p(n)\le\sqrt{n}}} (\sqrt{n}\ln n)^{m-1}\right) \\
&= \frac{x^m\zeta(m)}{m\ln x} + O\left(\frac{x^m}{\ln^2 x}\right).
\end{aligned}
$$

This completes the proof of the Theorem.

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